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ALGEBRAIC INDEPENDENCE OF CERTAIN MAHLER FUNCTIONS AND OF THEIR VALUES

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Abstract

This paper considers algebraic independence and hypertranscendence of functions satisfying Mahler-type functional equations $af(z^r) = f(z) + R(z)$, where a is a nonzero complex number, r an integer greater than 1, and $R(z)$ a rational function. Well-known results from the scope of Mahler's method then imply algebraic independence over the rationals of the values of these functions at algebraic points. As an application, algebraic independence results on reciprocal sums of Fibonacci and Lucas numbers are obtained.

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1. Introduction and main results

In the present paper, we are interested in the Mahler type functions

$$F_i(x,z) = \sum_{k=0}^{\infty} x^k \frac{A_i(z^{r^k})}{B_i(z^{r^k})}, \quad i = 0, 1, \ldots, m, \tag{1.1}$$

where $r \geq 2$ is an integer, $A_i(z), B_i(z) \in \mathbb{C}[z]\backslash\{0\}, A_i(0) = 0, B_0(z) \equiv 1$, and, for $i \geq 1$, $A_i(z)$ and $B_i(z)$ are coprime, and the $B_i(z)$ are distinct, nonconstant, and monic. Clearly

$$F_i(x,z) = xF_i(x,z^r) + \frac{A_i(z)}{B_i(z)}, \quad i = 0, 1, \ldots, m. \tag{1.2}$$

Now let $a \in \mathbb{C}$ and define $F_i(z) = F_i(a,z)$ for $i = 0, 1, \ldots, m$. Then these functions satisfy the functional equations

$$aF_i(z^r) = F_i(z) - \frac{A_i(z)}{B_i(z)}, \quad i = 0, 1, \ldots, m. \tag{1.3}$$

Further, for Fredholm series we define

$$F_0^{\langle\mu\rangle}(x,z) = \sum_{k=0}^{\infty} x^k z^{\mu r^k}, \quad \mu = 1,\dots,r-1, \tag{1.4}$$

and $F_0^{\langle\mu\rangle}(z) = F_0^{\langle\mu\rangle}(a,z)$, $\mu = 1,\dots,r-1$. Clearly, all these series $F_0^{\langle\mu\rangle}(z)$ and $F_i(z)$ converge in some disc U around the origin since $B_i(0) \neq 0$.

Let $\overline{\mathbb{Q}}$ denote the field of all complex algebraic numbers, and assume that $a \neq 0$ and the coefficients of the polynomials $A_i(z)$ and $B_i(z)$, $i = 0,1,\dots,m$, belong to $\overline{\mathbb{Q}}$. Then we may state the following theorems.

Theorem 1.1. *Let the polynomials $B_i(z)$, $i = 1,\dots,m$, satisfy the following conditions:*

(i) *if $\alpha \in \mathbb{C}$ satisfies $|\alpha| \neq 1$, then not all roots of $z^r = \alpha$ belong to $N_m = \{z : \prod_{i=1}^m B_i(z) = 0\}$;*

(ii) *if $\alpha \in \mathbb{C}$ satisfies $|\alpha| = 1$, then at most $r-2$ of the r roots of $z^r = \alpha$ belong to N_m;*

(iii) *for each $i = 1,\dots,m$, there exists γ_i such that*

$$B_i(\gamma_i) = 0 \quad \textit{and} \quad B_j(\gamma_i) \neq 0 \quad \textit{for any } j \neq i.$$

If $\zeta \in U\backslash\{0\}$ is an algebraic number such that $\zeta^{r^\nu} \notin N_m$ for $\nu = 0,1,\dots,$ then the numbers

$$F_0^{\langle 1\rangle}(\zeta),\dots,F_0^{\langle r-1\rangle}(\zeta), F_1(\zeta),\dots,F_m(\zeta)$$

are algebraically independent over $\mathbb{Q}$.

For example, if $B_1(z) = z-1$ and $B_2(z) = z+1$, then $N_2 = \{1,-1\}$. If $r \geq 3$, then Theorem 1.1 gives the algebraic independence of $F_0^{\langle 1\rangle}(\zeta),\dots,F_0^{\langle r-1\rangle}(\zeta), F_1(\zeta)$ and $F_2(\zeta)$ for all algebraic ζ with $0 < |\zeta| < 1$.

Theorem 1.2. *If the degrees of $A_1(z)$ and $B_1(z)$ are less than or equal to $r-1$, and $\zeta \in U\backslash\{0\}$ is an algebraic number such that $\zeta^{r^\nu} \notin N_1$ for $\nu = 0,1,\dots,$ then the numbers*

$$F_0^{\langle 1\rangle}(\zeta),\dots,F_0^{\langle r-1\rangle}(\zeta), F_1(\zeta)$$

are algebraically independent over $\mathbb{Q}$ except in the case

$$a = r, \quad B_1(z) = \frac{z^r - \alpha}{z-\alpha} \quad \textit{with } \alpha^{r-1} = 1, \quad A_1(z) = czB_1'(z) \quad \textit{with } c \in \overline{\mathbb{Q}}\backslash\{0\},$$

where, for all ζ with $|\zeta| < 1$,

$$\sum_{k=0}^{\infty} r^k \frac{\zeta^{r^k} B_1'(\zeta^{r^k})}{B_1(\zeta^{r^k})} = \frac{\zeta}{\alpha - \zeta}.$$

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We note that this special rational $F_1(z)$ is given in [4, Theorem 8]. By using Theorem 1.2, we may also consider the case $r = 2$ of the example after Theorem 1.1. For this, let

$$f_0(z) = f_0(a, z) = \sum_{k=0}^{\infty} a^k z^{2^k}, \quad f_1(z) = f_1(a, z) = \sum_{k=0}^{\infty} a^k \frac{z^{2^k}}{z^{2^k} - 1},$$
$$f_2(z) = f_2(a, z) = \sum_{k=0}^{\infty} a^k \frac{z^{2^k}}{z^{2^k} + 1}.$$

If ζ with $0 < |\zeta| < 1$ is an algebraic number, then Theorem 1.2 gives the algebraic independence of $f_0(\zeta)$ and $f_1(\zeta)$; the same holds for $f_0(\zeta)$ and $f_2(\zeta)$ if $a \neq 2$, but $f_2(2, \zeta) = \zeta/(1 - \zeta)$. Furthermore, $f_1(1, \zeta) - f_2(1, \zeta) = 2\zeta/(1 - \zeta)$, but if $a \neq 1$, we have no information on such relations.

Theorem 1.3. *If* $\deg A_1(z) \le r$, $\deg B_1(z) = r$, *and* $\zeta \in U \backslash \{0\}$ *is an algebraic number such that* $\zeta^{r^\nu} \notin N_1$ *for* $\nu = 0, 1, \ldots,$ *then the numbers*

$$F_0^{\langle 1 \rangle}(\zeta), \ldots, F_0^{\langle r-1 \rangle}(\zeta), F_1(\zeta)$$

are algebraically independent over $\mathbb{Q}$ *except in the following five cases:*

(1) $a \neq r$ *and*

$$B_1(z) = z^r - \alpha \quad \textit{with } \alpha^{r-1} = 1, \quad A_1(z) = c\alpha\Big(\sum_{j=1}^{r} (\alpha^{-1} z)^j - a(\alpha^{-1} z)^r\Big)$$

with $c \in \overline{\mathbb{Q}} \backslash \{0\}$;

(2) $r = 2, a = 2$ *and*

$$B_1(z) = 1 + z^2, \quad A_1(z) = cz^2 \quad \textit{with } c \in \overline{\mathbb{Q}} \backslash \{0\};$$

(3) $r = 2, a = 4$ *and*

$$B_1(z) = (1 + z)^2, \quad A_1(z) = cz \quad \textit{with } c \in \overline{\mathbb{Q}} \backslash \{0\};$$

(4) $r = 2, a = -2$ *and*

$$B_1(z) = 1 - z + z^2, \quad A_1(z) = cz \quad \textit{with } c \in \overline{\mathbb{Q}} \backslash \{0\};$$

(5) $r = 2, a = 2$ *and*

$$B_1(z) = 1 - z + z^2, \quad A_1(z) = cz(1 - 2z) \quad \textit{with } c \in \overline{\mathbb{Q}} \backslash \{0\}.$$

Moreover, the following equations hold for any ζ *with* $|\zeta| < 1$:

$$\sum_{k=0}^{\infty} \frac{\sum_{j=1}^{r} (\alpha^{-1}\zeta)^{jr^k} - a(\alpha^{-1}\zeta)^{r^{k+1}}}{\zeta^{r^{k+1}} - \alpha} = \frac{\zeta}{\alpha^{-1}\zeta - 1},$$

$$\sum_{k=0}^{\infty} 2^k \frac{\zeta^{2^{k+1}}}{\zeta^{2^{k+1}} + 1} = \frac{\zeta^2}{1 - \zeta^2}, \quad \sum_{k=0}^{\infty} \frac{4^k \zeta^{2^k}}{(1 + \zeta^{2^k})^2} = \frac{\zeta}{(1 - \zeta)^2},$$

$$\sum_{k=0}^{\infty} \frac{(-2)^k \zeta^{2^k}}{1 - \zeta^{2^k} + \zeta^{2^{k+1}}} = \frac{\zeta}{1 + \zeta + \zeta^2}, \quad \sum_{k=0}^{\infty} \frac{2^k \zeta^{2^k}(1 - 2\zeta^{2^k})}{1 - \zeta^{2^k} + \zeta^{2^{k+1}}} = \frac{\zeta(1 + 2\zeta)}{1 + \zeta + \zeta^2}.$$

Remark 1.4. The above cases of rational $F_1(z)$ are not all new; for the special case $a = 1$ of (1), see [4, Theorem 9], and for (3) and (4), see [3, Theorem 1.1].

Theorem 1.5. *Assume that t and u are positive integers. Let $\alpha_1, \ldots, \alpha_m, \alpha \in \overline{\mathbb{Q}}$ satisfy $|\alpha_i| \neq 1, |\alpha_i| \neq |\alpha_j|$ for all $i \neq j, |\alpha| = 1$, and let $B_i(z) = z^t - \alpha_i, i = 1, \ldots, m$, and $B_{m+1}(z) = z^u - \alpha$. If $\zeta \in U\backslash\{0\}$ is an algebraic number such that $\zeta^{r^\nu} \notin N_m$ for $\nu = 0, 1, \ldots,$ then the numbers*

$$F_0^{\langle 1\rangle}(\zeta), \ldots, F_0^{\langle r-1\rangle}(\zeta), F_1(\zeta), \ldots, F_m(\zeta)$$

are algebraically independent over $\mathbb{Q}$. Further, if $r \geq 3$ and u is not divisible by r, then these numbers together with $F_{m+1}(\zeta)$ are algebraically independent over $\mathbb{Q}$.

Assume now that $\deg A_0(z) \leq r - 1$, and denote by $g_m(z)$ the typical linear form

$$c_0F_0(z) + c_1F_1(z) + \cdots + c_mF_m(z) \tag{1.5}$$

in the functions $F_0(z), F_1(z), \ldots, F_m(z)$ with $(c_0, c_1, \ldots, c_m) \in \mathbb{C}^{m+1}\backslash\{\underline{0}\}$. We shall prove in Section 3 below that, under the assumptions of Theorem 1.1, the function $g_m(z)$ is hypertranscendental, that is, $g_m(z), g_m'(z), g_m''(z), \ldots$ are algebraically independent over $\mathbb{C}(z)$. The same holds for $g_1(z)$, if the assumptions of Theorem 1.2 or Theorem 1.3 are satisfied and we leave aside the exceptional cases of these theorems. Furthermore, if the assumptions of Theorem 1.5 are valid, then $g_m(z)$ is hypertranscendental as $g_{m+1}(z)$, if $r \geq 3$ and u is not divisible by r. For all these hypertranscendental functions, we can state the following result generalizing earlier ones (see [8–10]).

Theorem 1.6. *If $(c_0, c_1, \ldots, c_m) \in \overline{\mathbb{Q}}^{m+1}\backslash\{\underline{0}\}$, $g_m(z)$ is hypertranscendental, and $\zeta \in U\backslash\{0\}$ is an algebraic number such that $\zeta^{r^\nu} \notin N_m$ for $\nu = 0, 1, \ldots,$ then the numbers $g_m(\zeta), g_m'(\zeta), g_m''(\zeta), \ldots$ are algebraically independent over $\mathbb{Q}$.*

Remark 1.7. It will turn out in Section 2 that our function $g_m(z)$ introduced in (1.5) satisfies a functional equation of type $g(z^r) = a_1(z)g(z) + a_0(z)$ with $a_0, a_1 \in \mathbb{C}(z)$. For nonrational solutions of such equations, Nishioka [11] established conditions that guarantee the hypertranscendence of these solutions (and which we will use). It would be of great interest to have a similar hypertranscendence criterion for more general classes of Mahler-type functions, for example, for solutions of

$$g(z^r) = a_n(z)g(z)^n + \cdots + a_0(z) \tag{1.6}$$

with integer $n \geq 2$ and $a_0, \ldots, a_n \in \mathbb{C}(z)$. Note that here we know from [12] that all nonrational solutions are transcendental such that it is natural to ask the following open question.

Open Question. Are nonrational solutions of (1.6) always hypertranscendental?

The analogous problem could be posed for solutions of even more general Mahler-type functional equations as $P(z, f(z), f(z^r)) = 0$, P a polynomial in three variables. These questions are very much related to one asked by Hilbert [5] in his famous collection of 23 problems. There, in the text inserted between Problems 18 and 19, he suggests that we 'consider … the class of functions characterized by … algebraic differential equations'. And he continues: 'It should be observed that this class does not contain the functions that arise in number theory and whose investigation is of greatest importance. For example, the [Riemann zeta] function satisfies no algebraic differential equation'.

Our next result studies partial derivatives of (1.1) and (1.4), so let

$$F_{i,j}(x,z)=\left(\frac{\partial}{\partial x}\right)^{j}F_i(x,z),\qquad F_{0,j}^{\langle\mu\rangle}(x,z)=\left(\frac{\partial}{\partial x}\right)^{j}F_0^{\langle\mu\rangle}(x,z).$$

For distinct $a_1,\dots,a_s\in\overline{\mathbb{Q}}\backslash\{0\}$, we define

$$F_{i,j,k}(z)=F_{i,j}(a_k,z),\qquad F_{0,j,k}^{\langle\mu\rangle}(z)=F_{0,j}^{\langle\mu\rangle}(a_k,z). \tag{1.7}$$

Then we get the following generalization of Theorem 1.1.

THEOREM 1.8. *Assume that conditions (i)–(iii) of Theorem 1.1 are satisfied. If $\zeta\in U\backslash\{0\}$ is an algebraic number such that $\zeta^{r^\nu}\notin N_m$ for $\nu=0,1,\dots,$ then the numbers*

$$F_{0,j,k}^{\langle 1\rangle}(\zeta),\dots,F_{0,j,k}^{\langle r-1\rangle}(\zeta),F_{1,j,k}(\zeta),\dots,F_{m,j,k}(\zeta)\quad(j=0,\dots,\ell;\ k=1,\dots,s)$$

are algebraically independent over $\mathbb{Q}$.

As our final results of this section, we give some applications to the (ordinary) Fibonacci and Lucas sequences (F_n) and (L_n) defined by $F_0=0,F_1=1,F_n=F_{n-1}+F_{n-2}$ for $n\geq 2$, and $L_0=2,L_1=1,L_n=L_{n-1}+L_{n-2}$ for $n\geq 2$.

THEOREM 1.9. *Let $d\geq 1$ be an integer, and assume that $r\geq 3$. If $a_1,\dots,a_s$ are distinct nonzero algebraic numbers and $\rho_1=(1+\sqrt{5})/2$, then the numbers*

$$F_{0,j}^{\langle 1\rangle}(a_k,\rho_1^{-d}),\dots,F_{0,j}^{\langle r-1\rangle}(a_k,\rho_1^{-d}),\quad \varphi_{\nu,j,k}=\sum_{\substack{h=0\\ dr^h+\nu>0}}^{\infty}\frac{h^j a_k^h}{F_{dr^h+\nu}},$$

$$\lambda_{\nu,j,k}=\sum_{\substack{h=0\\ dr^h+\nu>0}}^{\infty}\frac{h^j a_k^h}{L_{dr^h+\nu}}\quad(j=0,\dots,\ell;\ k=1,\dots,s;\ \nu\in\mathbb{Z})$$

are algebraically independent over $\mathbb{Q}$. In particular, the numbers

$$\sum_{\substack{h=0\\ dr^h+\nu>0}}^{\infty}\frac{1}{F_{dr^h+\nu}},\qquad \sum_{\substack{h=0\\ dr^h+\nu>0}}^{\infty}\frac{1}{L_{dr^h+\nu}},\qquad \nu\in\mathbb{Z},$$

are algebraically independent over $\mathbb{Q}$.

THEOREM 1.10. *Let the assumptions be as in Theorem 1.9, but now with $r = 2$. Further, for each pair (ν, k) with $\nu \neq 0$, let $g_{\nu,j,k}$ denote either $\varphi_{\nu,j,k}$ or $\lambda_{\nu,j,k}$, $j = 0, \dots, \ell$. Then the numbers*

$$F_{0,j}^{\langle 1\rangle}(a_k, \rho_1^{-d}), \quad \lambda_{0,j,k}, g_{1,j,k}, g_{-1,j,k}, g_{2,j,k}, g_{-2,j,k}, \dots \quad (j = 0, \dots, \ell;\ k = 1, \dots, s)$$

are algebraically independent over $\mathbb{Q}$.

We briefly compare the statements of Theorems 1.9 and 1.10 concerning algebraic independence of series of type $\sum_{h\geq 0} h^j a_k^h / R_{dr^h+\nu}$, R the Fibonacci or Lucas sequence, with previous literature, where seemingly numbers of type $F_0^{\langle i\rangle}(a_k, \rho_1^{-d})$, $i = 1, \dots, r-1$, were never included. Most closely related to ours are, to the best of our knowledge, [6, Theorem 1.2] and the results in [14]. We quote here [6, Theorem 1.2] in a slightly abridged form as follows.

Let $(R_n)_{n=0,1,\dots}$ be a binary linear recurrence defined by $R_n = AR_{n-1} + BR_{n-2}$ $(n \geq 2)$ with integers A, B, R_0, R_1 satisfying $A^2 + 4B > 0$ and $(R_0, R_1) \neq (0, 0)$. Let $r \geq 2$ be an integer, assume $a_1, \dots, a_s \in \overline{\mathbb{Q}}\backslash\{0\}$ to be distinct, and let $b \in \overline{\mathbb{Q}}$. Then the numbers

$$\sum_{h=0}^{\infty}{}' \frac{a_k^h}{R_{r^h} + b} \quad (k = 1, \dots, s)$$

are algebraically independent except in eight precisely described cases (not to be quoted here).

The dash at $\sum$ indicates that those h producing a vanishing denominator have to be omitted. The authors claim that their proof still can be applied to similar series, where the subscript r^h of R in the denominator is replaced by $dr^h + \nu$ for fixed integers $d > 0$ and ν. In [14], it is proved that *if (R_n) is not a geometric progression, then the numbers*

$$\sum_{h=0}^{\infty}{}' \frac{h^j a_k^h}{R_{dr^h+\nu}} \quad (k = 1, \dots, s;\ \nu \in \mathbb{Z};\ j = 0, 1, \dots)$$

are algebraically independent except in two precisely described cases having $r = 2$.

It should be pointed out that our proofs of Theorems 1.9 and 1.10 would also allow inclusion of a fixed algebraic summand b in the denominator of each term of the series $\varphi_{\nu,j,k}, \lambda_{\nu,j,k}$. Moreover, our method could handle more general binary linear recurrences as well.

To conclude these remarks, we mention that the papers [15–17] contain more results on the algebraic independence of certain types of reciprocal sums of linear recurrences. Concerning 'only' transcendence of such sums, we refer the reader to [3, 4], [6, Theorem 1.1], and, in particular, to Kurosawa's paper [7] containing an extensive list of references.

2. Linear independence

Our subsequent proofs of Theorems 1.1–1.3, 1.5, 1.6 and 1.8 will be applications of [13, Theorem 4.2.1]. For these applications we need the algebraic independence over

$\mathbb{C}(z)$ of the corresponding functions. This will be obtained in Section 3 by using [13, Theorems 3.2.1 and 3.2.2], which reduce the problem of algebraic independence to the problem of rationality of linear combinations of these functions. This last problem will be studied in the present section. Finally, in Section 4, we will prove some general results implying Theorems 1.9 and 1.10 as special cases.

Here we shall study functional equations of the form

$$ag(z^r) = g(z) - \frac{A(z)}{B(z)}, \tag{2.1}$$

where $A(z)$ and $B(z)$ are coprime polynomials satisfying $A(0) = 0$. Obviously

$$g(z) = \sum_{k=0}^{\infty} a^k \frac{A(z^{r^k})}{B(z^{r^k})}$$

is an analytic solution of (2.1) in some neighbourhood of the origin. On the other hand, if

$$f(z) = \sum_{k=0}^{\infty} f_k z^k$$

is a solution of (2.1), analytic in some neighbourhood of the origin, then

$$a\sum_{k=0}^{\infty} f_k z^{kr} = \sum_{k=0}^{\infty} f_k z^k - \sum_{k=1}^{\infty} c_k z^k \quad \text{with } \frac{A(z)}{B(z)} = \sum_{k=1}^{\infty} c_k z^k,$$

and therefore $af_0 = f_0$, and all f_k with $k \geq 1$ are uniquely determined by the above equality. If $a \neq 1$, then $f_0 = 0$ and $f(z) = g(z)$. If $a = 1$, then we may choose f_0 arbitrarily and $f(z) - f_0 = g(z)$. Therefore, in considering the existence of rational solutions $w(z)$ of (2.1) it is enough to study solutions with $w(0) = 0$, and if such a solution exists, then $g(z) = w(z)$.

We first consider rationality of our typical linear combination (1.5),

$$g_m(z) = c_0F_0(z) + c_1F_1(z) + \cdots + c_mF_m(z),$$

where $(c_0, c_1, \ldots, c_m) \in \mathbb{C}^{m+1} \backslash \{\underline{0}\}$, and where $\deg A_0(z)$ is not generally bounded. We begin by proving the following result.

Lemma 2.1. *Let the polynomials $B_i(z)$, $i = 1, \ldots, m$, satisfy conditions (i)–(iii). Then $g_m(z)$ is rational if and only if $c_1 = \cdots = c_m = 0$ and $A_0(z)$ is of the form $C(z) - aC(z^r)$ with $C(z) \in \mathbb{C}[z]$.*

Proof. The 'if' part being trivial, we immediately suppose that

$$g_m(z) = \frac{P(z)}{Q(z)} \tag{2.2}$$

with coprime $P(z), Q(z) \in \mathbb{C}[z]$ and monic $Q(z)$. Since $g_m(0) = 0$, we have $P(0) = 0$. The definition of $g_m(z)$ with (1.3) implies

$$ag_m(z^r) = g_m(z) - c_0A_0(z) - c_1\frac{A_1(z)}{B_1(z)} - \cdots - c_m\frac{A_m(z)}{B_m(z)}. \tag{2.3}$$

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Thus, condition (iii) implies $P(z) \neq 0$. Further, by (2.2), Equation (2.3) is equivalent to

$$aP(z^r)Q(z)\prod_{i=1}^{m}B_i(z) = P(z)Q(z^r)\prod_{i=1}^{m}B_i(z) - c_0A_0(z)Q(z^r)Q(z)\prod_{i=1}^{m}B_i(z) - \sum_{i=1}^{m}c_iA_i(z)Q(z^r)Q(z)\prod_{j\neq i}B_j(z), \tag{2.4}$$

implying the divisibility relations

$$Q(z^r) \mid Q(z)\prod_{i=1}^{m}B_i(z), \quad Q(z) \mid Q(z^r)\prod_{i=1}^{m}B_i(z). \tag{2.5}$$

Assume that $Q(\alpha) = 0$ with $|\alpha| > 1$, and choose such an α with minimal $|\alpha|$. Then none of the zeros of $Q(z^r)$ with $z^r = \alpha$ is a zero of $Q(z)$. Therefore, by (2.5), all these r zeros must belong to N_m, contrary to condition (i). A similar contradiction is obtained if $Q(\alpha) = 0$ with some $|\alpha| < 1$.

By the above consideration, $|\alpha| = 1$ holds for all α satisfying $Q(\alpha) = 0$. Let $\alpha_1, \ldots, \alpha_t$ be all such distinct α arranged in such a way that $0 \leq \phi_1 < \cdots < \phi_t < 2\pi$, $\phi_i = \arg\alpha_i$. Then the distinct roots of $Q(z^r)$ belong to $T_1 \cup \cdots \cup T_t$ with $T_i = \{z : z^r = \alpha_i\}$. It is easily seen that the sets T_i are disjoint, and therefore there exists a set T_i containing at most one of $\alpha_1, \ldots, \alpha_t$. Then at least $r - 1$ of the roots of $z^r = \alpha_i$ belong to N_m which contradicts (ii). This implies $Q(z) = 1$, and so (2.4) gives

$$aP(z^r)\prod_{i=1}^{m}B_i(z) = P(z)\prod_{i=1}^{m}B_i(z) - c_0A_0(z)\prod_{i=1}^{m}B_i(z) - \sum_{i=1}^{m}c_iA_i(z)\prod_{j\neq i}B_j(z). \tag{2.6}$$

By condition (iii) and (2.6), we get $c_iA_i(\gamma_i) = 0$ for all $i = 1, \ldots, m$, implying $c_i = 0$ for all $i = 1, \ldots, m$, since $A_i(\gamma_i) \neq 0$ by the coprimality of $A_i(z), B_i(z)$. Thus $c_0 \neq 0$ and (2.6) reduces to

$$aP(z^r) = P(z) - c_0A_0(z).$$

This proves our lemma. □

From the above proof we get the following corollary.

Corollary 2.2. *If the assumptions of Lemma 2.1 hold, then the function* $c_1F_1(z) + \cdots + c_mF_m(z)$ *with* $(c_1, \ldots, c_m) \in \mathbb{C}^m \setminus \{\underline{0}\}$ *is not rational.*

Further, in the special case $m = 1$, we obtain the following corollaries.

Corollary 2.3. *Assume that* $B_1(z)$ *is a nonconstant polynomial satisfying conditions (i) and (ii) with* $m = 1$. *Then* $g_1(z) = c_0F_0(z) + c_1F_1(z)$ *is a rational function for* $(c_0, c_1) \in \mathbb{C}^2 \setminus \{\underline{0}\}$ *if and only if* $c_1 = 0$ *and* $A_0(z)$ *is of the form* $C(z) - aC(z^r)$, $C(z) \in \mathbb{C}[z]$. *In particular, conditions (i) and (ii) with* $m = 1$ *are satisfied if* $r \geq 3$ *and* $\deg B_1(z) \leq r - 2$.

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Corollary 2.4. *The function $F_0(z)$ is rational if and only if $A_0(z) = C(z) - aC(z^r)$, $C(z) \in \mathbb{C}[z]$. Further, $F_1(z)$ is not a rational function if conditions (i) and (ii) are valid with $m = 1$.*

If $\deg B_1(z) \geq r-1$, then Corollary 2.3 cannot be applied directly but, in some cases, it can be used to determine explicitly all rational possibilities for $g_1(z)$. We study such cases in Lemmas 2.7 and 2.10 after the following examples.

Example 2.5. The choice $B_1(z) = z-1$, $B_2(z) = z+1$ satisfies conditions (i)–(iii) for all $r \geq 3$, and therefore Lemma 2.1 implies that the function

$$c_0F_0(z) + c_1F_1(z) + c_2F_2(z)$$

is rational if and only if $c_1 = c_2 = 0$ and $A_0(z) = C(z) - aC(z^r)$ for some $C(z) \in \mathbb{C}[z]$. If $r = 2$, then condition (ii) is not satisfied, but see Example 2.9 for this case.

Example 2.6. Assume that the positive integers t and u are not divisible by r. Let $\alpha_1, \ldots, \alpha_m$, $\alpha \in \mathbb{C}\backslash\{0\}$ satisfy $|\alpha_i| \neq 1, |\alpha_i| \neq |\alpha_j|$ for all $i \neq j, |\alpha| = 1$. If $A_0(z)$ is not of the form $C(z) - aC(z^r)$ with some $C(z) \in \mathbb{C}[z]$, and if $B_i(z) = z^t - \alpha_i$ and $B_{m+1}(z) = z^u - \alpha$, then

$$c_0F_0(z) + c_1F_1(z) + \cdots + c_mF_m(z)$$

is rational only if $c_0 = \cdots = c_m = 0$. Further, for $r \geq 3$,

$$c_0F_0(z) + c_1F_1(z) + \cdots + c_{m+1}F_{m+1}(z)$$

is rational only if $c_0 = \cdots = c_{m+1} = 0$. It is enough to prove that conditions (i) and (ii) are valid ((iii) is clear). To verify (i), we show that the set $T = \{\arg z : z^t - b = 0\}$, where $|b| \neq 1$, does not contain all $\arg z$ with $z^r - \beta = 0$. For this, note that if

$$\frac{\arg\beta}{r} + \frac{2j\pi}{r} = \frac{\arg b}{t} + \frac{2i\pi}{t}, \quad \frac{\arg\beta}{r} + \frac{2(j+1)\pi}{r} = \frac{\arg b}{r} + \frac{2k\pi}{t}$$

with integers i, j, k, then $r \mid t$ contrary to the assumption. Further, if $r \geq 3$, $U = \{\arg z : z^u - \alpha = 0\}$, $|\beta| = 1$ and

$$\frac{\arg\beta}{r} + \frac{2j\pi}{r} \in U,$$

then

$$\frac{\arg\beta}{r} + \frac{2(j\pm 1)\pi}{r} \notin U,$$

since r is not a factor of u. This implies (ii).

Lemma 2.7. *If the polynomials $A_0(z), A_1(z)$, and $B_1(z)$ have degrees less than or equal to $r-1$, then $g_1(z)$ is rational if and only if $a = r$, $c_0 = 0$ and, moreover,*

$$B_1(z) = \frac{z^r - \alpha}{z - \alpha} \quad \textit{with } \alpha^{r-1} = 1, \quad A_1(z) = czB_1'(z) \quad \textit{with } c \in \mathbb{C}\backslash\{0\}, \tag{2.7}$$

in which case

$$\sum_{k=0}^{\infty} r^k \frac{z^{r^k} B_1'(z^{r^k})}{B_1(z^{r^k})} = \frac{z}{\alpha - z}.$$

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Remark 2.8. We note that [4, Theorem 8] implies that $F_1(z)$ with $\deg B_1(z) = r-1$ is rational only in the case of Lemma 2.7. The particular case $a = 1$ follows also from [18, Lemma 1]. In the case $r = 2$, one has

$$\sum_{k=0}^{\infty} 2^k \frac{z^{2^k}}{z^{2^k}+1} = \frac{z}{1-z},$$

see also [3, Theorem 1.1]. The final equality in Lemma 2.7 is apparent, noticing that

$$\frac{zB_1'(z)}{B_1(z)} = \frac{-z}{z-\alpha} + \frac{rz^r}{z^r-\alpha}.$$

Proof of Lemma 2.7. Under the assumptions of Lemma 2.7, conditions (i) and (ii) with $m = 1$ of Corollary 2.3 are satisfied except in the case

$$B_1(z) = \frac{z^r-\alpha}{z-\gamma}, \quad |\alpha| = 1,\ \gamma^r = \alpha.$$

Therefore we now assume this and use the proof of Lemma 2.1.

By (2.5), $\deg Q(z) \le 1$. If $Q(z) = 1$, then (2.4) gives

$$aP(z^r)B_1(z) = P(z)B_1(z) - c_0A_0(z)B_1(z) - c_1A_1(z).$$

Thus $c_1A_1(\delta) = 0$ with some δ satisfying $B_1(\delta) = 0$. Since $A_1(\delta) \ne 0$, we get $c_1 = 0$, $c_0 \ne 0$. By the last equation, we have

$$c_0A_0(z) = P(z) - aP(z^r),$$

but this is not possible, since $1 \le \deg A_0(z) \le r-1$.

If $\deg Q(z) = 1$, say $Q(z) = z - \beta$, then (2.5) leads to

$$z^r - \beta = B_1(z)(z-\beta).$$

Therefore $|\beta| = |\alpha| = 1$, and $\alpha = \beta = \gamma,\ Q(z^r) = B_1(z)(z-\alpha)$. Substituting this in (2.4), we obtain

$$aP(z^r) = P(z)B_1(z) - c_0A_0(z)B_1(z)(z-\alpha) - c_1A_1(z)(z-\alpha). \tag{2.8}$$

This gives first $\deg P(z) = 1,\ P(z) = c^*z,\ c^* \in \mathbb{C}\setminus\{0\}$. Further, since $\deg A_0(z)B_1(z)(z-\alpha) \ge r+1$ and the degrees of the other terms are less than or equal to r, we find $c_0 = 0,\ c_1 \ne 0$. We now put $A_1(z) = zA^*(z)$ and note that $\gamma = \alpha$ implies $\alpha^{r-1} = 1$, $B_1(\alpha) = r$. Then (2.8) takes the form

$$ac^*z^{r-1} = c^*\frac{z^r-\alpha}{z-\alpha} - c_1A^*(z)(z-\alpha).$$

At $z = \alpha$ we now get $a = r$. Then we find, after some calculations, that

$$A^*(z) = -\frac{c^*}{c_1}B_1'(z),$$

which proves Lemma 2.7. □

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EXAMPLE 2.9. Let $f_0(z)$, $f_1(z)$ and $f_2(z)$ be the functions defined after Theorem 1.2. By Lemma 2.7, the linear form $c_0f_0(z) + c_1f_1(z)$ is rational for any a, r only if $c_0 = c_1 = 0$. The same holds for $f_0(z)$ and $f_2(z)$ if $r \geq 3$; but if $r = 2$, then $f_2(2, z) = z/(1 - z)$ holds. Furthermore, by (2.10) in the following lemma, we see that $f_1(1, z) - f_2(1, z) = 2z/(1 - z)$ if $r = 2$.

LEMMA 2.10. *If* $\deg A_0(z) \leq r - 1$, $\deg A_1(z) \leq r$, *and* $\deg B_1(z) = r$, *then* $g_1(z)$ *is rational if and only if* $c_0 = 0$ *and one of the following cases holds:*

(1) $a \neq r$ *and*

$$B_1(z) = z^r - \alpha, \quad \alpha^{r-1} = 1, \quad A_1(z) = c\alpha\Big(\sum_{j=1}^{r}(\alpha^{-1}z)^j - a(\alpha^{-1}z)^r\Big), \quad c \in \mathbb{C}\setminus\{0\}; \tag{2.9}$$

(2) $r = 2, a = 2$ *and*

$$B_1(z) = 1 + z^2, \quad A_1(z) = cz^2, \quad c \in \mathbb{C}\setminus\{0\};$$

(3) $r = 2, a = 4$ *and*

$$B_1(z) = (1 + z)^2, \quad A_1(z) = cz, \quad c \in \mathbb{C}\setminus\{0\};$$

(4) $r = 2, a = -2$ *and*

$$B_1(z) = 1 - z + z^2, \quad A_1(z) = cz, \quad c \in \mathbb{C}\setminus\{0\};$$

(5) $r = 2, a = 2$ *and*

$$B_1(z) = 1 - z + z^2, \quad A_1(z) = cz(1 - 2z), \quad c \in \mathbb{C}\setminus\{0\}.$$

Moreover,

$$\sum_{k=0}^{\infty}\frac{\sum_{j=1}^{r}(\alpha^{-1}z)^{jr^k} - a(\alpha^{-1}z)^{r^{k+1}}}{z^{r^{k+1}} - \alpha} = \frac{z}{\alpha^{-1}z - 1}, \tag{2.10}$$

$$\sum_{k=0}^{\infty}2^k\frac{z^{2^{k+1}}}{z^{2^{k+1}} + 1} = \frac{z^2}{1 - z^2}, \quad \sum_{k=0}^{\infty}\frac{4^kz^{2^k}}{(1 + z^{2^k})^2} = \frac{z}{(1 - z)^2},$$

$$\sum_{k=0}^{\infty}\frac{(-2)^kz^{2^k}}{1 - z^{2^k} + z^{2^{k+1}}} = \frac{z}{1 + z + z^2}, \quad \sum_{k=0}^{\infty}\frac{2^kz^{2^k}(1 - 2z^{r^k})}{1 - z^{2^k} + z^{2^{k+1}}} = \frac{z(1 + 2z)}{1 + z + z^2}.$$

REMARK 2.11. The case $a = 1$ of (1) follows also from [4, Theorem 9], and cases (3) and (4) are given in [3, Theorem 1.1]. Note also that (2) is a special case of Lemma 2.7 if we replace z by z^2 there.

PROOF OF LEMMA 2.10. We again use the proof of Lemma 2.1. Now (2.5) implies that $\deg Q(z) \leq r/(r - 1)$, so $\deg Q(z) \leq 1$, if $r \geq 3$, and $\deg Q(z) \leq 2$, if $r = 2$.

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If $Q(z) = 1$, we get a contradiction as in the proof of Lemma 2.7. Assume now that $\deg Q(z) = 1$ or 2 (if $r = 2$). By (2.4),

$$aP(z^r)Q(z)B_1(z) = P(z)Q(z^r)B_1(z) - c_0A_0(z)Q(z^r)Q(z)B_1(z) \\ - c_1A_1(z)Q(z^r)Q(z). \tag{2.11}$$

Assume first that $\deg Q(z) = 1$. If $\deg P(z) \geq 2$, then the degree of the left-hand side is greater than the degrees of the terms on the right-hand side, therefore $\deg P(z) = 1$, $P(z) = cz$, $c \in \mathbb{C}\setminus\{0\}$ (remember that $P(0) = 0$). But this means that $\deg A_0(z)Q(z^r)Q(z)B_1(z) \geq 2r + 2$ is greater than the degree of any other term in the above equality. Thus $c_0 = 0$, $c_1 \neq 0$. In the case $r = \deg Q(z) = 2$ we get $\deg P(z) \leq 2$, and we have the same conclusion $c_0 = 0$, $c_1 \neq 0$. This implies that $F_1(z)$ is a rational function in all cases under consideration, and we may assume in the following that $c_1 = 1$.

We now consider (2.11) with $c_0 = 0$, $c_1 = 1$. Clearly, it is enough to study those $B_1(z)$ which do not satisfy conditions (i) and (ii) with $m = 1$ of Corollary 2.3. So we have to consider the following two cases:

$$B_1(z) = z^r - \alpha, \tag{2.12}$$

$$B_1(z) = \frac{z^r - \alpha}{z - \gamma}(z - \delta), \quad |\alpha| = 1, \gamma^r = \alpha, \delta \neq \gamma. \tag{2.13}$$

Assume first that $\deg Q(z) = 1$, say $Q(z) = z - \beta$. We saw above that then $P(z) = cz$ holds with some $c \in \mathbb{C}\setminus\{0\}$. Equation (2.11) now takes the form

$$acz^r(z - \beta)B_1(z) = cz(z^r - \beta)B_1(z) - A_1(z)(z^r - \beta)(z - \beta). \tag{2.14}$$

If $B_1(z)$ is of the form (2.12), then

$$(z^r - \beta) \mid (z - \beta)(z^r - \alpha), \quad (z - \beta) \mid (z^r - \beta)(z^r - \alpha). \tag{2.15}$$

This implies $\alpha = \beta$, and then the second relation in (2.15) yields $\alpha^{r-1} = 1$. By (2.14),

$$acz^r = cz\frac{z^r - \alpha}{z - \alpha} - A_1(z).$$

This gives

$$A_1(z) = c\alpha\Big(\sum_{j=1}^{r}(\alpha^{-1}z)^j - a(\alpha^{-1}z)^r\Big)$$

which is case (1), if $a \neq r$. If $a = r$, then $A_1(\alpha) = 0$, which contradicts the coprimality of $A_1(z)$ and $B_1(z)$ (in fact we get the special case of Lemma 2.7, if we cancel out the common factor $z - \alpha$).

Secondly, we consider $B_1(z)$ from (2.13). By (2.5),

$$(z^r - \beta) \,\Big|\, \frac{z^r - \alpha}{z - \gamma}(z - \delta)(z - \beta), \tag{2.16}$$

or $B_1(z)(z - \beta) = (z^r - \beta)T(z)$, where $T(z)$ is a monic polynomial of degree 1. This result with $P(z) = cz$ and (2.11) implies

$$acz^rT(z) = czB_1(z) - A_1(z)(z - \beta). \tag{2.17}$$

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If $r \geq 3$, then we get, by (2.16) and $\delta \neq \gamma$,

$$\alpha = \beta = \gamma, \quad T(z) = z - \delta.$$

Thus the use of (2.17) leads to $A_1(\delta)(\delta - \beta) = 0$. Since $\delta \neq \gamma = \beta$ and $A_1(\delta) \neq 0$ by coprimality of $A_1(z)$ and $B_1(z)$, we get a contradiction. If $r = 2$ and $\alpha = \beta$, we have the same contradiction as above. In the case $r = 2$ and $\alpha \neq \beta$ we get $z^2 - \beta = (z - \delta)(z - \beta)$, $\beta = 1$, $\delta = -1$, $B_1(z) = (z + \gamma)(z + 1)$ and $T(z) = z + \gamma$. By using these results with (2.17), we now obtain $A_1(-\gamma)(-\gamma - 1) = 0$, which again gives a contradiction. Thus, the case $\deg Q(z) = 1$ is proved.

Assume now that $r = 2 = \deg Q(z)$. Let $Q(z) = z^2 + q_1 z + q_0$ and $B_1(z) = z^2 - \alpha$ as in (2.12). By (2.5), $Q(z^2) = Q(z)B_1(z)$, or

$$z^4 + q_1 z^2 + q_0 = (z^2 + q_1 z + q_0)(z^2 - \alpha).$$

By comparing coefficients on both sides, we obtain $\alpha = q_0 = -1$, $q_1 = 0$. Substituting this in (2.11), we get

$$aP(z^2) = P(z)(z^2 + 1) - A_1(z)(z^2 - 1).$$

Here $P(z) = p_1 z + p_2 z^2$, $A_1(z) = a_1 z + a_2 z^2$, and, comparing coefficients again, we have $a = 2$, $A_1(z) = cz^2$, $P(z) = -cz^2$, $c \in \mathbb{C}\backslash\{0\}$. This is case (2).

Assume then that $B_1(z)$ is of the form (2.13),

$$B_1(z) = \frac{z^2 - \alpha}{z - \gamma}(z - \delta) = (z + \gamma)(z - \delta) =: z^2 + b_1 z + b_0.$$

Note that $b_1 \neq 0$ since $\gamma \neq \delta$. If $Q(z) = z^2 + q_1 z + q_0$, then, by (2.5),

$$z^4 + q_1 z^2 + q_0 = (z^2 + q_1 z + q_0)(z^2 + b_1 z + b_0). \tag{2.18}$$

Comparison of the coefficients in this equation gives $q_1 = -b_1 \neq 0$, $q_0 = b_0 = 1$. Therefore,

$$B_1(z) = z^2 + bz + 1, \quad Q(z) = z^2 - bz + 1, \quad b = \gamma - \delta, \quad \gamma\delta = -1,$$

and therefore $Q(z) = (z - \gamma)(z + \delta)$. Consequently, (2.18) has the form

$$(z^2 - \gamma)(z^2 + \delta) = (z^2 - \gamma^2)(z^2 - \delta^2).$$

Now there are two possibilities, either $\gamma = \gamma^2$, $\delta = -\delta^2$ or $\gamma = \delta^2$, $\delta = -\gamma^2$. In the first case $\gamma = 1$, $\delta = -1$ and $b = 2$, and in the second $\gamma^3 = 1$ $(\gamma \neq 1)$, $\delta = -\gamma^2$ and $b = -1$.

By the above information, (2.11) has the form

$$aP(z^2) = P(z)(z^2 + bz + 1) - A_1(z)(z^2 - bz + 1),$$

where $P(z) = p_1 z + p_2 z^2$ and $A_1(z) = a_1 z + a_2 z^2$. By comparing coefficients we get the following system of equations:

$$p_1 - a_1 = 0, \quad p_2 + bp_1 - a_2 + ba_1 = ap_1, \quad bp_2 + p_1 + ba_2 - a_1 = 0, \quad p_2 - a_2 = ap_2.$$

These imply three possibilities. The first is $p_2 = a_2 = 0$, $p_1 = a_1 = c \neq 0$, $b = 2$, $a = 4$, leading to (3). Then we may have $p_2 = a_2 = 0$, $p_1 = a_1 = c \neq 0$, $b = -1$, $a = -2$, which gives (4). Finally, the case $p_1 = a_1 = c \neq 0$, $p_2 = -a_2 = 2c$, $b = -1$, $a = 2$ leads to (5). Thus, Lemma 2.10 is proved. □

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We note that Corollary 2.3 and Lemmas 2.7 and 2.10 can be used to study the rationality of the function

$$F(z) = \sum_{k=0}^{\infty} a^k \frac{A(z^{r^k})}{B(z^{r^k})}$$

with coprime polynomials $A(z)$ and $B(z)$ satisfying $A(0) = 0$, and $1 \le \deg B(z) \le r$. Namely, we may write

$$A(z) = A_0(z)B(z) + A_1(z),$$

where $A_0(0) = A_1(0) = 0$, $A_1(z)$, $B(z)$ are coprime and $\deg A_1(z) \le \deg B(z)$, and therefore

$$F(z) = \sum_{k=0}^{\infty} a^k A_0(z^{r^k}) + \sum_{k=0}^{\infty} a^k \frac{A_1(z^{r^k})}{B(z^{r^k})}.$$

Here $A_0(z) \equiv 0$ if $\deg A(z) \le \deg B(z)$, and $\neq 0$ otherwise. Thus $F(z)$ is of the form $F(z) = c_0F_0(z) + c_1F_1(z)$ with $(c_0, c_1) = (0, 1)$ or $(1, 1)$.

As our final result of this section we give a generalization of Example 2.6, where we apply (2.3) following the lines of the proof in [14, Lemma 6], which studies the case where $A_i(z)$ are powers of z and all t_i below are equal.

Lemma 2.12. *Assume that $t_1, \ldots, t_m$ and u are positive integers. Let $\alpha_1, \ldots, \alpha_m, \alpha \in \mathbb{C}\backslash\{0\}$ satisfy $|\alpha_i| \neq 1$, $\sqrt[t_i]{|\alpha_i|} \neq \sqrt[t_j]{|\alpha_j|}$ for all $i \neq j$, $|\alpha| = 1$. If $B_i(z) = z^{t_i} - \alpha_i$, $i = 1, \ldots, m$, $B_{m+1}(z) = z^u - \alpha$, and $(r-1)t_i \ge t_j$ for all i, j, then*

$$g(z) := c_0F_0(z) + \cdots + c_mF_m(z) + c_{m+1}F_{m+1}(z)$$

with nontrivial $(c_0, \ldots, c_{m+1})$ is rational if and only if $c_1 = \cdots = c_m = 0$ and $g_1(z) := c_0F_0(z) + c_{m+1}F_{m+1}(z)$ is rational. In particular, if $A_0(z)$ is not of the form $C(z) - aC(z^r)$ with $C(z) \in \mathbb{C}[z]$ and either $c_{m+1} = 0$ or $r \ge 3$ and u is not divisible by r, then $g(z)$ is not a rational function.

Proof. Assuming that $g(z) = P(z)/Q(z)$ as in the proof of Lemma 2.1, we have an analogue of (2.3),

$$ag(z^r) = g(z) - c_0A_0(z) - c_1\frac{A_1(z)}{B_1(z)} - \cdots - c_m\frac{A_m(z)}{B_m(z)} - c_{m+1}\frac{A_{m+1}(z)}{B_{m+1}(z)}. \tag{2.19}$$

If $(c_1, \ldots, c_m) \neq \underline{0}$, then $g(z)$ must have a pole of absolute value not equal to 1. If there exists a pole of absolute value greater than 1, let p be such a pole with maximal absolute value. Now p is not a pole of $g(z^r)$, so the assumptions of our lemma together with (2.19) imply that $p^{t_i} - \alpha_i = 0$ with exactly one $1 \le i \le m$. But then all $\sqrt[t_i]{1}p$ are poles of $g(z)$. This implies that rt_i numbers $\sqrt[r]{\sqrt[t_i]{1}p}$ are poles of $g(z^r)$. By the assumption $\sqrt[t_i]{|\alpha_i|} \neq \sqrt[t_j]{|\alpha_j|}$ for $i \neq j$, the function $ag(z^r) - g(z)$ has exactly t_j poles of the same absolute value $\sqrt[t_j]{|\alpha_j|}$ (if $c_j \neq 0$), and therefore at least $rt_i - t_j \ge t_i$ (with some $j \neq i$) of the above $\sqrt[r]{\sqrt[t_i]{1}p}$ are poles of $g(z)$. Let these be $q_1, \ldots, q_v$ with $v \ge t_i$. The

rv numbers $\sqrt[r]{q_i}$ are poles of $g(z^r)$, and again at least $rv - t_k \geq t_i$ of these are poles of $g(z)$. By repeating this, we get a contradiction, and therefore $(c_1, \ldots, c_m) = \underline{0}$. The same argument works also if $g(z)$ has a pole of absolute value less than 1. Then the final claim of the lemma follows from Example 2.6, and Lemma 2.12 is proved. □

3. Algebraic independence and hypertranscendence

Our subsequent algebraic independence considerations are based on [13, Theorems 3.2.1 and 3.2.2] saying that algebraic independence of the functions satisfying functional equations of type (2.1) can be reduced to rationality considerations of linear combinations (over $\mathbb{C}$) of these functions, and here we may use the results in Section 2.

For the proof of Theorem 1.1, we need the following analogue for functions.

Theorem 3.1. *Assume that the polynomials $B_i(z)$, $i = 1, \ldots, m$, satisfy conditions (i)–(iii). Then the functions*

$$F_0^{\langle 1\rangle}(z), \ldots, F_0^{\langle r-1\rangle}(z), F_1(z), \ldots, F_m(z)$$

are algebraically independent over $\mathbb{C}(z)$.

Proof. Assume that the above functions are algebraically dependent. Then [13, Theorems 3.2.2] implies that these functions are linearly dependent over $\mathbb{C}$ mod $\mathbb{C}(z)$. Thus there exist nontrivial constants $c_{0,\mu}$ and c_i such that

$$g(z) := c_{0,1}F_0^{\langle 1\rangle}(z) + \cdots + c_{0,r-1}F_0^{\langle r-1\rangle}(z) + c_1F_1(z) + \cdots + c_mF_m(z) \in \mathbb{C}(z).$$

If all $c_{0,\mu} = 0$, then at least one of the c_i is nonzero, and we have a contradiction to Corollary 2.2. If $c_{0,\mu} \neq 0$ for some μ, then

$$g(z) = F_0(z) + c_1F_1(z) + \cdots + c_mF_m(z) \in \mathbb{C}(z),$$

where $A_0(z) = c_{0,1}z + \cdots + c_{0,r-1}z^{r-1}$, but this contradicts Lemma 2.1, proving Theorem 3.1. □

Analogously, Lemmas 2.7, 2.10 and 2.12 give the following results.

Theorem 3.2. *If the degrees of $A_1(z)$ and $B_1(z)$ are less than or equal to $r-1$, then the functions*

$$F_0^{\langle 1\rangle}(z), \ldots, F_0^{\langle r-1\rangle}(z), F_1(z)$$

are algebraically independent over $\mathbb{C}(z)$ except if $a = r$ and (2.7) holds.

For example, Theorem 3.2 applied to the functions of Example 2.9 implies, that, for $r \geq 3$, the functions $f_0(z)$, $f_1(z)$ and $f_2(z)$ are algebraically independent. In the case $r = 2$, $f_0(z)$ and $f_1(z)$ are algebraically independent, $f_0(z)$ and $f_2(z)$ are algebraically independent for all $a \neq 2$, but $f_2(2, z) = z/(1-z)$. See also [8, Theorem 2] for these functions.

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THEOREM 3.3. *If* $\deg A_1(z) \le r$ *and* $\deg B_1(z) = r$, *then the functions*

$$F_0^{\langle 1\rangle}(z), \dots, F_0^{\langle r-1\rangle}(z), F_1(z)$$

are algebraically independent over $\mathbb{C}(z)$ *except in cases (1)–(5) of Lemma 2.10.*

THEOREM 3.4. *Let the assumptions of Theorem 1.5 be satisfied. Then the functions*

$$F_0^{\langle 1\rangle}(z), \dots, F_0^{\langle r-1\rangle}(z), F_1(z), \dots, F_m(z)$$

are algebraically independent over $\mathbb{C}(z)$. *Further, if* $r \ge 3$ *and* u *is not divisible by* r, *then these functions together with* $F_{m+1}(z)$ *are algebraically independent over* $\mathbb{C}(z)$.

We next study hypertranscendence of the solutions of (2.1),

$$ag(z^r) = g(z) - \frac{A(z)}{B(z)},$$

with coprime polynomials $A(z)$ and $B(z)$ satisfying $A(0) = 0$.

THEOREM 3.5. *If (2.1) does not have a rational solution,* $F(z) \in \mathbb{C}[[z]]$ *converges in some neighbourhood of the origin, and satisfies (2.1), then* $F(z)$ *is hypertranscendental.*

PROOF. We assume that there is some nonnegative integer m such that $F(z)$, $F'(z), \dots, F^{(m)}(z)$ are algebraically dependent over $\mathbb{C}(z)$, which is equivalent to the algebraic dependence of $F(z), DF(z), \dots, D^mF(z)$, where $D := z(d/dz)$. Then we apply [11, Theorem 3] or [13, Theorem 4.2.3] to $F(z)$. By (2.1), we have to take $u(z) = a^{-1}$, $v(z) = a^{-1}A(z)/B(z)$, whence $M = 0 = Q$, $u_1(z) = 1$ using the notation of [13, Theorem 4.2.3]. By this theorem, there exists some $w(z) \in \mathbb{C}(z)$ satisfying

$$w(z^r) = a^{-1}w(z) + a^{-1}\frac{A(z)}{B(z)} \quad \text{or} \quad w(z^r) = a^{-1}w(z) + a^{-1}\frac{A(z)}{B(z)} - \frac{\gamma}{u_2(z)}, \tag{3.1}$$

where $u_2 \in \mathbb{C}(z)\backslash\{0\}$ fulfils the condition $u_2(z^r) = u_2(z)$ (leading to $u_2 \in \mathbb{C}^\times$), and $\gamma \in \mathbb{C}$ is the constant term in the z-expansion of $A(z)/(B(z)u_2(z))$ which vanishes since $A(0) = 0$, $B(0) \ne 0$. Thus, (3.1) reduces to the single equation

$$aw(z^r) = w(z) + \frac{A(z)}{B(z)}$$

and this contradicts an assumption of Theorem 3.5. □

COROLLARY 3.6. *All functions* $g_m(z)$ *and* $g_1(z)$ *of the previous section, which are not rational functions, are hypertranscendental. For example, if the assumptions of Lemma 2.7 are valid, then* $g_1(z)$ *is hypertranscendental except if* $a = r$ *and (2.7) hold.*

REMARK 3.7. Very recently, Coons [2] considered functional equations of type (2.1) with $a = 1$ and $\deg A(z), \deg B(z) \le r - 1$, $A(0) = 0$. In his Theorem 2.2, he established the transcendence over $\mathbb{C}(z)$ of the solutions $F(z) \in \mathbb{C}[[z]]$ of such equations converging on some U as above. Note that our Corollary 3.6 implies even the

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hypertranscendence of these functions $F(z)$. Under the additional hypotheses that $F(0)$ and the coefficients of $A(z)$ and $B(z)$ are algebraic, one obtains the algebraic independence of $F(\zeta), F'(\zeta), F''(\zeta), \ldots$ for any algebraic $\zeta \in U\backslash\{0\}$ with $B(\zeta^{r^\nu}) \neq 0$ for $\nu = 0, 1, \ldots$ (compare also our Theorem 1.6 above). This fairly generalizes the transcendence results on $F(\zeta)$ for very particular examples F given in [2].

Finally, we consider the functions (1.7). By (1.2), we have

$$\begin{aligned} F_{i,0}(x,z) &= xF_{i,0}(x,z^r) + \frac{A_i(z)}{B_i(z)}, \\ F_{i,1}(x,z) &= xF_{i,1}(x,z^r) + F_{i,0}(x,z^r), \\ F_{i,2}(x,z) &= xF_{i,2}(x,z^r) + 2F_{i,1}(x,z^r), \\ &\vdots \\ F_{i,\ell}(x,z) &= xF_{i,\ell}(x,z^r) + \ell F_{i,\ell-1}(x,z^r). \end{aligned}$$

This implies that, for any pair (i,k), the functions $F_{i,0,k}(z), \ldots, F_{i,\ell,k}(z)$ satisfy a system of functional equations

$$\begin{aligned} F_{i,0,k}(z^r) &= a_k^{-1}F_{i,0,k}(z) + b_{i,k,0}(z), \\ F_{i,1,k}(z^r) &= -a_k^{-2}F_{i,0,k}(z) + a_k^{-1}F_{i,1,k}(z) + b_{i,k,1}(z), \\ F_{i,2,k}(z^r) &= a_{2,0}^{i,k}F_{i,0,k}(z) - 2a_k^{-2}F_{i,1,k}(z) + a_k^{-1}F_{i,2,k}(z) + b_{i,k,2}(z), \\ &\vdots \\ F_{i,\ell,k}(z^r) &= a_{\ell,0}^{i,k}F_{i,0,k}(z) + \cdots + a_{\ell,\ell-2}^{i,k}F_{i,\ell-2,k}(z) - \ell a_k^{-2}F_{i,\ell-1,k}(z) + a_k^{-1}F_{i,\ell,k}(z) + b_{i,k,\ell}(z), \end{aligned}$$

where the $a_{j,t}^{i,k}$ are complex constants and

$$b_{i,k,j}(z) = (-1)^{j-1}j!a_k^{-j-1}\frac{A_i(z)}{B_i(z)}.$$

An analogous system is obtained for $F_{0,0,k}^{\langle\mu\rangle}(z), \ldots, F_{0,\ell,k}^{\langle\mu\rangle}(z)$, where, in this case, $A_i(z)$ should be replaced by z^μ, and $B_i(z)$ by 1.

Theorem 3.8. *If conditions (i)–(iii) are satisfied, then the functions*

$$F_{0,j,k}^{\langle 1\rangle}(z), \ldots, F_{0,j,k}^{\langle r-1\rangle}(z), F_{1,j,k}(z), \ldots, F_{m,j,k}(z) \quad (j = 0, \ldots, \ell;\ k = 1, \ldots, s)$$

are algebraically independent over $\mathbb{C}(z)$.

Proof. Assume that the functions under consideration are algebraically dependent. We can apply [13, Theorem 3.2.1] to the above system of functional equations. Since the a_k are distinct, only the (i,k) pairs $(1,k), \ldots, (m,k)$ and the $(0,\mu,k)$ triples $(0,1,k), \ldots, (0,r-1,k)$ have the same a_k, and therefore there exists a nontrivial set of constants $c_1, \ldots, c_m; c_{0,1,k}, \ldots, c_{0,r-1,k}$ such that, for some k, the function

$$\begin{aligned} c_{0,1,k}F_{0,0,k}^{\langle 1\rangle}(z) + \cdots + c_{0,r-1,k}F_{0,0,k}^{\langle r-1\rangle}(z) + c_1F_{1,0,k}(z) + \cdots + c_mF_{m,0,k}(z) \\ = F_0(a_k,z) + c_1F_1(a_k,z) + \cdots + c_mF_m(a_k,z) \end{aligned}$$

is rational, where $A_0(z) = c_{0,1,k}z + \cdots + c_{0,r-1,k}z^{r-1}$. This is impossible by Theorem 3.1 (take there $a = a_k$). $\square$

Obviously Theorems 3.2–3.4 can be analogously generalized.

The proofs of Theorems 1.1–1.3, 1.5, 1.6 and 1.8 now follow immediately by applying [13, Theorem 4.2.1] with the above Theorems 3.1–3.5, 3.8 and Corollary 3.6, very much parallel to the proof of [1, Theorem 2].

4. Applications

To prove Theorems 1.9 and 1.10, we first recall that

$$F_n = \frac{1}{\sqrt{5}}(\rho_1^n - \rho_2^n), \quad L_n = \rho_1^n + \rho_2^n \quad \text{where } \rho_1 = \frac{1+\sqrt{5}}{2},\ \rho_2 = -\rho_1^{-1}.$$

Therefore we have, for all integers $d \geq 1$, $v \in \mathbb{Z}$ and $k \geq 0$ satisfying $dr^k + v > 0$,

$$\frac{1}{F_{dr^k+v}} = C_{1,v}\frac{A_{1,v}(\rho_1^{-dr^k})}{B_{1,v}(\rho_1^{-dr^k})}, \quad \frac{1}{L_{dr^k+v}} = C_{2,v}\frac{A_{2,v}(\rho_1^{-dr^k})}{B_{2,v}(\rho_1^{-dr^k})}, \tag{4.1}$$

where $A_{1,v}(z) = A_{2,v}(z) = z$, $B_{1,v}(z) = z^2 - \alpha_v$, $B_{2,v}(z) = z^2 + \alpha_v$ with $\alpha_v = (-1)^{dr+v}\rho_1^{2v}$, and certain $C_{1,v}, C_{2,v} \in \mathbb{Q}(\sqrt{5})\backslash\{0\}$.

We now define

$$\varphi_v(x,z) = \sum_{k=0}^{\infty} x^k \frac{z^{r^k}}{z^{2r^k} - \alpha_v}, \quad \lambda_v(x,z) = \sum_{k=0}^{\infty} x^k \frac{z^{r^k}}{z^{2r^k} + \alpha_v},$$

assuming, for the moment, only that $\alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{C}\backslash\{0\}$ satisfy $|\alpha_0| = 1$, $|\alpha_i| \neq |\alpha_j|$ for $i \neq j$. Further, let $\varphi_v(z) = \varphi_v(a,z)$, $\lambda_v(z) = \lambda_v(a,z)$ for some nonzero complex a, and let

$$G_m(z) := c_1 F_0^{\langle 1\rangle}(z) + \cdots + c_{r-1}F_0^{\langle r-1\rangle}(z) + \sum_{v=0}^{m}(c_{1,v}\varphi_v(z) + c_{2,v}\lambda_v(z))$$

with nontrivial constants $c_i, c_{1,v}, c_{2,v}$. Then the following lemma holds.

Lemma 4.1. *If $r \geq 3$, then $G_m(z)$ is not a rational function.*

Proof. We assume that $G_m(z) \in \mathbb{C}(z)$ and deduce a contradiction. Clearly, $G_m(z)$ satisfies the functional equation

$$aG_m(z^r) = G_m(z) - c_1 z - \cdots - c_{r-1}z^{r-1} - \sum_{v=0}^{m}\Big(c_{1,v}\frac{z}{z^2 - \alpha_v} + c_{2,v}\frac{z}{z^2 + \alpha_v}\Big). \tag{4.2}$$

If $(c_{1,v}, c_{2,v}) \neq (0,0)$, we may write

$$c_{1,v}\frac{z}{z^2 - \alpha_v} + c_{2,v}\frac{z}{z^2 + \alpha_v} =: \frac{A_v(z)}{B_v(z)}$$

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with coprime $A_\nu(z), B_\nu(z)$, namely

$$A_\nu(z) = z((c_{1,\nu} + c_{2,\nu})z^2 + (c_{1,\nu} - c_{2,\nu})\alpha_\nu), \quad B_\nu(z) = z^4 - \alpha_\nu^2 \quad \text{if } c_{1,\nu}c_{2,\nu} \neq 0,$$

and

$$A_\nu(z) = c_{j,\nu}z, \quad B_\nu(z) = z^2 + (-1)^j\alpha_\nu \quad \text{if } c_{j,\nu} \neq 0, c_{j+1,\nu} = 0$$

for $j \in \{1, 2\}$, adopting the convention $c_{3,\nu} := c_{1,\nu}$. Thus, we may use Lemma 2.12 with $t_\nu = 2$ or $t_\nu = 4$; note that $\sqrt[4]{|\alpha_\nu^2|} = \sqrt{|\alpha_\nu|}$. Since $(r-1)2 \geq 4$ for all $r \geq 3$ we get a contradiction if $(c_{1,\nu}, c_{2,\nu}) \neq (0, 0)$ holds for some $\nu \in \{1, \ldots, m\}$. Therefore a rational $G_m(z)$ satisfies a simplified Equation (4.2), namely

$$aG_m(z^r) = G_m(z) - c_1z - \cdots - c_{r-1}z^{r-1} - \frac{A_0(z)}{B_0(z)}.$$

If $c_{1,0}c_{2,0} \neq 0$, then $B_0(z) = z^4 - \alpha_0^2$ and $A_0(z) = z((c_{1,0} + c_{2,0})z^2 + (c_{1,0} - c_{2,0})\alpha_0)$. In the case $r = 3$ or $r \geq 5$, the use of Lemma 2.1 gives a contradiction as in Example 2.6. In the case $r = 4$, Lemma 2.10 implies that $G_m(z)$ could be rational only if $c_1 = c_2 = c_3 = 0$, $a = 1$, and $A_0(z)$ were of the form given in (2.9), but this is not the case. Thus we must have $c_{1,0}c_{2,0} = 0$.

If $c_{1,0} \neq 0$ and $c_{2,0} = 0$, then $B_0(z) = z^2 - \alpha_0$, $A_0(z) = c_{1,0}z$, and we get a contradiction from Corollary 2.3 (if $r \geq 4$), or from Lemma 2.7 (if $r = 3$). A similar contradiction follows if $c_{1,0} = 0$ and $c_{2,0} \neq 0$. Thus $c_{1,0} = c_{2,0} = 0$. But then Corollary 2.4 gives a final contradiction, proving Lemma 4.1. □

The above proof does not work in the case $r = 2$. However, in this case, Lemma 2.12 immediately gives the following lemma. Note that the case $\alpha_0 = 1$ leads to exception (1) of Lemma 2.10.

Lemma 4.2. *If $r = 2$ and $\alpha_0 \neq 1, \alpha_1, \ldots, \alpha_m$ are as in Lemma 4.1, then, for any nontrivial choice of $(c_1, c_{1,0}, \ldots, c_{1,m})$, the linear form*

$$c_1F_0^{\langle 1\rangle}(z) + \sum_{\nu=0}^{m} c_{1,\nu}\varphi_\nu(z)$$

is not a rational function.

We now define

$$\varphi_{\nu,j}(x, z) = \left(\frac{\partial}{\partial x}\right)^j \varphi_\nu(x, z), \quad \lambda_{\nu,j}(x, z) = \left(\frac{\partial}{\partial x}\right)^j \lambda_\nu(x, z),$$

and, for distinct $a_1, \ldots, a_s \in \mathbb{C}\backslash\{0\}$,

$$\varphi_{\nu,j,k}(z) = \varphi_{\nu,j}(a_k, z), \quad \lambda_{\nu,j,k}(z) = \lambda_{\nu,j}(a_k, z).$$

Analogously to the proof of Theorem 3.8, we now get the following theorems.

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THEOREM 4.3. *If $r \geq 3$ and $\alpha_0, \alpha_1, \ldots, \alpha_m \in \mathbb{C}\backslash\{0\}$ satisfy $|\alpha_0| = 1$, and $|\alpha_i| \neq |\alpha_j|$ for $i \neq j$, then the functions*

$$F^{\langle 1\rangle}_{0,j,k}(z), \ldots, F^{\langle r-1\rangle}_{0,j,k}(z), \varphi_{0,j,k}(z), \lambda_{0,j,k}(z), \ldots, \varphi_{m,j,k}(z), \lambda_{m,j,k}(z)$$
$$(j = 0, \ldots, \ell;\ k = 1, \ldots, s)$$

are algebraically independent over $\mathbb{C}(z)$.

THEOREM 4.4. *If $r = 2$ and $\alpha_0 \neq 1, \alpha_1, \ldots, \alpha_m$ are as in Theorem 4.3, then the functions*

$$F^{\langle 1\rangle}_{0,j,k}(z), \varphi_{0,j,k}(z), \ldots, \varphi_{m,j,k}(z) \quad (j = 0, \ldots, \ell;\ k = 1, \ldots, s)$$

are algebraically independent over $\mathbb{C}(z)$.

This implies, as in the previous section, the following results for function values.

THEOREM 4.5. *Let $r \geq 3$ and $\alpha_0, \alpha_1, \ldots, \alpha_m$ be nonzero algebraic numbers satisfying $|\alpha_0| = 1$, and $|\alpha_i| \neq |\alpha_j|$ for $i \neq j$. If $\zeta \in U\backslash\{0\}$ is an algebraic number satisfying $\zeta^{2r^\nu} \neq \pm\alpha_i$ for $\nu = 0, 1, \ldots$ and $i = 0, 1, \ldots, m$, then the numbers*

$$F^{\langle 1\rangle}_{0,j,k}(\zeta), \ldots, F^{\langle r-1\rangle}_{0,j,k}(\zeta), \varphi_{0,j,k}(\zeta), \lambda_{0,j,k}(\zeta), \ldots, \varphi_{m,j,k}(\zeta), \lambda_{m,j,k}(\zeta)$$
$$(j = 0, \ldots, \ell;\ k = 1, \ldots, s)$$

are algebraically independent over $\mathbb{Q}$.

THEOREM 4.6. *Let $r = 2$ and $\alpha_0 \neq 1, \alpha_1, \ldots, \alpha_m$ be nonzero algebraic numbers satisfying $|\alpha_0| = 1$, and $|\alpha_i| \neq |\alpha_j|$ for $i \neq j$. If $\zeta \in U\backslash\{0\}$ is an algebraic number satisfying $\zeta^{2r^\nu} \neq \alpha_i$ for $\nu = 0, 1, \ldots$ and $i = 0, 1, \ldots, m$, then the numbers*

$$F^{\langle 1\rangle}_{0,j,k}(\zeta), \varphi_{0,j,k}(\zeta), \ldots, \varphi_{m,j,k}(\zeta) \quad (j = 0, \ldots, \ell;\ k = 1, \ldots, s)$$

are algebraically independent over $\mathbb{Q}$.

We now apply the above Theorem 4.5 with the special choice $m = 2M$, $\alpha_\nu = (-1)^{dr+\nu}\rho_1^{2\nu}$, $\nu = 0, 1, \ldots, M$, $\alpha_{M-\nu} = (-1)^{dr+\nu}\rho_1^{2\nu}$, $\nu = -1, \ldots, -M$, and $\zeta = \rho_1^{-dr^t}$ with $dr^t > M$. Here M is an arbitrary positive integer, fixed for the moment. By defining

$$\Phi_\nu(x, z) = x^t\varphi_\nu(x, z), \Phi_{\nu,j}(x, z) = \left(\frac{\partial}{\partial x}\right)^j \Phi_\nu(x, z),$$
$$\Lambda_\nu(x, z) = x^t\lambda_\nu(x, z), \Lambda_{\nu,j}(x, z) = \left(\frac{\partial}{\partial x}\right)^j \Lambda_\nu(x, z),$$

we obtain, by Theorem 4.5, the algebraic independence of the numbers

$$F^{\langle 1\rangle}_{0,j}(a_k, \rho_1^{-dr^t}), \ldots, F^{\langle r-1\rangle}_{0,j}(a_k, \rho_1^{-dr^t}), \Phi_{\nu,j}(a_k, \rho_1^{-dr^t}), \Lambda_{\nu,j}(a_k, \rho_1^{-dr^t})$$
$$(j = 0, \ldots, \ell;\ k = 1, \ldots, s;\ -M \leq \nu \leq M).$$

Since, by (4.1),

$$C_{1,\nu}\Phi_{\nu,j}(a_k, \rho_1^{-dr^t}) = \varphi^*_{\nu,j,k} - \sum_{\substack{h=0 \\ dr^h+\nu>0}}^{t-1} h\cdots(h-j+1)a_k^{h-j}\frac{1}{F_{dr^h+\nu}}$$

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with

$$\varphi^*_{\nu,j,k} = \sum_{\substack{h=0 \\ dr^h+\nu>0}}^{\infty} h\cdots(h-j+1)a_k^{h-j}\frac{1}{F_{dr^h+\nu}},$$

and the corresponding formula holds also for $\Lambda_{\nu,j}(a_k, \rho_1^{-dr^t})$, we obtain the algebraic independence of

$$F_{0,j}^{\langle 1\rangle}(a_k, \rho_1^{-d}), \dots, F_{0,j}^{\langle r-1\rangle}(a_k, \rho_1^{-d}),\ \varphi_{\nu,j,k}, \lambda_{\nu,j,k}$$
$$(j = 0, \dots, \ell;\ k = 1, \dots, s;\ -M \le \nu \le M)$$

for $r \ge 3$. This proves Theorem 1.9. Furthermore, by noting that the change of α_ν to $-\alpha_\nu$ in $\varphi_\nu(z)$ leads to $\lambda_\nu(z)$, we finally get Theorem 1.10 by using Theorem 4.6.

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