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Measures of maximal and full dimension for smooth maps

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Abstract. For a C^1 non-conformal repeller, this paper proves that there exists an ergodic measure of full Carathéodory singular dimension. For an average conformal hyperbolic set of a C^1 diffeomorphism, this paper constructs a Borel probability measure (with support strictly inside the repeller) of full Hausdorff dimension. If the average conformal hyperbolic set is of a $C^{1+\alpha}$ diffeomorphism, this paper shows that there exists an ergodic measure of maximal dimension.

Key words: measure of full dimension, measure of maximal dimension, repeller, hyperbolic sets
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1. *Introduction*

Let $f : M \to M$ be a $C^{1+\alpha}$ expanding map on a compact smooth Riemannian manifold M with a conformal repeller Λ. Let μ be the unique equilibrium measure corresponding to the Hölder continuous function $-s \log \|D_x f\|$, where s is the unique solution of Bowen's equation

$$P(f|_\Lambda, -s \log \|D_x f\|) = 0,$$

then the following properties hold:

(1) $\dim_H \Lambda = \dim_B \Lambda = s$;
(2) the s-Hausdorff dimension of Λ is positive and finite, moreover, it is equivalent to the equilibrium measure μ;
(3) $\dim_H \mu = \dim_H \Lambda$,

where $\dim_H$ and $\dim_B$ denote the Hausdorff dimension and the box dimension, respectively. The first property of Hausdorff dimension was first established by Bowen in a special case [8]. Ruelle showed the general case in [24], where his proof consists of showing the second property. Falconer [15] obtained the equality between Hausdorff dimension and box dimension. Later, the smoothness $C^{1+\alpha}$ was eventually relaxed to C^1 by Gatzouras and Peres [18]. The third property is clear from the variational principle of topological pressure. Such a measure is called the *measure of full dimension*.

By the variational principle of topological pressure, there exists an equilibrium measure provided that the entropy map $\mu \mapsto h_\mu(f)$ is upper semi-continuous. The existence of measures of full dimension could be regarded as a dimensional version of the existence of an equilibrium state. However, the map $\mu \mapsto \dim_H \mu$ enjoys no continuity property even if the entropy map is upper semi-continuous. This is the crucial difference between dimension and pressure/entropy.

As one can see, a C^1 conformal repeller admits a measure of full dimension. How about the non-conformal and hyperbolic case? The answer is usually negative, although a certain special non-conformal repeller–average conformal repeller, which is introduced in [2], does have an ergodic measure of full dimension (see [10, Theorem E]). So it is natural to generalize the question into two parts: one is to consider a general quantity of dimension type; the other one is to consider the existence of *measures of maximal dimension*, that is, try to find an invariant measure which attains the supremum of the following quantity:

$$\delta(f) = \sup\{\dim_H \mu : \mu \text{ is } f\text{-invariant}\}.$$

This quantity was introduced by Denker and Urbański [14] in the context of one-dimensional complex dynamics, where they considered the supremum over the ergodic measures of positive entropy. Later, this quantity has been intensively studied in one-dimensional complex dynamics (see [25] for more details).

For non-conformal repellers, we consider a substitute quantity of dimension type called Carathéodory singular dimension (see §2.3.2 for a detailed definition) which was introduced by Cao, Pesin, and Zhao [11]. They proved its continuity for $C^{1+\alpha}$ maps under C^1 topology. Later, following the approach described in [21], the authors introduced the Carathéodory singular dimension of invariant measures in [12], and proved that the unique zero of the measure theoretic pressure function equals the Carathéodory singular dimension of ergodic measures. For a general C^1 non-conformal repeller, in this paper, we will prove that there exists an ergodic measure of full Carathéodory singular dimension.

For the existence of measures of maximal dimension for hyperbolic diffeomorphisms, it was shown by Barreira and Wolf [5] that if $f : M \to M$ is a $C^{1+\alpha}$ surface diffeomorphism and Λ is a topological mixing locally maximal hyperbolic set, then there exists an ergodic measure of maximal dimension, see [4, Ch. 5] for the hyperbolic conformal case. In [22], Rams proved the existence of a measure of maximal dimension for piecewise linear horseshoe maps by computing $\delta(f)$ explicitly. In [30], Wolf showed that there exist finitely many measures of maximal dimension for polynomial automorphisms of C^2. However, these are not guaranteed to be measures of full dimension unless the automorphism is volume preserving. It should be noted that only a minimal lack of hyperbolicity may yield no measure of maximal dimension. Urbański and Wolf [26] constructed nonlinear horseshoes of a surface, which are hyperbolic except at one parabolic fixed point, and which do not have any measure of maximal dimension.

For an average conformal hyperbolic (ACH for short) set Λ of a C^1 diffeomorphism, which is introduced in [29], we construct a Borel probability measure μ (not necessarily invariant) on Λ and show that this measure has full Hausdorff dimension, that is, $\dim_H \mu = \dim_H \Lambda$. Our method consists of using the weak Gibbs measure of a continuous function. The desired measure is the product of two weak Gibbs measure with respect to the future and past behavior of the derivative continuous function along unstable and stable directions. The measure constructed in this way has support strictly inside the repeller. If Λ is an average conformal hyperbolic set of a $C^{1+\alpha}$ diffeomorphism, following Barreira and Wolf's approach in [5], this paper proves that there exists an ergodic measure of maximal dimension, which extends the result [5] for average conformal hyperbolic sets.

The paper is organized as follows. In §2, we recall some necessary concepts, such as average conformal hyperbolic set and Carathéodory singular dimension, and give the statement of the main result in this paper. Section 3 presents the detailed proofs of the results in the previous section. Namely, we prove that there exists an ergodic measure of full Carathéodory singular dimension on a C^1 non-conformal repeller; and we construct a Borel probability measure of full Hausdorff dimension on a C^1 average conformal hyperbolic set; finally, we show that there exists an ergodic measure of maximal dimension on a $C^{1+\alpha}$ average conformal hyperbolic set.

2. *Preliminaries and statements*

In this section, we recall some notions in dimension theory and smooth dynamical systems, and give the statement of the main results in this paper. The proofs will be postponed to the next section.

2.1. *Topological pressure.*

Let $f : X \to X$ be a continuous transformation on a compact metric space X equipped with metric d. A subset $F \subset X$ is called an $(n, \epsilon)-$separated set with respect to f if for any two different points $x, y \in F$, we have $d_n(x, y) := \max_{0\le k\le n-1} d(f^k(x), f^k(y)) > \epsilon$. A sequence of continuous functions $\Phi = \{\phi_n\}_{n\ge 1}$ is called *sub-additive*, if

$$\phi_{m+n} \le \phi_n + \phi_m \circ f^n \quad \text{for all } n, m \in \mathbb{N}.$$

Given a sub-additive potential $\Phi = \{\phi_n\}_{n\ge 1}$ on X, put

$$P_n(f, \Phi, \epsilon) = \sup \left\{ \sum_{x\in F} e^{\phi_n(x)} | F \text{ is an } (n, \epsilon)\text{-separated subset of } X \right\}.$$

Definition 2.1. We call the following quantity

$$P(f, \Phi) = \lim_{\epsilon\to 0} \limsup_{n\to\infty} \frac{1}{n} \log P_n(f, \Phi, \epsilon) \tag{2.1}$$

the *sub-additive topological pressure* of (f, Φ).

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Remark 2.1. If $\Phi = \{\phi_n\}_{n\ge 1}$ is *additive* in the sense that $\phi_n(x) = \phi(x) + \phi(fx) + \cdots + \phi(f^{n-1}x)$ for some continuous function $\phi : X \to \mathbb{R}$, we simply write $P(f, \Phi)$ as $P(f, \phi)$. If it is clear from the context of the dynamics, we will simplify the topological pressure as $P(\phi)$.

Cao, Feng, and Huang [9] proved the following variational principle:

$$P(f, \Phi) = \sup\{h_\mu(f) + \mathcal{L}_*(\Phi, \mu) : \mu \in \mathcal{M}_f(X),\ \mathcal{L}_*(\Phi, \mu) \ne -\infty\}, \tag{2.2}$$

where $\mathcal{M}_f(X)$ denotes the space of all f-invariant measures on X, $h_\mu(f)$ denotes the metric entropy of f with respect to μ (see [27] for details of metric entropy), and

$$\mathcal{L}_*(\Phi, \mu) = \lim_{n\to\infty} \frac{1}{n} \int \phi_n d\mu$$

for every $\mu \in \mathcal{M}_f(X)$. The previous limit is well defined. A standard sub-additive argument yields the existence of this limit. A measure $\mu \in \mathcal{M}_f(X)$ that attains the supermum in equation (2.2) is called an equilibrium state of the topological pressure $P(f, \Phi)$.

2.2. *Dimensions of sets and measures.* Now we recall the definitions of Hausdorff and box dimensions of subsets and measures. Given a subset $Z \subset X$, for any $s \ge 0$, let

$$\mathcal{H}^s_\delta(Z) = \inf\left\{ \sum_{i=1}^{\infty} (\text{diam} U_i)^s : \{U_i\}_{i\ge 1} \text{ is a cover of } Z \text{ with } \text{diam} U_i \le \delta, \text{ for all } i \ge 1 \right\}$$

and

$$\mathcal{H}^s(Z) = \lim_{\delta\to 0} \mathcal{H}^s_\delta(Z).$$

The above limit exists, though the limit may be infinity. We call $\mathcal{H}^s(Z)$ the s-Hausdorff measure of Z.

Definition 2.2. The following jump-up value of $\mathcal{H}^s(Z)$

$$\dim_H Z = \inf\{s : \mathcal{H}^s(Z) = 0\} = \sup\{s : \mathcal{H}^s(Z) = \infty\}$$

is called the *Hausdorff dimension* of Z. The *lower and upper box dimension* of Z are defined respectively by

$$\underline{\dim}_B Z = \liminf_{\delta\to 0} \frac{\log N(Z, \delta)}{-\log \delta} \quad \text{and} \quad \overline{\dim}_B Z = \limsup_{\delta\to 0} \frac{\log N(Z, \delta)}{-\log \delta},$$

where $N(Z, \delta)$ denotes the least number of balls of radius δ that are needed to cover the set Z. If $\underline{\dim}_B Z = \overline{\dim}_B Z$, we will denote the common value by $\dim_B Z$ and call it the *box dimension* of Z.

Given a Borel probability measure μ on X, the following quantity

$$\begin{aligned} \dim_H \mu &= \inf\{\dim_H Z : Z \subset X \text{ and } \mu(Z) = 1\} \\ &= \lim_{\delta\to 0} \inf\{\dim_H Z : Z \subset X \text{ and } \mu(Z) \ge 1 - \delta\} \end{aligned}$$

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is called the *Hausdorff dimension of the measure* μ. Similarly, we call the following two quantities

$$\underline{\dim}_B \mu = \lim_{\delta\to 0} \inf\{\underline{\dim}_B Z : Z \subset X \text{ and } \mu(Z) \geq 1-\delta\}$$

and

$$\overline{\dim}_B \mu = \lim_{\delta\to 0} \inf\{\overline{\dim}_B Z : Z \subset X \text{ and } \mu(Z) \geq 1-\delta\}$$

the *lower box dimension* and *upper box dimension* of μ, respectively.

If μ is a finite measure on *X*, the following quantities

$$\bar{d}_\mu(x) = \limsup_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \quad \text{and} \quad \underline{d}_\mu(x) = \liminf_{r\to 0} \frac{\log \mu(B_r(x))}{\log r}$$

are called *lower and upper point-wise dimensions* of μ at point x, respectively, where $B_r(x) = \{y \in X : d(x, y) < r\}$. We recall two basic properties relating these quantities with the Hausdorff dimension of subsets and measures (see [21] for details):

(1) if $\underline{d}_\mu(x) \geqslant a$ for μ almost every $x \in \Lambda$, then $\dim_H \mu \geqslant a$;

(2) if $\underline{d}_\mu(x) \leqslant a$ for every $x \in Z \subset \Lambda$, then $\dim_H Z \leqslant a$.

2.3. *Measures of full Carathéodory singular dimension for repellers.* In this section, we will recall the concept of Carathéodory singular dimension of subsets and invariant measures. For a non-conformal repeller of a C^1 map, we will show that there exists an ergodic measure of full Carathéodory singular dimension.

2.3.1. *Singular valued potentials.* Let $f : M \to M$ be a C^1 transformation of a m_0-dimensional compact smooth Riemannian manifold *M*, and let Λ be a compact *f*-invariant subset of *M*. Denote by $\mathcal{M}(f|_\Lambda)$ and $\mathcal{E}(f|_\Lambda)$ the set of all *f*-invariant measures and ergodic measures on Λ, respectively.

If a compact *f*-invariant subset Λ satisfies the following two properties:

(1) there exists an open neighborhood U of Λ such that $\Lambda = \{x \in U : f^n(x) \in U$ for all $n \geq 0\}$;

(2) there is $\kappa > 1$ such that

$$\|D_x f v\| \geq \kappa \|v\| \quad \text{for all } x \in \Lambda, \quad \text{and } v \in T_x M,$$

where $\|\cdot\|$ is the norm induced by the Riemannian metric on *M*, and $D_x f : T_x M \to T_{f(x)} M$ is the differential operator,

then we call Λ a *repeller* for *f* or *f* is *expanding* on Λ.

Given $x \in \Lambda$ and $n \geq 1$, denote the singular values of $D_x f^n$ (square roots of the eigenvalues of $(D_x f^n)^* D_x f^n$) in the decreasing order by

$$\alpha_1(x, f^n) \geq \alpha_2(x, f^n) \geq \cdots \geq \alpha_{m_0}(x, f^n). \tag{2.3}$$

For $t \in [0, m_0]$, set

$$\varphi^t(x, f^n) := \sum_{i=m_0-[t]+1}^{m_0} \log \alpha_i(x, f^n) + (t - [t]) \log \alpha_{m_0-[t]}(x, f^n). \tag{2.4}$$

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The functions $x \mapsto \alpha_i(x, f^n)$, $x \mapsto \varphi^t(x, f^n)$ are continuous for every $n \geq 1$, since f is smooth. It is easy to see that for all $n, \ell \in \mathbb{N}$,

$$\varphi^t(x, f^{n+\ell}) \geq \varphi^t(x, f^n) + \varphi^t(f^n(x), f^\ell).$$

Hence, the sequence of functions $\Phi_f(t) := \{-\varphi^t(\cdot, f^n)\}_{n\geq 1}$ is sub-additive, which is called the *sub-additive singular valued potentials*.

2.3.2. *Carathéodory singular dimension of sets and measures.* The Carathéodory singular dimension of a repeller is introduced in [11]. Following the approach in [11], we will introduce the notions of Carathéodory singular dimension of subsets and measures.

Let $B_n(x, r) := \{x \in M : d_n(x, y) < r\}$. Given a subset $Z \subseteq \Lambda$, for each small number $r > 0$, let

$$m(Z, t, r) := \lim_{N\to\infty} \inf \left\{ \sum_i \exp\left(\sup_{y\in B_{n_i}(x_i,r)} -\varphi^t(y, f^{n_i})\right)\right\},$$

where the infimum is taken over all collections $\{B_{n_i}(x_i, r)\}$ of Bowen's balls with $x_i \in \Lambda$, $n_i \geq N$ that cover Z. It is easy to see that there is a critical point

$$\dim_{C,r} Z := \inf\{t : m(Z, t, r) = 0\} = \sup\{t : m(Z, t, r) = +\infty\}. \tag{2.5}$$

Consequently, we call the following quantity

$$\dim_C Z := \liminf_{r\to 0} \dim_{C,r} Z$$

the *Carathéodory singular dimension of Z*. In particular, the Carathéodory singular dimension of the repeller Λ is independent of sufficiently small $r > 0$ (see [11, Theorem 4.1]).

For each f-invariant measure μ supported on Λ, let

$$\dim_{C,r} \mu := \inf\{\dim_{C,r} Z : \mu(Z) = 1\},$$

and the following quantity

$$\dim_C \mu := \liminf_{r\to 0} \dim_{C,r} \mu$$

is called the *Carathéodory singular dimension* of the measure μ.

THEOREM A. *Let $f : M \to M$ be a C^1 transformation of an m_0-dimensional compact smooth Riemannian manifold M, and Λ a repeller of f. Then there exists an f-invariant ergodic measure μ such that*

$$\dim_C \mu = \dim_C \Lambda = \sup\{\dim_C \nu : \nu \in \mathcal{M}(f|_\Lambda)\}.$$

2.4. *Measures of full and maximal dimension for ACH sets.* In this section, we first recall the concept of an average conformal hyperbolic set which is introduced in [29]. By modifying the methods in [13], we construct a Borel probability measure (not necessarily invariant) of full Hausdorff dimension for ACH sets of C^1 diffeomorphisms. Following Barreira and Wolf's approach [5], we prove that there exists an ergodic measure of maximal Hausdorff dimension for ACH sets of $C^{1+\alpha}$ diffeomorphisms.

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2.4.1. *Definition of ACH.* Let $f : M \to M$ be a C^1 diffeomorphism on a m_0-dimensional compact Riemannian manifold. For each $x \in M$, the following quantities

$$\|D_x f\| = \sup_{0 \neq v \in T_x M} \frac{\|D_x f(v)\|}{\|v\|}, \quad m(D_x f) = \inf_{0 \neq v \in T_x M} \frac{\|D_x f(v)\|}{\|v\|}$$

are respectively called the maximal norm and minimum norm of the differentiable operator $D_x f : T_x M \to T_{f(x)} M$, where $\|\cdot\|$ is the norm induced by the Riemannian metric on M. A compact f-invariant subset $\Lambda \subset M$ is called a *locally maximal hyperbolic set* if there exists an open neighborhood U such that $\Lambda = \bigcap_{n \in \mathbb{Z}} f^n U$, and a continuous splitting of the tangent bundle $T_x M = E_x^s \oplus E_x^u$, and constants $0 < \lambda < 1$, $C > 0$ such that for every $x \in \Lambda$:

(1) $D_x f(E_x^u) = E_{f(x)}^u$, $D_x f(E_x^s) = E_{f(x)}^s$;
(2) for every $n \in \mathbb{N}$, one has $\|D_x f^{-n}(v)\| \leqslant C\lambda^n \|v\|$ for all $v \in E_x^u$, and $\|D_x f^n(v)\| \leqslant C\lambda^n \|v\|$ for all $v \in E_x^s$.

For $x \in M$ and $v \in T_x M$, the Lyapunov exponent of v at x is the limit

$$\lambda(x, v) = \lim_{n \to \infty} \frac{1}{n} \log \|D_x f^n(v)\|$$

whenever the limit exists. Given an invariant measure $\mu \in \mathcal{M}(f|_\Lambda)$, by the Oseledec multiplicative ergodic theorem [20], for μ-almost every x, every vector $v \in T_x M$ has a Lyapunov exponent, and they can be denoted by $\lambda_1(x) \geqslant \lambda_2(x) \geqslant \cdots \geqslant \lambda_{m_0}(x)$. Furthermore, if μ is ergodic, since the Lyapunov exponents are f-invariant, we write the Lyapunov exponents as $\lambda_1(\mu) \geqslant \lambda_2(\mu) \geqslant \cdots \geqslant \lambda_{m_0}(\mu)$. Notice that $\|D_x f\| = \|D_x f|_{E_x^u}\|$, $\|D_x f^{-1}\| = \|D_x f^{-1}|_{E_x^s}\|$ in the hyperbolic setting.

A hyperbolic set $\Lambda \subset M$ is called an *average conformal hyperbolic set* if for each $\mu \in \mathcal{E}(f|_\Lambda)$, one has $\lambda_1(\mu) = \lambda_2(\mu) = \cdots = \lambda_{d_u}(\mu) > 0$ and $\lambda_{d_u+1}(\mu) = \lambda_{d_u+2}(\mu) = \cdots = \lambda_{m_0}(\mu) < 0$, where $d_u = \dim E^u$ and $d_s = \dim E^s = m_0 - d_u$. In other words, it has only two Lyapunov exponents $\lambda_u(\nu) > 0$ and $\lambda_s(\nu) < 0$ with respect to each $\nu \in \mathcal{E}(f|_\Lambda)$.

2.4.2. *Statements of main results.* Although we cannot obtain an invariant measure of full dimension even in the case of conformal hyperbolic dynamical systems (see [4, Ch. 5] for a detailed description), the following result shows that there exists a measure (not necessarily invariant) of full dimension for ACH sets of a C^1 diffeomorphism.

THEOREM B. *Let $f : M \to M$ be a C^1 diffeomorphism on a m_0-dimensional compact Riemannian manifold, and let $\Lambda \subset M$ be a compact locally maximal hyperbolic set such that $f|_\Lambda$ is topologically mixing and average conformal. Then there exists a Borel probability measure μ on Λ with support strictly inside Λ such that*

$$\dim_H \mu = \dim_H \Lambda.$$

Under the setting of the above theorem, if f is of $C^{1+\alpha}$ smoothness, then there exists an f-invariant ergodic measure of maximal dimension.

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THEOREM C. *Let $f : M \to M$ be a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact Riemannian manifold, and let $\Lambda \subset M$ be a compact locally maximal hyperbolic set such that $f|_\Lambda$ is topologically mixing and average conformal. Then there exists an f-invariant ergodic probability measure μ on Λ such that*

$$\dim_H \mu = \sup\{\dim_H \nu : \nu \in \mathcal{M}(f|_\Lambda)\}.$$

3. *Proofs*

In this section, we provide the detailed proof of the main results in this paper.

3.1. *Proof of Theorem A*.

By the definition of *Carathéodory singular dimension* of subsets and measures, for every $\mu \in \mathcal{M}(f|_\Lambda)$, we have that

$$\dim_C \Lambda = \dim_{C,r} \Lambda \geq \dim_{C,r} \mu$$

for all sufficiently small $r > 0$, see [11, Theorem 4.1] for the first equality. Letting $r \to 0$, one has $\dim_C \Lambda \geq \dim_C \mu$ for every $\mu \in \mathcal{M}(f|_\Lambda)$. Hence, we have that

$$\dim_C \Lambda \geq \sup\{\dim_C \mu : \mu \in \mathcal{M}(f|_\Lambda)\}.$$

However, for each f-invariant measure μ, let

$$P_\mu(f|_\Lambda, \Phi_f(t)) := h_\mu(f) + \mathcal{L}_*(\Phi_f(t), \mu).$$

It is easy to see that $P_\mu(f|_\Lambda, \Phi_f(t)) = 0$ has a unique root, since $P_\mu(f|_\Lambda, \Phi_f(t))$ is strictly decreasing and continuous with respect to t. If $\mu \in \mathcal{E}(f|_\Lambda)$, it follows from [12, Theorem A] that

$$\dim_C \mu = t_\mu,$$

where t_μ is the unique solution of the equation $P_\mu(f|_\Lambda, \Phi_f(t)) = 0$. Let t^* be the unique zero of Bowen's equation $P(f|_\Lambda, \Phi_f(t)) = 0$. Then $\dim_C \Lambda = t^*$ (see [11, Theorem 4.1] for details).

Since f is expanding on Λ, the map $\mu \mapsto P_\mu(f|_\Lambda, \Phi_f(t^*))$ is upper semi-continuous on $\mathcal{M}(f|_\Lambda)$. It follows from the variational principle of sub-additive topological pressure that there exists an f-invariant ergodic measure $\widetilde{\mu}$ so that

$$0 = P(f|_\Lambda, \Phi_f(t^*)) = P_{\widetilde{\mu}}(f|_\Lambda, \Phi_f(t^*)).$$

Hence,

$$\dim_C \Lambda = t^* = t_{\widetilde{\mu}} = \dim_C \widetilde{\mu}.$$

This completes the proof of the theorem. □

3.2. *Proof of Theorem B*.

We first recall some facts of average conformal hyperbolic sets. For each $x \in \Lambda$, denote

$$\phi_u(x) = -\log|\det D_x f|_{E^u_x}|^{1/d_u}, \quad \phi_s(x) = \log|\det D_x f|_{E^s_x}|^{1/d_s}, \tag{3.1}$$

where $d_u = \dim E_x^u$, $d^s = \dim E_x^s$, and $d_u + d_s = m_0$. It is clear that ϕ_u and ϕ_s are continuous functions, since f is a C^1 diffeomorphism. Furthermore, the following properties hold:

(1) for any $n \in \mathbb{Z}$, one has $m(D_x f^n|_{E_x^i}) \leqslant |\det D_x f^n|_{E_x^i}|^{1/d_i} \leqslant \|D_x f^n|_{E_x^i}\|$ and

$$\lim_{n\to\pm\infty} \frac{1}{|n|}(\log \|D_x f^n|_{E_x^i}\| - \log m(D_x f^n|_{E_x^i})) = 0 \tag{3.2}$$

uniformly on Λ, for $i \in \{u, s\}$. In fact, let $\psi_n(x) = \log \|D_x f^n|_{E_x^u}\| - \log m(D_x f^n|_{E_x^u})$. Then the sequence of continuous functions $\Psi := \{\psi_n\}_{n\geq 1}$ is sub-additive. By [19, Theorem A.3], one has

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \max_{x\in\Lambda} \psi_n(x) &= \sup\{\mathcal{L}_*(\Psi, \mu) : \mu \in \mathcal{E}(f|_\Lambda)\}\\ &= \sup\{\lambda_1(\mu) - \lambda_{d_u}(\mu) : \mu \in \mathcal{E}(f|_\Lambda)\}\\ &= 0.\end{aligned}$$

The case of $i = s$ can be proven in a similar fashion. This yields the uniformly convergence in equation (3.2). See [2, Theorem 4.2] for the detailed proof of the case of average conformal repellers;

(2) let t_u and t_s denote the unique root of $P(f|_\Lambda, t\phi_u) = 0$ and $P(f|_\Lambda, t\phi_s) = 0$, respectively. Then

$$\dim_H \Lambda = \dim_B \Lambda = t_u + t_s, \tag{3.3}$$

see [29, Theorem A and Remark 7] for a detailed description.

Since f is hyperbolic on Λ, it is expansive, so we let $0 < c < 1$ be an expansive constant of $f|_\Lambda$. In the rest of the proof of Theorem B, we fix a small number $\delta > 0$. According to equation (3.2), there exists $N(\delta)$ such that

$$1 \leqslant \frac{\|D_x f^n|_{E_x^u}\|}{\exp\{-\sum_{i=0}^{n-1} \phi_u(f^i x)\}} \leqslant e^{n\delta}, \quad e^{-\ell\delta} \leqslant \frac{\|D_x f^{-\ell}|_{E_x^s}\|}{\exp\{-\sum_{i=0}^{\ell-1} \phi_s(f^{-i} x)\}} \leqslant 1$$

for any $n, \ell \geqslant N(\delta)$ and any $x \in \Lambda$. Fix a positive integer $L \geqslant N(\delta)$. It follows from the uniform continuity of the map $x \mapsto \|D_x f^L\|$ that there exists $0 < \varepsilon_0 < c/4$ such that

$$e^{-\delta} \leqslant \frac{\|D_z f^L\|}{\|D_y f^L\|} \leqslant e^{\delta} \quad \text{and} \quad e^{-\delta} \leqslant \frac{\|D_z f^{-L}\|}{\|D_y f^{-L}\|} \leqslant e^{\delta}$$

for any $y, z \in \Lambda$ with $d(y, z) < \varepsilon_0$.

Choose a Markov partition $\mathcal{R} = \{R_1, \ldots, R_s\}$ of Λ such that

$$\operatorname{diam} \mathcal{R} := \max\{\operatorname{diam} R_i \mid i = 1, \ldots, s\} < \varepsilon_0$$

and $\#\{1 \leqslant q \leqslant s : R_p \cap R_q = \varnothing\} > 0$ for every $p \in \{1, \ldots, s\}$ (see [7]). Let $A = (a_{ij})_{1\leqslant i,j\leqslant s}$ be the structure matrix of $\mathcal{R}$ and (Σ_A, σ) be the corresponding Markov subshift. We denote the set of all words of length n of Σ_A by $\Sigma(n)$ and let

$$_{-k}[a_{-k} \ldots a_0 \ldots a_\ell]_\ell = \{\mathbf{b} = (b_i) \in \Sigma_A \mid a_i = b_i, \quad \text{for all } i = -k, \ldots, 0, \ldots, \ell\}$$

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for each $\mathbf{a} = (a_i) \in \Sigma_A$ and $k, \ell \in \mathbb{N}$. Define a coding map $h : \Sigma_A \to \Lambda$ by

$$h({}_{-k}[a_{-k} \ldots a_0 \ldots a_\ell]_\ell) = \bigcap_{j=-k}^{\ell} f^{-j} R_{a_j} \quad \text{for all } (a_{-k} \ldots a_\ell) \in \Sigma(k + \ell + 1),$$

then the map h is a continuous surjection which satisfies $h \circ \sigma = f \circ h$. Let

$$\mathcal{R}(-k, \ell) = \bigg\{ \bigcap_{j=-k}^{\ell} f^{-j} R_{a_j} : (a_{-k} \ldots a_\ell) \in \Sigma(k + \ell + 1)\bigg\}.$$

Since h is bounded finite to one (see [1]), there is an integer $e_0 > 0$ such that

$$\#\{W \in \mathcal{R}(-k, \ell) \mid x \in W\} \leqslant e_0 \tag{3.4}$$

for every $x \in \Lambda$ and $k, \ell \in \mathbb{N}$.

We construct the desired Borel probability measure as follows. Since $t_u\phi_u \circ h$ and $t_s\phi_s \circ h$ are continuous functions and

$$P(\sigma, t_u\phi_u \circ h) = P(f|_\Lambda, t_u\phi_u) = 0, \quad P(\sigma, t_s\phi_s \circ h) = P(f|_\Lambda, t_s\phi_s) = 0,$$

there exist two weak Gibbs measures m_+, m_- on Σ_A in the sense that there exist two sequences of positive constants $\{A_\ell\}_{\ell\in\mathbb{N}}$ and $\{B_k\}_{k\in\mathbb{N}}$ satisfying $\lim_{\ell\to\infty}(1/\ell) \log A_\ell = 0$ and $\lim_{k\to\infty}(1/k) \log B_k = 0$ such that

$$\frac{1}{A_{l+1}} \leqslant \frac{m_+({}_0[a_0 \ldots a_l]_l)}{\exp\bigg\{ \sum_{j=0}^{l} t_u\phi_u \circ h(\sigma^j \mathbf{a})\bigg\}} \leqslant A_{l+1}$$

and

$$\frac{1}{B_{k+1}} \leqslant \frac{m_-({}_{-k}[a_{-k} \ldots a_0]_0)}{\exp\bigg\{ \sum_{j=0}^{k} t_s\phi_s \circ h(\sigma^{-j} \mathbf{a})\bigg\}} \leqslant B_{k+1}$$

for every $\mathbf{a} = (a_i) \in \Sigma_A$ and $k, \ell \in \mathbb{N}$ (see [3, pp. 289]).

Let m be a Borel probability measure on Σ_A such that

$$m({}_{-k}[a_{-k} \ldots a_0 \ldots a_\ell]_\ell) = \begin{cases} C_1 m_+({}_0[a_0 \ldots a_\ell]_\ell) m_-({}_{-k}[a_{-k} \ldots a_0]_0), & a_0 = 1, \\ 0, & a_0 \neq 1, \end{cases}$$

for every $\mathbf{a} = (a_i) \in \Sigma_A$, $k, \ell \in \mathbb{N}$, and $C_1 = m_+({}_0[1]_0)^{-1} m_-({}_0[1]_0)^{-1}$.

Define a Borel probability measure μ on Λ by

$$\mu(A) = m(h^{-1}(A))$$

for each Borel subset $A \subset \Lambda$. It is clear from the definition that the support of μ is R_1. Furthermore, if $a_0 = 1$ and $(a_{-k} \ldots a_0 \ldots a_\ell) \in \Sigma(k + \ell + 1)$, $k, \ell \in \mathbb{N}$, then we have that for $x \in \bigcap_{j=-k}^{\ell} f^{-j} R_{a_j}$,

$$\frac{C_1}{A_{\ell+1} B_{k+1}} \leqslant \frac{\mu(\bigcap_{j=-k}^{\ell} f^{-j} R_{a_j})}{\exp\{\sum_{i=0}^{\ell} t_u\phi_u(f^i x) + \sum_{j=0}^{k} t_s\phi_s(f^{-j} x)\}} \leqslant C_1 A_{\ell+1} B_{k+1}. \tag{3.5}$$

To prove Theorem B, we first prove some auxiliary results.

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LEMMA 3.1. *Let $f : M \to M$ be a C^1 diffeomorphism on an m_0-dimensional compact Riemannian manifold, and let $\Lambda \subset M$ be a compact locally maximal hyperbolic set such that $f|_\Lambda$ is topologically mixing and average conformal. There exists $\varepsilon_0 > 0$ (make ε_0 small if necessarily) such that for any $n, m \in \mathbb{N}$, if $x, z \in \Lambda$ satisfy $\max_{0\leqslant j\leqslant n-1} d(f^j x, f^j z) < \varepsilon_0$ and $y, w \in \Lambda$ satisfy $\max_{0\leqslant j\leqslant m-1} d(f^{-j} y, f^{-j} w) < \varepsilon_0$, then*

$$e^{-n\delta} < \frac{|\det D_z f^n|_{E^u_z}|^{1/d_u}}{|\det D_x f^n|_{E^u_x}|^{1/d_u}} < e^{n\delta}$$

and

$$e^{-m\delta} < \frac{|\det D_w f^{-m}|_{E^s_w}|^{1/d_s}}{|\det D_y f^{-m}|_{E^s_y}|^{1/d_s}} < e^{m\delta}.$$

Proof. Since $\log|\det D_x f|$ is uniformly continuous on Λ, there exists $\varepsilon_0 > 0$ such that if $d(x, z) < \varepsilon_0$, then

$$|\log|\det D_z f| - \log|\det D_x f|| < \delta.$$

Hence, if $x, z \in \Lambda$ satisfy $\max_{0\leqslant j\leqslant n-1} d(f^j x, f^j z) < \varepsilon_0$, then

$$\left|\log \frac{|\det D_z f^n|_{E^u_z}|^{1/d_u}}{|\det D_x f^n|_{E^u_x}|^{1/d_u}}\right| \leq \sum_{j=0}^{n-1} \left|\log \frac{|\det D_{f^j z} f|_{E^u_{f^j z}}|^{1/d_u}}{|\det D_{f^j x} f|_{E^u_{f^j x}}|^{1/d_u}}\right| < n\delta.$$

The other one can be proven in a similar fashion. This completes the proof of the lemma. □

Set $\delta_0 = \inf\{d(x, y) : \ x \in R_p, \ \ y \in R_q, \ \ R_p \cap R_q = \emptyset, \ \ 1 \leqslant p < q \leqslant s\} > 0$, $r_0 = \min\{\delta_0/2, \varepsilon_0/2\} > 0$, and $C_2 = \max_{1\leqslant i\leqslant L-1} \max_{x\in M}\{\|D_x f^i\|\}$.

LEMMA 3.2. *For $n, \ell > L \geqslant N(\delta)$, if $x \in \Lambda$ and $0 < r < r_0$ satisfy that $C_2 r \exp\{2n\delta - \sum_{i=0}^{n-1} \phi_u(f^i x)\} < r_0$ and $C_2 r \exp\{2\ell\delta - \sum_{i=0}^{\ell-1} \phi_s(f^{-i} x)\} < r_0$, then*

$$B_r(x) \subset B^f_{r_0}(x, -\ell, n),$$

where $B^f_{r_0}(x, -\ell, n) = \{z \in M : d(f^j x, f^j z) < r_0$, for all $j = -\ell, \ldots, n\}$.

Proof. Take $y \in B_r(x)$, we first show that $d(f^j x, f^j y) < r_0$ for every $j = 1, \ldots, n$ if $C_2 r \exp\{2n\delta - \sum_{i=0}^{n-1} \phi_u(f^i x)\} < r_0$.

Choose $0 < \varepsilon_1 < \varepsilon_0/2$ so small that $C_2(r + \varepsilon_1) \exp\{2n\delta - \sum_{i=0}^{n-1} \phi_u(f^i x)\} < r_0$. According to the definition of the Riemannian metric, there exists a smooth curve $\xi : [0, 1] \to M$ such that

$$\xi(0) = x, \ \xi(1) = y \quad \text{and} \quad \int_0^1 \|\dot{\xi}(s)\|\, ds \leq r + \varepsilon_1.$$

Since

$$d(x, \xi(t)) \leqslant \int_0^t \|\dot{\xi}(s)\|\, ds \leqslant r + \varepsilon_1 < \varepsilon_0 \ (0 \leqslant t \leqslant 1),$$

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we have

$$\begin{aligned} d(f^L x, f^L \xi(t)) &\le \int_0^t \|D_{\xi(s)} f^L\| \cdot \|\dot{\xi}(s)\|\, ds \\ &\leqslant e^{\delta} \|D_x f^L\| \int_0^t \|\dot{\xi}(s)\|\, ds \\ &= e^{\delta} \|D_x f^L|_{E_x^u}\| \int_0^t \|\dot{\xi}(s)\|\, ds \\ &\leqslant (r+\varepsilon_1) e^{(L+1)\delta} \cdot \prod_{i=0}^{L-1} \|\det D_{f^i x} f|_{E^u_{f^i x}}\|^{1/d_u} \\ &= (r+\varepsilon_1) \exp\left\{(L+1)\delta - \sum_{i=0}^{L-1} \phi_u(f^i x)\right\} \\ &< r_0 (< \varepsilon_0). \end{aligned}$$

Furthermore, we have that

$$\begin{aligned} d(f^{2L} x, f^{2L} \xi(t)) &\leqslant \int_0^t \|D_{\xi(s)} f^L\| \cdot \|D_{f^L \xi(s)} f^L\| \cdot \|\dot{\xi}(s)\|\, ds \\ &\leqslant e^{2\delta} \|D_x f^L\| \cdot \|D_{f^L x} f^L\| \int_0^t \|\dot{\xi}(s)\|\, ds \\ &= e^{2\delta} \|D_x f^L|_{E_x^u}\| \cdot \|D_{f^L x} f^L|_{E^u_{f^L x}}\| \int_0^t \|\dot{\xi}(s)\|\, ds \\ &\leqslant (r+\varepsilon_1) e^{2(L+1)\delta} \cdot \prod_{i=0}^{2L-1} \|\det D_{f^i x} f|_{E^u_{f^i x}}\|^{1/d_u} \\ &= (r+\varepsilon_1) \exp\left\{2(L+1)\delta - \sum_{i=0}^{2L-1} \phi_u(f^i x)\right\}. \end{aligned}$$

Therefore, for every $j = 1, \ldots, n$, write $j = g_j L + t_j$, where $g_j \in \mathbb{N}$ and $0 \leqslant t_j < L$, we have

$$\begin{aligned} d(f^j x, f^j \xi(t)) &\leqslant \int_0^t \prod_{i=0}^{g_j-1} \|D_{f^{iL}\xi(s)} f^L\| \cdot \|D_{f^{g_j L}\xi(s)} f^{t_j}\| \cdot \|\dot{\xi}(s)\|\, ds \\ &\leqslant C_2 e^{g_j \delta} \cdot \prod_{i=0}^{g_j-1} \|D_{f^{iL} x} f^L\| \cdot \int_0^t \|\dot{\xi}(s)\|\, ds \\ &\leqslant C_2 (r+\varepsilon_1) \exp\left\{g_j (L+1)\delta - \sum_{i=0}^{g_j L-1} \phi_u(f^i x)\right\} \\ &\leqslant C_2 (r+\varepsilon_1) \exp\left\{2j\delta - \sum_{i=0}^{j-1} \phi_u(f^i x)\right\} \\ &\leqslant C_2 (r+\varepsilon_1) \exp\left\{2n\delta - \sum_{i=0}^{n-1} \phi_u(f^i x)\right\} \\ &< r_0 \; (0 \leqslant t \leqslant 1). \end{aligned}$$

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Setting $t = 1$, we obtain $d(f^j x, f^j y) \leqslant r_0$ for every $1 \leqslant j \leqslant n$. Analogously, one can show that $d(f^{-j}x, f^{-j}y) < r_0$ for $j = 1, \ldots, \ell$ if $C_2 r \exp\{2\ell\delta - \sum_{i=0}^{\ell-1} \phi_s(f^{-i}x)\} < r_0$. This completes the proof of the lemma. □

For each $x \in \Lambda$ and sufficiently small $0 < r < r_0$, let

$$n_1 = \min\left\{n \in \mathbb{Z}^+ : C_2 r \exp\left\{2(n+1)\delta - \sum_{i=0}^{n} \phi_u(f^i x)\right\} \geqslant r_0\right\},$$

$$n_2 = \min\left\{n \in \mathbb{Z}^+ : C_2 r \exp\left\{2(n+1)\delta - \sum_{i=0}^{n} \phi_s(f^{-i} x)\right\} \geqslant r_0\right\}.$$

It follows from the definition of n_1, n_2 that $n_1, n_2 \to +\infty$ $(r \to 0)$. Recall that $\mathcal{R} = \{R_1, \ldots, R_s\}$ is a Markov partition of Λ and e_0 is defined in equation (3.4).

LEMMA 3.3. *For each $x \in \Lambda$, take a sufficiently small $0 < r < r_0$ so that n_1, $n_2 > L$. Then there exist $W_1, \ldots, W_m \in \mathcal{R}(-n_2, n_1)$ with $m = m(x, r_0, -n_2, n_1) \leq s^2 e_0$ such that*

$$W_k \cap B^f_{r_0}(x, -n_2, n_1) \neq \varnothing \quad \textit{for all } k = 1, \ldots, m,$$

and $B_r(x) \cap \Lambda \subset \bigcup_{k=1}^m W_k$.

Proof. For $y \in \Lambda$, let $\mathcal{R}_y = \{Q \in \mathcal{R} \mid R \cap Q \neq \varnothing, y \in R \in \mathcal{R}\}$ and $P_y = \bigcup_{\mathcal{R}_y} Q$, then $P_y \subset B_{2\varepsilon_0}(y) \cap \Lambda$. From the definition of δ_0, if $z \in \Lambda$ and $d(y, z) < \delta_0$, then for any $Q \in \mathcal{R}$ containing z, $Q \in \mathcal{R}_y$.

Let

$$\mathcal{P}(x, -n_2, n_1) = \left\{W = \bigcap_{j=-n_2}^{n_1} f^{-j} R_{a_j} \in \mathcal{R}(-n_2, n_1) : R_{a_j} \subset P_{f^j x} \text{ for all } j = -n_2, \ldots, n_1\right\}$$

and $\mathcal{P}(x, r_0, -n_2, n_1) = \{W \in \mathcal{P}(x, -n_2, n_1) : B^f_{r_0}(x, -n_2, n_1) \cap W \neq \varnothing\}$. By Lemma 3.2 and the choice of n_1, n_2, we have

$$B_r(x) \cap \Lambda \subset B^f_{r_0}(x, -n_2, n_1) \cap \Lambda \subset \bigcup_{W \in \mathcal{P}(x, r_0, -n_2, n_1)} W.$$

Let $m = \#\mathcal{P}(x, r_0, -n_2, n_1)$ and $\mathcal{P}(x, r_0, -n_2, n_1) = \{W_1, \ldots, W_m\}$. To complete the proof of the lemma, it suffices to show that

$$m = m(x, r_0, -n_2, n_1) \leq s^2 e_0.$$

To prove this, set $\ell_1 = \#\mathcal{R}_{f^{n_1}x}$, $\ell_2 = \#\mathcal{R}_{f^{-n_2}x}$ and take $\alpha_1, \ldots, \alpha_{\ell_1}, \beta_1, \ldots, \beta_{\ell_2} \in \{1, \ldots, s\}$ so that

$$P_{f^{n_1}x} = \bigcup_{p=1}^{\ell_1} R_{\alpha_p}, \quad P_{f^{-n_2}x} = \bigcup_{q=1}^{\ell_2} R_{\beta_q}.$$

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Then,

$$\mathcal{P}(x, r_0, -n_2, n_1) = \bigcup_{p=1}^{l_1} \bigcup_{q=1}^{l_2} Q_{p,q},$$

where $Q_{p,q} = \{W \in \mathcal{P}(x, r_0, -n_2, n_1) : W \subset f^{n_2} R_{\beta_q} \cap f^{-n_1} R_{\alpha_p}\}$. Fix $1 \leqslant p \leqslant \ell_1$, $1 \leqslant q \leqslant \ell_2$ such that $Q_{p,q} \neq \varnothing$. Put $t = \#Q_{p,q}$ and let

$$Q_{p,q} = \{W^1_{p,q}, \ldots, W^t_{p,q}\}.$$

Since (Σ_A, σ) is topologically mixing, there is a $K_0 \geqslant 2$ such that $A^K > 0$ for each $K \geqslant K_0$. We choose $(\alpha_p \omega_1 \omega_2 \ldots \omega_{K_0-1} \beta_q) \in \Sigma(K_0 + 1)$ and take

$$z^i_{p,q} \in W^i_{p,q} \cap \left(\bigcap_{k=1}^{K_0-1} f^{-n_1-k} R_{\omega_k} \right) \quad \text{such that } f^{n_1+n_2+K_0} z^i_{p,q} = z^i_{p,q}$$

for each $i = 1, 2, \ldots, t$.

Hence, for $1 \leq i, \; j \leq t$, one has

$$\begin{aligned} d(f^k z^i_{p,q}, f^k z^j_{p,q}) &\leqslant d(f^k z^i_{p,q}, f^k x) + d(f^k x, f^k z^j_{p,q}) \\ &\leqslant 2\varepsilon_0 + 2\varepsilon_0 \\ &< c \end{aligned}$$

for each $k = -n_2, \ldots, 0, \ldots, n_1$, where c is the expansive constant of f. Moreover, for $k = 1, \ldots, K_0 - 1$, we have that

$$d(f^{n_1+k} z^i_{p,q}, f^{n_1+k} z^j_{p,q}) \leqslant \operatorname{diam} R_{\omega_k} < c.$$

This implies that $d(f^m z^i_{p,q}, f^m z^j_{p,q}) < c$ for every $m \in \mathbb{Z}$. Thus, $z^i_{p,q} = z^j_{p,q}$ for each $1 \leqslant i, j \leqslant t$. Hence, we have that

$$z^1_{p,q} \in \bigcap_{i=1}^{t} W^i_{p,q} = \bigcap_{W \in Q_{p,q}} W.$$

Furthermore, one has $\#Q_{p,q} \leq e_0$ since $\#\{W \in \mathcal{R}(-n_2, n_1) : z^1_{p,q} \in W\} \leqslant e_0$. Hence, we have

$$\begin{aligned} m &= \#\mathcal{P}(x, r_0, -n_2, n_1) \\ &= \sum_{p=1}^{l_1} \sum_{q=1}^{l_2} \#Q_{p,q} \\ &\leqslant s^2 e_0. \end{aligned}$$

This completes the proof of the lemma. □

Using the previous results, we proceed with the proof of Theorem B.

Proof of Theorem B. For every $x \in \operatorname{supp}\mu$ $(= R_1)$ and sufficiently small $0 < r < r_0$, choose $n_1, \; n_2 > L \geqslant N(\delta)$ and $W_1, \ldots, W_m \in \mathcal{R}(-n_2, n_1)$ with $m \leqslant s^2 e_0$, as in Lemma 3.3. For each $k = 1, \ldots, m$, pick

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$$y_k \in W_k \cap B^f_{r_0}(x, -n_2, n_1).$$

By Lemma 3.1, we have

$$\exp\left\{\sum_{i=0}^{n_1} \phi_u(f^i y_k)\right\} \le \exp\left\{(n_1+1)\delta + \sum_{i=0}^{n_1} \phi_u(f^i x)\right\}$$

and

$$\exp\left\{\sum_{j=0}^{n_2} \phi_s(f^{-j} y_k)\right\} \leqslant \exp\left\{(n_2+1)\delta + \sum_{j=0}^{n_2} \phi_s(f^{-j} x)\right\}.$$

Furthermore, for each $k = 1, \ldots, m$, it follows from equation (3.5) that

$$\begin{aligned}
\mu(W_k) &\leqslant C_1 A_{n_1+1} B_{n_2+1} \exp\left\{\sum_{i=0}^{n_1} t_u\phi_u(f^i y_k) + \sum_{j=0}^{n_2} t_s\phi_s(f^{-j} y_k)\right\} \\
&\le C_1 A_{n_1+1} B_{n_2+1} \exp\left\{(t_u(n_1+1) + t_s(n_2+1))\delta + \sum_{i=0}^{n_1} t_u\phi_u(f^i x)\right. \\
&\quad \left. + \sum_{j=0}^{n_2} t_s\phi_s(f^{-j} x)\right\} \\
&\le C_1 A_{n_1+1} B_{n_2+1} \exp\{3(t_u(n_1+1) + t_s(n_2+1))\delta\} \\
&\quad \cdot \exp\left\{-2t_u(n_1+1)\delta + \sum_{i=0}^{n_1} t_u\phi_u(f^i x) - 2t_s(n_2+1)\delta + \sum_{j=0}^{n_2} t_s\phi_s(f^{-j} x)\right\} \\
&\leqslant C_1 A_{n_1+1} B_{n_2+1}\left(\frac{C_2 r}{r_0}\right)^{\dim_H \Lambda} \cdot \exp\{3(t_u(n_1+1) + t_s(n_2+1))\delta\},
\end{aligned}$$

where we use the fact that $\dim_H \Lambda = t_u + t_s$. Hence, we have

$$\begin{aligned}
&\mu(B_r(x) \cap \Lambda) \\
&\quad \leqslant \sum_{k=1}^{m} \mu(W_k) \\
&\quad \leqslant C A_{n_1+1} B_{n_2+1} r^{\dim_H \Lambda} \cdot \exp\{3(t_u(n_1+1) + t_s(n_2+1))\delta\},
\end{aligned} \tag{3.6}$$

where $C = s^2 e_0 C_1 C_2^{\dim_H \Lambda} r_0^{-\dim_H \Lambda}$.

By the definition of $n_1,\ n_2$, we have

$$2n_1\delta - \sum_{i=0}^{n_1-1} \phi_u(f^i x) < \log r_0 - \log C_2 - \log r, \tag{3.7}$$

$$2n_2\delta - \sum_{i=0}^{n_2-1} \phi_s(f^{-i} x) < \log r_0 - \log C_2 - \log r. \tag{3.8}$$

Let $M := \max_{x\in\Lambda}\{\phi_u(x), \phi_s(x)\} < 0$. Thus, by equation (3.7), we conclude

$$\liminf_{r\to 0} \frac{2n_1\delta - n_1 M}{\log r} \geqslant \liminf_{r\to 0} \frac{\log r_0 - \log C_2 - \log r}{\log r},$$

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that is,

$$\liminf_{r\to 0} \frac{n_1}{\log r} \geqslant \frac{-1}{2\delta - M}.$$

Similarly, by equatoin (3.8), one has

$$\liminf_{r\to 0} \frac{n_2}{\log r} \geqslant \frac{-1}{2\delta - M}.$$

Therefore, by equation (3.6), we have

$$\liminf_{r\to 0} \frac{\log \mu(B_r(x))}{\log r} \geqslant \dim_H \Lambda - \frac{3(t_u + t_s)\delta}{2\delta - M}$$

for every $x \in \operatorname{supp}\mu$. The arbitrariness of δ implies

$$\underline{d}_\mu(x) \geqslant \dim_H \Lambda$$

for every $x \in \operatorname{supp}\mu$. Thus, we have that

$$\dim_H \mu \geqslant \dim_H \Lambda.$$

However, the reverse inequality $\dim_H \mu \leqslant \dim_H \Lambda$ clearly follows from the definitions. This completes the proof of Theorem B. □

3.3. *Proof of Theorem C.* Assume that $f : M \to M$ is a $C^{1+\alpha}$ diffeomorphism on an m_0-dimensional compact Riemannian manifold, and let $\Lambda \subset M$ be a compact locally maximal hyperbolic set such that $f|_\Lambda$ is topologically mixing and average conformal. Recall that $\mathcal{M}(f|_\Lambda)$ and $\mathcal{E}(f|_\Lambda)$ denote the set of f-invariant measures and ergodic measures on Λ, respectively.

Recall that the topological pressure $P(\phi)$ of a continuous function $\phi : \Lambda \mapsto \mathbb{R}$ (with respect to $f|_\Lambda$) satisfies the following variational principle:

$$P(\phi) = \sup_{\mu\in\mathcal{M}(f|_\Lambda)} \bigg\{h_\mu(f) + \int \phi \, d\mu\bigg\}. \tag{3.9}$$

Remark that $P(0) = h_{\mathrm{top}}(f)$ is the topological entropy of $f|_\Lambda$. A measure $\mu \in \mathcal{M}(f|_\Lambda)$ which attains the supermum in equation (3.9) is called an *equilibrium measure* of ϕ, and two functions $\phi, \psi : \Lambda \to \mathbb{R}$ are said to be *cohomologous* if $\phi - \psi = \eta - \eta \circ f$ for some continuous function $\eta : \Lambda \to \mathbb{R}$. Denote by $C^\alpha(\Lambda)$ the space of Hölder continuous functions $\varphi : \Lambda \to \mathbb{R}$ with Hölder exponent α. We list several properties of the topological pressure in the following (see [23] for details):

(1) the map $\phi \mapsto P(\phi)$ is analytic in $C^\alpha(\Lambda)$;
(2) each function $\phi \in C^\alpha(\Lambda)$ has a unique equilibrium measure $\nu_\phi \in \mathcal{E}(f|_\Lambda)$;
(3) for each $\phi, \psi \in C^\alpha(\Lambda)$, we have $\nu_\phi = \nu_\psi$ if and only if $\phi - \psi$ is cohomologous to a constant;
(4) for each $\phi, \psi \in C^\alpha(\Lambda)$ and $t \in \mathbb{R}$, we have

$$\frac{d}{dt} P(\phi + t\psi) \geqslant 0,$$

with equality if and only if ψ is cohomologous to a constant.

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Note that the functions ϕ_u and ϕ_s defined in equation (3.1) are α-Hölder continuous in this case. For each $\nu \in \mathcal{M}(f|_\Lambda)$, put

$$\lambda_u(\nu) = -\int \phi_u \, d\nu, \quad \lambda_s(\nu) = \int \phi_s \, d\nu \quad \text{and} \quad d(\nu) = h_\nu(f)\bigg(\frac{1}{\lambda_u(\nu)} - \frac{1}{\lambda_s(\nu)}\bigg),$$

then $\lambda_u(\nu) > 0, \ \lambda_s(\nu) < 0$. Furthermore, if $\nu \in \mathcal{E}(f|_\Lambda)$, then

$$\dim_H \nu = d(\nu). \tag{3.10}$$

See [28] for the detailed proofs, which can be viewed as an extension of Young's results in [31] to the case of an average conformal hyperbolic setting. If $\mu \in \mathcal{M}(f|_\Lambda)$, Fang, Cao, and Zhao [16, Theorem 4.4] proved that

$$\dim_H \mu = \text{ess sup}\{\dim_H \nu : \nu \in \mathcal{E}(f|_\Lambda)\},$$

with the essential supremum taken with respect to the ergodic decomposition τ of μ. See [6, Theorem 2] for the case of hyperbolic surface diffeomorphisms. Their approach extends without change to general conformal hyperbolic diffeomorphisms (see [4, Theorem 13.2.4]). Consequently, to prove Theorem C, it suffices to show that there exists $\mu \in \mathcal{E}(f|_\Lambda)$ so that

$$\dim_H \mu = \sup\{\dim_H \nu : \nu \in \mathcal{E}(f|_\Lambda)\}.$$

Next, one can show the desired result by following *mutatis mutandis* Barreira and Wolf's proof [5] (see also [4, Ch. 5]). We outline some key steps for the reader's convenience.

Consider the following bivariate function:

$$Q : \mathbb{R}^2 \to \mathbb{R}, \quad Q(p, q) = P(p\phi_u + q\phi_s).$$

Since $\phi_u, \phi_s \in C^\alpha(\Lambda)$, $p\phi_u + q\phi_s$ has a unique equilibrium measure $\nu_{p,q} \in \mathcal{E}(f|_\Lambda)$ for each $(p, q) \in \mathbb{R}^2$. Let

$$\lambda_u(p, q) = \lambda_u(\nu_{p,q}), \quad \lambda_s(p, q) = \lambda_s(\nu_{p,q}), \quad h(p, q) = h_{\nu_{p,q}}(f)$$

and $Q(p, q) = h(p, q) - p\lambda_u(p, q) + q\lambda_s(p, q)$. By properties (1)–(4) of the topological pressure, one can show λ_u, λ_s and h as functions in $\mathbb{R}^2$ are real-analytic. Furthermore, let

$$d_u(p, q) = \frac{h(p, q)}{\lambda_u(p, q)}, \quad d_s(p, q) = -\frac{h(p, q)}{\lambda_s(p, q)}.$$

Then d_s and d_u are also real-analytic.

Since the maps $\nu \mapsto \lambda_u(\nu)$ and $\nu \mapsto \lambda_s(\nu)$ are continuous on the compact space $\mathcal{M}(f|_\Lambda)$, put

$$\lambda_u^{\min} = \min_{\mu \in \mathcal{M}(f|_\Lambda)} \lambda_u(\mu), \quad \lambda_u^{\max} = \max_{\mu \in \mathcal{M}(f|_\Lambda)} \lambda_u(\mu),$$
$$\lambda_s^{\min} = \min_{\mu \in \mathcal{M}(f|_\Lambda)} \lambda_s(\mu), \quad \lambda_s^{\max} = \max_{\mu \in \mathcal{M}(f|_\Lambda)} \lambda_s(\mu). \tag{3.11}$$

Set

$$I_u = (\lambda_u^{\min}, \ \lambda_u^{\max}), \quad I_s = (\lambda_s^{\min}, \ \lambda_s^{\max}).$$

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Note that $I_u \neq \varnothing$ (respectively $I_s \neq \varnothing$) if and only if ϕ_u (respectively ϕ_s) is not cohomologous to a constant.

Let $\{\nu_n\}_{n\geqslant 1}$ be a sequence of measures in $\mathcal{E}(f|_\Lambda)$ such that

$$\lim_{n\to\infty} \dim_H \nu_n = \sup\{\dim_H \nu : \nu \in \mathcal{E}(f|_\Lambda)\}.$$

Without loss of generality, assume that $\{\nu_n\}_{n\geqslant 1}$ converges to some measure $m \in \mathcal{M}(f|_\Lambda)$. By equation (3.10) and the upper semi-continuity of the entropy map $\nu \mapsto h_\nu(f)$, one has

$$\limsup_{n\to\infty} \dim_H \nu_n = \limsup_{n\to\infty} d(\nu_n) \leq d(m).$$

To prove the desired result, it suffices to show that there exists $\mu \in \mathcal{E}(f|_\Lambda)$ such that

$$\dim_H \mu = d(m). \tag{3.12}$$

We also note that when m is ergodic, it follows from equation (3.10) that $\dim_H m = d(m)$, this completes the proof. However, m may be non-ergodic.

As in [4, Lemmas 5.24, 5.25, and 5.26], one can show the following properties:

(i) if $\lambda_s(m) \in I_s$, then there exists $p \in [0, h_m(f)/\lambda_u(m)]$ such that $\lambda_u(p, \gamma_s(p)) = \lambda_u(m)$;

(ii) assume that neither ϕ_u nor ϕ_s are cohomologous to a constant, then $\lambda_u(m) \in I_u$ if and only if $\lambda_s(m) \in I_s$;

(iii) if $\lambda_u(p, q) = \lambda_u(m)$ and $\lambda_s(p, q) = \lambda_s(m)$ for some $p, q \in \mathbb{R}$, then $m = \nu_{p,q}$.

Item (ii) implies that it is sufficient to consider the following four cases:

(I) $\lambda_u(m) \in I_u$ and $\lambda_s(m) \in I_s$;

(II) $\lambda_s(m) \in I_s$ and ϕ_u is cohomologous to a constant;

(III) $\lambda_u(m) \in I_u$ and ϕ_s is cohomologous to a constant;

(IV) $\lambda_u(m) \notin I_u$ and $\lambda_s(m) \notin I_s$.

For case (I), as in [5, Lemma 4], one can prove that there exists $(p, q) \in \mathbb{R}^2$ so that $m = \nu_{p,q}$. For cases (II) and (III), following the proof of [5, Lemmas 5 and 6], one can also show that $m = \nu_{p,q}$ for some $(p, q) \in \mathbb{R}^2$. For the last case, one can show that: (1) $\lambda_u(m) = \lambda_u^{\min}$ and $\lambda_s(m) = \lambda_s^{\max}$; (2) there exists $\nu \in \mathcal{E}(f|_\Lambda)$ such that

$$\lambda_u(\nu) = \lambda_u(m), \quad \lambda_s(\nu) = \lambda_s(m) \quad \text{and} \quad h_\nu(f) = h_m(f).$$

This completes the proof of equation (3.12).

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