NUMERICAL STUDIES ON THE STABILITY OF THE SOLAR SYSTEM

Paul Nacozy
The University of Texas at Austin

In this paper, recent numerical and analytical results of the author and others, concerning the stability of the solar system are presented. The results indicate that the actual planetary system could possibly have stable and bounded motion.

In order to clearly distinguish the type of stability that will be referred to here, we will first discuss and define various types of stability of solutions of dynamical systems and, in particular, of gravitational systems. Perhaps one of the most stringent types of stability is asymptotic stability. This type of stability requires that the distance between two particles on neighboring solutions or orbits approaches zero as time approaches infinity. A weaker type of stability is Liapunoff stability. This type of stability requires that the distance between two particles on neighboring solutions can be made as small as desired as time goes to infinity, by adjustment of the initial conditions of the two solutions. An even weaker type of stability, often referred to as orbital stability, merely requires that the distance between two neighboring solutions or orbits can be made to remain as small as desired as time approaches infinity, by adjustment of the initial conditions.

The first two of these types of stability are generally not encountered in gravitational systems. The third type is encountered in some gravitational systems, but in discussions on the stability of the major planetary system we usually do not need to require stability as stringent as orbital stability.

Three somewhat weaker and less stringent types of stability may be defined as follows. The first type is often referred to as Laplacian stability. This type of stability requires that a solution for a system of particles have mutual distances bounded both from below and from above and that the particles have no close approaches or collisions and that no particle escapes to infinity. A second type of stability is referred to as the Komolgoroff-Arnol'd-Moser (KAM) stability. The KAM type of stability requires that a solution be represented by a quasi-periodic function, possessing a finite number of non-commensurable basic frequencies. (If the frequencies were commensurable, then the

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solution would be periodic.) A third type of stability is referred to as Poisson stability. This type of stability requires certain restrictions be placed on three of the osculating orbital elements that represent the orbits of the particles of the system. These three orbital elements are the semi-major axis, the eccentricity, and the inclination. One definition of Poisson stability places the restriction on these three elements that they have no secular trends. Another closely related deformation is that the functions of time that represent these three osculating elements be bounded functions both below and above. Some definitions of Poisson stability restrict their discussion to the semi-major axes, assuming that the eccentricity and inclination behave in a similar fashion to the semi-major axes.

Laplacian stability, KAM stability, and Poisson stability, as they are defined above, are closely related in that they all insure bounded motion, and it is this property of bounded motion that is usually referred to in discussions of the stability of the solar system. The KAM stability is more stringent than the other two for it requires that the motion be represented by purely quasi-periodic functions. We will restrict our discussion here to bounded motion in either the Laplacian or Poisson sense.

Many recent numerical results appear to show that for orbits of the planetary type there exists initial conditions for which the orbits of the planets have bounded motion with a lack of secular trends in the planetary semi-major axes, eccentricities and inclinations, at least for very long times. Cohen, Hubbard and Oesterwinter (1972) have recently provided a solution for the five outer planets by numerical integration over a period of one million years. Their solution shows no noticeable secular trends in these orbital elements, indicating that the motion of the outer planets is apparently bounded, exhibiting Laplacian and Poisson stability, at least for the one million year period. If secular trends do in fact exist, the one million year solution shows that they are so small that their detection would require a numerical integration over times <u>very much longer</u> than the one million year period.

Using the method of surface of sections, numerical calculations have recently been made by Henon and Heiles (1964), Jefferys (1966) and Contopolous (1967). Their results confirm that the KAM type of stability exists for certain gravitational systems, and their results indicate that quasiperiodic motion is possible with masses that are at least as large as the planetary masses and perhaps <u>much</u> larger.

The recent works of Harrington, Szebehely, Ovenden, Birn, Lecar and Franklin are related and all show that for systems of the planetary type, Laplacian stability can occur. Harrington (1972) and Szebehely (1972) show, separately, that if the planetary masses are small enough and their mutual distances are large enough, Laplacian stability can exist. Ovenden, et al (1974) shows that for small enough planetary masses and with mutual distances satisfying certain properties related to near-commensurabilities, Laplacian stability is possible. Birn (1973) and Lecar and Franklin (1974) show that for planetary masses small enough there are certain regions in the phase space which allow stable motion in the sense of Laplace. Within these regions planetary orbits appear stable and outside these regions they appear unstable.

The study that we have undertaken extends the solution of Cohen, Hubbard and Oesterwinter by increasing the values of the planetary masses (Nacozy, 1976 and 1977). It is hoped that a system having increased planetary masses would partially and in some sense simulate the solutions with the actual masses over much longer intervals of time. For the solution with increased masses, the amplitudes of some of the periodic perturbations and the trends of the secular perturbations, if they exist, will be increased as long as the general character of the motion remains unaltered. In particular, third-order secular terms in the semi-major axes, factored by the third power of the masses, if they exist, should be apparent sooner if the planetary masses are increased. If third-order secular terms do exist they would occur in 1/1000 of the time if the planetary masses were increased by ten times. Since, as Kuiper (1973) has pointed out, the stability of the Jupiter-Saturn system is obviously the dominant criterion for the continued existence of the planetary system, we undertook a study in a general three-body problem consisting of Jupiter, Saturn and the Sun. The system utilized the actual mutual inclination of Jupiter and Saturn and the actual osculating initial conditions. The masses of Jupiter and Saturn were increased by a factor γ so that the ratio of the mass of Jupiter to the mass of Saturn remained constant. The factor γ was increased from $\gamma = 1$ (the actual planetary masses) to $\gamma = 1000$ (which gives Jupiter the mass of the Sun). Solutions were obtained by a numerical integration wherein regularization was employed for the closest pair and two different types of integration methods were used; one being an eighth-order Runge-Kutta and the second being a variable-order recurrent power series method. In the integration, various stepsizes were used in order to obtain an estimate of the global truncation error. In addition the energy integral of the system was monitored and its constancy was held to one part in 10^9 (Nacozy, 1977).

For the mass parameter γ less than about 29, our results show that the motions of Jupiter and Saturn are qualitatively similar to the one million year solution of the outer planets by Cohen, Hubbard, and Oesterwinter. As γ is increased beyond 29, the system is altered significantly and unbounded motion is immediately apparent. For γ between 29 and 100, Saturn is ejected from the system after about one or two thousand years, on a highly elliptical orbit. For γ greater than about 100, Saturn is ejected from the system much quicker on a hyperbolic orbit. The crucial result here is that there is apparently a range of values of γ ($\gamma < 29$) that provide only bounded and stable motion and a range of large values ($\gamma > 29$) that provide only unbounded and unstable motion with a very sharp transition from the stable character to the unstable character for the value of $\gamma = 29$. These results are related to the results of Henon and Heiles, Jefferys, Contopoulos and others using the method of surface of section where they show that quasi-periodic and bounded motion exist for small values of the mass parameter and as the mass parameter is increased past a certain value they also show a sudden disruption of the system and sudden transition from quasi-periodic motion to non-quasi-periodic motion.

All of the above mentioned numerical results indicate that the actual Jupiter-Saturn-Sun system has masses of Jupiter and Saturn that are much smaller than those causing breakup of the system. The results might imply that the actual Jupiter-Saturn-Sun system is stable in the sense of Laplace (and Poisson) probably at least for many millions of years and perhaps for very much longer times.

Our results also may be placed in context with a result given by Szebehely (1977) and an additional analytical result that has been obtained recently by Nacozy and Kwok (1978) and Hadjidimetriou (1978). Szebehely has found that the Hill curve corresponding to the general three-body problem considering the Jupiter-Saturn-Sun system is closed around Jupiter and the Sun, excluding Saturn for the mass parameter $\gamma < 14$. As the mass parameter is increased beyond 14 the Hill curve opens allowing Jupiter, Saturn and the Sun to have the possibility of an interchange and hence a subsequent Saturn ejection. Our result, showing that Saturn is ejected when the mass parameter γ is increased beyond 29 shows that even after the Hill curve opens, bounded motion can still persist for larger values of the mass parameter. This result is analogous and possibly an extension of similar results in the circular restricted problem of three bodies where a zero-velocity curve around one of the primaries can open as the mass parameter μ increases while bounded motion persists for the infinitesimal particle until the mass parameter is increased further. The additional result recently found by Nacozy and Kwok, and separately by Hadjidemetriou, shows that the periodic orbit that is close to the actual Jupiter-Saturn-Sun system having the commensurability ratio of 5:2 and having the actual masses of Jupiter, Saturn and the Sun, is stable in the linear sense. As we increase the mass parameter and obtain the family of periodic orbits with the mass parameter as the parameter of the family, we obtain stability up to the mass parameter $\gamma = 39$. For $\gamma > 39$, instability occurs. This result shows that a periodic orbit can persist and remain stable, for larger values of the mass parameter than that which causes breakup for a non-periodic (but possibly quasi-periodic) orbit.

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