

ON SOME SUM-TO-PRODUCT IDENTITIES

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To George Szekeres on his ninetieth birthday

We give new proofs of some sum-to-product identities due to Blecksmith, Brillhart and Gerst, as well as some other such identities found recently by us.

1. INTRODUCTION AND STATEMENT OF RESULTS

In the first two of a sequence of papers, Blecksmith, Brillhart and Gerst [2, 3] give five pairs of simple and beautiful sum-to-product identities, Theorems 2–6 below. We give another such in Theorem 1. We do several new things — we prove Theorem 1 (Section 2), we show that Theorem 2 follows from Theorem 1 (Section 3), we present new proofs of Theorems 3 and 4 (Section 4) and we show that Theorems 5 and 6 follow from Theorems 3 and 4 (Section 5).

In order to state our results, we must explain some by now fairly standard notation.

$$\begin{aligned} (a; q)_\infty &= (1 - a)(1 - aq)(1 - aq^2) \dots, \\ (a_1, a_2, \dots, a_n; q)_\infty &= (a_1; q)_\infty (a_2; q)_\infty \dots (a_n; q)_\infty, \\ \left( \begin{matrix} a_1, a_2, \dots, a_m \\ b_1, b_2, \dots, b_n \end{matrix}; q \right)_\infty &= \frac{(a_1, a_2, \dots, a_m; q)_\infty}{(b_1, b_2, \dots, b_n; q)_\infty}, \\ (q)_\infty &= (q; q)_\infty. \end{aligned}$$

We make use of Jacobi’s triple product identity [1, (2.2.10)]

$$(aq, a^{-1}, q; q)_\infty = \sum_{-\infty}^{\infty} (-1)^n a^n q^{(n^2+n)/2}$$

and the quintuple product identity [4, 5]

$$\left( \begin{matrix} a^2q, a^{-2}, q \\ aq, a^{-1} \end{matrix}; q \right)_\infty = \sum_{-\infty}^{\infty} (-1)^n a^{3n} q^{(3n^2+n)/2} + \sum_{-\infty}^{\infty} (-1)^n a^{3n-1} q^{(3n^2-n)/2}.$$

The theorems referred to above are:

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Received 28th August, 2000

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**THEOREM 1.**

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{2n^2} = 2 \left( q^3, q^5, q^8; q^8 \right)_{\infty},$$

$$\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{2n^2} = 2q \left( q, q^7, q^8; q^8 \right)_{\infty}.$$

**THEOREM 2.** [2, Theorem 1]

$$\sum_{-\infty}^{\infty} (-1)^n q^{2n^2} + \sum_{-\infty}^{\infty} (-1)^n q^{n^2} = 2(q)_{\infty} / (q^4, q^6, q^8, q^{10}, q^{22}, q^{24}, q^{26}, q^{28}; q^{32})_{\infty},$$

$$\sum_{-\infty}^{\infty} (-1)^n q^{2n^2} - \sum_{-\infty}^{\infty} (-1)^n q^{n^2} = 2q(q)_{\infty} / (q^2, q^8, q^{12}, q^{14}, q^{18}, q^{20}, q^{24}, q^{30}; q^{32})_{\infty}.$$

**THEOREM 3.** [2, Theorem 3]

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{3n^2} = 2 \left( q^2, q^6, q^{10}, q^{12}; q^{12} \right)_{\infty},$$

$$\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{3n^2} = 2q \left( q^2, q^6, q^{10}, q^{12}; q^{12} \right)_{\infty}.$$

**THEOREM 4.** [3, Theorem 3]

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{5n^2} = 2 \left( q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20}; q^{20} \right)_{\infty},$$

$$\sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{5n^2} = 2q \left( q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20}; q^{20} \right)_{\infty}.$$

**THEOREM 5.** [3, Theorem 1]

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} + \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left( q^2, q^5, q^7, q^{12}, q^{17}, q^{19}, q^{22}, q^{24}; q^{24} \right)_{\infty},$$

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} - \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left( q, q^{10}, q^{11}, q^{12}, q^{13}, q^{14}, q^{23}, q^{24}; q^{24} \right)_{\infty}.$$

and

**THEOREM 6.** [3, Theorem 4]

$$\sum_{-\infty}^{\infty} q^{n(n+1)} + \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2 \left( q^3, q^7, q^{10}, q^{13}, q^{17}, q^{20}; q^{20} \right)_{\infty},$$

$$\sum_{-\infty}^{\infty} q^{n(n+1)} - \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2 \left( q, q^9, q^{10}, q^{11}, q^{19}, q^{20}; q^{20} \right)_{\infty}.$$

2. PROOF OF THEOREM 1

We shall start by obtaining the 2-dissections

$$(1) \quad (q^3, q^5, q^8; q^8)_\infty^2 = \sum_{u,v=-\infty}^\infty q^{8u^2-2u+8v^2} - q^3 \sum_{u,v=-\infty}^\infty q^{8u^2+6u+8v^2+8v},$$

$$(q, q^7, q^8; q^8)_\infty^2 = \sum_{u,v=-\infty}^\infty q^{8u^2-6u+8v^2} - q \sum_{u,v=-\infty}^\infty q^{8u^2+2u+8v^2+8v}.$$

We have

$$(q^3, q^5, q^8; q^8)_\infty^2 = \left\{ \sum_{-\infty}^\infty (-1)^n q^{4n^2-n} \right\}^2$$

$$= \sum_{m,n=-\infty}^\infty (-1)^{m+n} q^{4m^2-m+4n^2-n}$$

$$= \sum_{m \equiv n \pmod{2}} q^{4m^2-m+4n^2-n} - \sum_{m \not\equiv n \pmod{2}} q^{4m^2-m+4n^2-n}$$

$$= \sum_{u,v=-\infty}^\infty q^{4(u+v)^2-(u+v)+4(u-v)^2-4(u-v)}$$

$$- \sum_{u,v=-\infty}^\infty q^{4(u+v+1)^2-4(u+v+1)+4(u-v)^2-(u-v)}$$

$$= \sum_{u,v=-\infty}^\infty q^{8u^2-2u+8v^2} - q^3 \sum_{u,v=-\infty}^\infty q^{8u^2+6u+8v^2+8v}$$

as claimed. The other identity is proved similarly.

It follows that

$$(2) \quad (q^3, q^5, q^8; q^8)_\infty^2 + q(q, q^7, q^8; q^8)_\infty^2 = \frac{(q^2)_\infty^4}{(q)_\infty (q^4)_\infty},$$

$$(q^3, q^5, q^8; q^8)_\infty^2 - q(q, q^7, q^8; q^8)_\infty^2 = \frac{(q)_\infty (q^4)_\infty^6}{(q^2)_\infty^3 (q^8)_\infty^2}.$$

For,

$$(q^3, q^5, q^8; q^8)_\infty^2 + q(q, q^7, q^8; q^8)_\infty^2$$

$$= \sum_{u,v=-\infty}^\infty q^{8u^2-2u+8v^2} - q^2 \sum_{u,v=-\infty}^\infty q^{8u^2+2u+8v^2+8v}$$

$$\begin{aligned}
 &+ q \sum_{u,v=-\infty}^{\infty} q^{8u^2-6u+8v^2} - q^3 \sum_{u,v=-\infty}^{\infty} q^{8u^2+6u+8v^2+8v} \\
 = &\sum_{-\infty}^{\infty} q^{8u^2-2u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\
 &+ q \sum_{-\infty}^{\infty} q^{8u^2-6u} \left\{ \sum_{-\infty}^{\infty} q^{8v^2} - \sum_{-\infty}^{\infty} q^{8v^2+8v+2} \right\} \\
 = &\sum_{-\infty}^{\infty} (-1)^v q^{2v^2} \left\{ \sum_{-\infty}^{\infty} q^{8u^2-2u} + \sum_{-\infty}^{\infty} q^{8u^2-6u+1} \right\} \\
 = &\sum_{-\infty}^{\infty} (-1)^v q^{2v^2} \sum_{-\infty}^{\infty} q^{2u^2-u} \\
 = &\prod_{n \geq 1} \frac{1-q^{2n}}{1+q^{2n}} \prod_{n \geq 1} \frac{(1-q^{2n})^2}{1-q^n} \\
 = &\frac{(q^2)_{\infty}^4}{(q)_{\infty}(q^4)_{\infty}}
 \end{aligned}$$

as claimed, and the other identity is proved similarly.

If we divide (2) by

$$(q, q^3, q^4, q^5, q^7, q^8; q^8)_{\infty} = (q, q^3, q^4; q^4)_{\infty} = \frac{(q)_{\infty}(q^4)_{\infty}}{(q^2)_{\infty}}$$

we obtain

$$(3) \quad \left( \begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7; q^8 \end{matrix} \right)_{\infty} + q \left( \begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5; q^8 \end{matrix} \right)_{\infty} = \frac{(q^2)_{\infty}^5}{(q)_{\infty}^2 (q^4)_{\infty}^2} = \sum_{-\infty}^{\infty} q^{n^2}$$

and

$$\left( \begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7; q^8 \end{matrix} \right)_{\infty} - q \left( \begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5; q^8 \end{matrix} \right)_{\infty} = \frac{(q^4)_{\infty}^5}{(q^2)_{\infty}^2 (q^8)_{\infty}^2} = \sum_{-\infty}^{\infty} q^{2n^2}.$$

Theorem 1 follows. □

### 3. PROOF OF THEOREM 2

We start by proving

**COROLLARY TO THEOREM 1.**

$$(4) \quad \begin{aligned}
 \left( \begin{matrix} q^3, q^5, q^8 \\ q, q^4, q^7; q^8 \end{matrix} \right)_{\infty} &= \left( \begin{matrix} q^6, q^{10}, q^{16} \\ q^2, q^8, q^{14}; q^{16} \end{matrix} \right)_{\infty} + q \left( \begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_{\infty}, \\
 \left( \begin{matrix} q, q^7, q^8 \\ q^3, q^4, q^5; q^8 \end{matrix} \right)_{\infty} &= \left( \begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_{\infty} - q \left( \begin{matrix} q^2, q^{14}, q^{16} \\ q^6, q^8, q^{10}; q^{16} \end{matrix} \right)_{\infty}.
 \end{aligned}$$

PROOF: We have

$$\begin{aligned}
 2 \left( q^3, q^5, q^8; q^8 \right)_\infty &= \sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{2n^2} \\
 &= \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{(2n+1)^2} + \sum_{-\infty}^{\infty} q^{2n^2} \\
 &= \left\{ \sum_{-\infty}^{\infty} q^{2n^2} + \sum_{-\infty}^{\infty} q^{4n^2} \right\} + 2q \sum_{n \geq 0} q^{4n^2+4n} \\
 &= 2 \left( q^6, q^{10}, q^{16}; q^{16} \right)_\infty + 2q \left( q^{16}; q^8 \right)_\infty
 \end{aligned}$$

as claimed. The other identity is proved similarly. □

Next we show that

$$\begin{aligned}
 (5) \quad (q)_\infty &= (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty \\
 &\quad - q(q^4, q^6, q^{10}, q^{16}, q^{22}, q^{26}, q^{28}, q^{32}, q^{32})_\infty.
 \end{aligned}$$

We have

$$\begin{aligned}
 (q)_\infty &= \sum_{-\infty}^{\infty} (-1)^n q^{(3n^2-n)/2} \\
 &= \sum_{-\infty}^{\infty} q^{(3(4n)^2-(4n))/2} - \sum_{-\infty}^{\infty} q^{(3(4n-1)^2-(4n-1))/2} \\
 &\quad - \sum_{-\infty}^{\infty} q^{(3(4n+1)^2-(4n+1))/2} + \sum_{-\infty}^{\infty} q^{(3(4n+2)^2-(4n+2))/2} \\
 &= \left\{ \sum_{-\infty}^{\infty} q^{24n^2-2n} - \sum_{-\infty}^{\infty} q^{24n^2-14n+2} \right\} - \left\{ \sum_{-\infty}^{\infty} q^{24n^2+10n+1} - \sum_{-\infty}^{\infty} q^{24n^2+22n+5} \right\} \\
 &= \left( q^4, q^{12}, q^{16}; -q^2, -q^{14}; q^{16} \right)_\infty - q \left( q^4, q^{12}, q^{16}; -q^6, -q^{10}; q^{16} \right)_\infty \\
 &= (q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32})_\infty \\
 &\quad - q(q^4, q^6, q^{10}, q^{16}, q^{22}, q^{26}, q^{28}, q^{32}, q^{32})_\infty.
 \end{aligned}$$

PROOF OF THEOREM 2: From (5) and (4),

$$\begin{aligned}
 2(q)_\infty / (q^4, q^6, q^8, q^{10}, q^{22}, q^{24}, q^{26}, q^{28}, q^{32})_\infty \\
 = 2 \left( q^2, q^{12}, q^{14}, q^{16}, q^{18}, q^{20}, q^{30}, q^{32}; q^{32} \right)_\infty - 2q \left( q^{16}, q^{32}; q^8, q^{24}; q^{32} \right)_\infty
 \end{aligned}$$

$$\begin{aligned}
 &= 2 \left( \begin{matrix} q^{12}, q^{20} \\ q^4, q^{28}; q^{32} \end{matrix} \right)_\infty \left( \begin{matrix} q^2, q^{14}, q^{16} \\ q^6, q^8, q^{10}; q^{16} \end{matrix} \right)_\infty - 2q \left( \begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_\infty \\
 &= 2 \left( \begin{matrix} q^{12}, q^{20} \\ q^4, q^{28}; q^{32} \end{matrix} \right)_\infty \left\{ \left( \begin{matrix} q^{32} \\ q^{16}; q^{32} \end{matrix} \right)_\infty - q^2 \left( \begin{matrix} q^4, q^{28}, q^{32} \\ q^{12}, q^{16}, q^{20}; q^{32} \end{matrix} \right)_\infty \right\} \\
 &\qquad\qquad\qquad - 2q \left( \begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_\infty \\
 &= 2 \left( \begin{matrix} q^{12}, q^{20}, q^{32} \\ q^4, q^{16}, q^{28}; q^{32} \end{matrix} \right)_\infty - 2q^2 \left( \begin{matrix} q^{32} \\ q^{16}; q^{32} \end{matrix} \right)_\infty - 2q \left( \begin{matrix} q^{16} \\ q^8; q^{16} \end{matrix} \right)_\infty \\
 &= \left\{ \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{8n^2} \right\} - \sum_{-\infty}^{\infty} q^{2(2n+1)^2} - \sum_{-\infty}^{\infty} q^{(2n+1)^2} \\
 &= \left\{ \sum_{-\infty}^{\infty} q^{4n^2} - \sum_{-\infty}^{\infty} q^{(2n+1)^2} \right\} + \left\{ \sum_{-\infty}^{\infty} q^{8n^2} - \sum_{-\infty}^{\infty} q^{2(2n+1)^2} \right\} \\
 &= \sum_{-\infty}^{\infty} (-1)^n q^{n^2} + \sum_{-\infty}^{\infty} (-1)^n q^{2n^2}.
 \end{aligned}$$

as claimed. The other identity is proved in similar fashion. □

#### 4. PROOFS OF THEOREMS 3 AND 4

**PROOF OF THEOREM 3:** We start by obtaining the 3-dissections

$$\begin{aligned}
 (6) \quad &\left( \begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11}; q^{12} \end{matrix} \right)_\infty = \left( \begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^9, q^{27}, q^{33}; q^{36} \end{matrix} \right)_\infty + q \left( \begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33}; q^{36} \end{matrix} \right)_\infty, \\
 &\left( \begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q^3, q^5, q^7, q^9; q^{12} \end{matrix} \right)_\infty = \left( \begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33}; q^{36} \end{matrix} \right)_\infty - q^2 \left( \begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^9, q^{15}, q^{21}, q^{27}; q^{36} \end{matrix} \right)_\infty
 \end{aligned}$$

We have

$$\begin{aligned}
 &\left( \begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11}; q^{12} \end{matrix} \right)_\infty = \left( \begin{matrix} q^6 \\ q^3, q^9; q^{12} \end{matrix} \right)_\infty \left( \begin{matrix} q^2, q^{10}, q^{12}, q^{14}, q^{22}, q^{24} \\ q, q^{11}, q^{13}, q^{23}; q^{24} \end{matrix} \right)_\infty \\
 &= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \prod_{n \geq 1} (1 + q^{12n-11})(1 + q^{12n-1})(1 - q^{12n}) \\
 &\qquad\qquad\qquad \times \prod_{n \geq 1} (1 - q^{24n-14})(1 - q^{24n-10})(1 - q^{24n}) \\
 &= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \sum_{-\infty}^{\infty} q^{6n^2-5n} \sum_{-\infty}^{\infty} (-1)^n q^{12n^2-2n} \\
 &= \frac{(-q^3; q^6)_\infty}{(q^{24})_\infty} \sum_{m, n = -\infty}^{\infty} (-1)^n q^{6m^2-5m+12n^2-2n}.
 \end{aligned}$$

We now split the sum into three, according to the residue modulo 3 of  $m + n$ .

If  $m + n \equiv 0 \pmod{3}$ ,  $m - 2n \equiv 0$ , let  $s = (m + n)/3$ ,  $t = (m - 2n)/3$ ,  $m = 2s + t$ ,  $n = s - t$ ;

if  $m + n \equiv 1$ ,  $s = (m + n - 1)/3$ ,  $t = (m - 2n - 1)/3$ ,  $m = 2s + t + 1$ ,  $n = s - t$ ;

if  $m + n \equiv -1$ ,  $s = (m + n + 1)/3$ ,  $t = (m - 2n + 1)/3$ ,  $m = 2s + t - 1$ ,  $n = s - t$ .

We obtain

$$\begin{aligned} & \left( \begin{matrix} q^2, q^6, q^{10}, q^{12} \\ q, q^3, q^9, q^{11}; q^{12} \end{matrix} \right)_{\infty} \\ &= \frac{(-q^3; q^6)_{\infty}}{(q^{24})_{\infty}} \left\{ \sum_{s,t=-\infty}^{\infty} (-1)^{s-t} q^{6(2s+t)^2 - 5(2s+t) + 12(s-t)^2 - 2(s-t)} \right. \\ & \quad + \sum_{s,t=-\infty}^{\infty} (-1)^{s-t} q^{6(2s+t+1)^2 - 5(2s+t+1) + 12(s-t)^2 - 2(s-t)} \\ & \quad \left. + \sum_{s,t=-\infty}^{\infty} (-1)^{s-t} q^{6(2s+t-1)^2 - 5(2s+t-1) + 12(s-t)^2 - 2(s-t)} \right\} \\ &= \frac{(-q^3; q^6)_{\infty}}{(q^{24})_{\infty}} \left\{ \sum_{s,t=-\infty}^{\infty} (-1)^{s+t} q^{36s^2 - 12s + 18t^2 - 3t} \right. \\ & \quad + \sum_{s,t=-\infty}^{\infty} (-1)^{s+t} q^{36s^2 + 12s + 18t^2 + 9t + 1} \\ & \quad \left. + \sum_{s,t=-\infty}^{\infty} (-1)^{s+t} q^{36s^2 - 36s + 18t^2 - 15t + 11} \right\} \\ &= (-q^3; q^6)_{\infty} \left\{ (q^{15}, q^{21}, q^{36}; q^{36})_{\infty} + q(q^9, q^{27}, q^{36}; q^{36})_{\infty} \right\} \\ &= \left( \begin{matrix} q^6, q^{18}, q^{30} \\ q^3, q^9, q^{15}, q^{21}, q^{27}, q^{33}; q^{36} \end{matrix} \right)_{\infty} \left\{ (q^{15}, q^{21}, q^{36}; q^{36})_{\infty} + q(q^9, q^{27}, q^{36}; q^{36})_{\infty} \right\} \\ &= \left( \begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^9, q^{27}, q^{33}; q^{36} \end{matrix} \right)_{\infty} + q \left( \begin{matrix} q^6, q^{18}, q^{30}, q^{36} \\ q^3, q^{15}, q^{21}, q^{33}; q^{36} \end{matrix} \right)_{\infty}, \end{aligned}$$

as claimed. The other identity is proved in similar fashion.

Now

$$\begin{aligned} q \left( \frac{q^6, q^{18}, q^{30}, q^{36}}{q^3, q^{15}, q^{21}, q^{33}}; q^{36} \right)_{\infty} &= q \prod_{n \geq 1} (1 + q^{18n-15})(1 + q^{18n-3})(1 - q^{18n}) \\ &= q \sum_{-\infty}^{\infty} q^{9n^2-6n} = \sum_{-\infty}^{\infty} q^{(3n-1)^2} = \frac{1}{2} \sum_{3 \nmid n} q^{n^2} \\ &= \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2} \right\}. \end{aligned}$$

If we write

$$P_1(q) = \left( \frac{q^2, q^6, q^{10}, q^{12}}{q, q^3, q^9, q^{11}}; q^{12} \right)_{\infty}, \quad P_2(q) = \left( \frac{q^2, q^6, q^{10}, q^{12}}{q^3, q^5, q^7, q^9}; q^{12} \right)_{\infty},$$

then (6) becomes

$$\begin{aligned} P_1(q) &= P_1(q^3) + \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2} \right\}, \\ qP_2(q) &= \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{9n^2} \right\} - q^3 P_2(q^3). \end{aligned}$$

Theorem 3 follows by iteration, since  $P_1(q^{3^n}), P_2(q^{3^n}) \rightarrow 1$  as  $n \rightarrow \infty$ . □

PROOF OF THEOREM 4: We start by obtaining the 5-dissections

$$\begin{aligned} (7) \quad \left( \frac{q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20}}{q, q^4, q^9, q^{11}, q^{16}, q^{19}}; q^{20} \right)_{\infty} &= \left( \frac{q^{10}, q^{40}, q^{50}, q^{60}, q^{90}, q^{100}}{q^5, q^{20}, q^{45}, q^{55}, q^{80}, q^{95}}; q^{100} \right)_{\infty} \\ &\quad + q \left( \frac{q^{30}, q^{50}, q^{70}, q^{100}}{q^{15}, q^{35}, q^{65}, q^{85}}; q^{100} \right)_{\infty} \\ &\quad + q^4 \left( \frac{q^{10}, q^{50}, q^{90}, q^{100}}{q^5, q^{45}, q^{55}, q^{95}}; q^{100} \right)_{\infty}, \\ \left( \frac{q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20}}{q^3, q^7, q^8, q^{12}, q^{13}, q^{17}}; q^{20} \right)_{\infty} &= \left( \frac{q^{30}, q^{50}, q^{70}, q^{100}}{q^{15}, q^{35}, q^{65}, q^{85}}; q^{100} \right)_{\infty} \\ &\quad + q^3 \left( \frac{q^{10}, q^{50}, q^{90}, q^{100}}{q^5, q^{45}, q^{55}, q^{95}}; q^{100} \right)_{\infty} \\ &\quad - q^4 \left( \frac{q^{20}, q^{30}, q^{50}, q^{70}, q^{80}, q^{100}}{q^{15}, q^{35}, q^{40}, q^{60}, q^{65}, q^{85}}; q^{100} \right)_{\infty} \end{aligned}$$



We have

$$\begin{aligned}
 & \left( q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} ; q^{20} \right)_{\infty} \\
 &= \frac{1}{(q^{20})_{\infty}} \left( \begin{matrix} q^8, q^{12}, q^{20} \\ q^4, q^{16} \end{matrix} ; q^{20} \right)_{\infty} \left( q^2, q^{10}, q^{18}, q^{20} ; q^{20} \right)_{\infty} \\
 &= \frac{1}{(q^{20})_{\infty}} \left( \begin{matrix} q^8, q^{12}, q^{20} \\ q^4, q^{16} \end{matrix} ; q^{20} \right)_{\infty} \prod_{n \geq 1} (1 + q^{10n-9})(1 + q^{10n-1})(1 - q^{10n}) \\
 &= \frac{1}{(q^{20})_{\infty}} \left\{ \sum_{-\infty}^{\infty} (-1)^n q^{30n^2-2n} + \sum_{-\infty}^{\infty} (-1)^n q^{30n^2-22n+4} \right\} \sum_{-\infty}^{\infty} q^{5n^2-4n} \\
 &= \frac{1}{(q^{20})_{\infty}} \left\{ \sum_{m,n=-\infty}^{\infty} (-1)^m q^{30m^2-2m+5n^2-4n} \right. \\
 & \qquad \qquad \qquad \left. + \sum_{m,n=-\infty}^{\infty} (-1)^m q^{30m^2-22m+5n^2-4n+4} \right\}.
 \end{aligned}$$

We now split each sum into five, according to the residue modulo 5 of  $m + 2n$ .

If  $m + 2n \equiv 0 \pmod{5}$ ,  $2m - n \equiv 0$ ,  $3m + n \equiv 0$ , let

$$s = \frac{1}{5}(3m + n), \quad t = \frac{1}{5}(2m - n), \quad m = s + t, \quad n = 2s - 3t;$$

if  $m + 2n \equiv 1$ ,  $s = (3m + n + 2)/5$ ,  $t = (2m - n - 2)/5$ ,  $m = s + t$ ,  $n = 2s - 3t - 2$ ;

if  $m + 2n \equiv 2$ ,  $s = (3m + n - 1)/5$ ,  $t = (2m - n + 1)/5$ ,  $m = s + t$ ,  $n = 2s - 3t + 1$ ;

if  $m + 2n \equiv -1$ ,  $s = (3m + n - 2)/5$ ,  $t = (2m - n + 2)/5$ ,  $m = s + t$ ,  $n = 2s - 3t + 2$ ;

if  $m + 2n \equiv -2$ ,  $s = (3m + n + 1)/5$ ,  $t = (2m - n - 1)/5$ ,  $m = s + t$ ,  $n = 2s - 3t - 1$ .

If we make these substitutions into the two sums, we obtain ten sums, each of which separates into the product of two sums, each summable via the triple product identity. Two of the ten sums are zero and two cancel, leaving six which can be grouped in pairs in an obvious way, and each pair summed by the quintuple product identity, leading to the stated identity. The other identity is proved in similar fashion.

Now,

$$\begin{aligned}
 & q \left( \begin{matrix} q^{30}, q^{50}, q^{70}, q^{100} \\ q^{15}, q^{35}, q^{65}, q^{85} \end{matrix} ; q^{100} \right)_{\infty} + q^4 \left( \begin{matrix} q^{10}, q^{50}, q^{90}, q^{100} \\ q^5, q^{45}, q^{55}, q^{95} \end{matrix} ; q^{100} \right)_{\infty} \\
 &= q \prod_{n \geq 1} (1 + q^{50n-35})(1 + q^{50n-15})(1 - q^{50n}) \\
 & \qquad \qquad \qquad + q^4 \prod_{n \geq 1} (1 + q^{50n-45})(1 + q^{50n-5})(1 - q^{50n})
 \end{aligned}$$

$$\begin{aligned}
 &= q \sum_{-\infty}^{\infty} q^{25n^2-10n} + q^4 \sum_{-\infty}^{\infty} q^{25n^2-20n} = \sum_{-\infty}^{\infty} q^{(5n-1)^2} + \sum_{-\infty}^{\infty} q^{(5n-2)^2} = \frac{1}{2} \sum_{5 \nmid n} q^{n^2} \\
 &= \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2} \right\}.
 \end{aligned}$$

If we write

$$P_1(q) = \left( q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20}; q^{20} \right)_{\infty}, \quad P_2(q) = \left( q^4, q^6, q^{10}, q^{14}, q^{16}, q^{20}; q^{20} \right)_{\infty}$$

then (7) becomes

$$\begin{aligned}
 P_1(q) &= P_1(q^5) + \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2} \right\}, \\
 qP_2(q) &= \frac{1}{2} \left\{ \sum_{-\infty}^{\infty} q^{n^2} - \sum_{-\infty}^{\infty} q^{25n^2} \right\} - q^5 P_2(q^5).
 \end{aligned}$$

Theorem 4 follows by iteration, since  $P_1(q^{5^n}), P_2(q^{5^n}) \rightarrow 1$  as  $n \rightarrow \infty$ . □

### 5. PROOFS OF THEOREMS 5 AND 6

PROOF OF THEOREM 5: We start by proving

$$\begin{aligned}
 (8) \quad &\prod_{n \geq 1} (1 + q^{12n-11})(1 + q^{12n-9})(1 + q^{12n-3})(1 + q^{12n-1})(1 - q^{12n})^2 \\
 &= \prod_{n \geq 1} (1 + q^{24n-20})(1 + q^{24n-14})(1 + q^{24n-10})(1 + q^{24n-4})(1 - q^{24n})^2 \\
 &\quad + q \prod_{n \geq 1} (1 + q^{24n-22})(1 + q^{24n-16})(1 + q^{24n-8})(1 + q^{24n-2})(1 - q^{24n})^2.
 \end{aligned}$$

We have

$$\begin{aligned}
 &\prod_{n \geq 1} (1 + q^{12n-11})(1 + q^{12n-9})(1 + q^{12n-3})(1 + q^{12n-1})(1 - q^{12n})^2 \\
 &= \sum_{m, n = -\infty}^{\infty} q^{6m^2-5m+6n^2-3n} \\
 &= \sum_{m \equiv n \pmod{2}} q^{6m^2-5m+6n^2-3n} + \sum_{m \not\equiv n \pmod{2}} q^{6m^2-5m+6n^2-3n}
 \end{aligned}$$

$$\begin{aligned}
 &= \sum_{u,v=-\infty}^{\infty} q^{6(u+v)^2-5(u+v)+6(u-v)^2-3(u-v)} \\
 &\quad + \sum_{u,v=-\infty}^{\infty} q^{6(u+v+1)^2-5(u+v+1)+6(u-v)^2-3(u-v)} \\
 &= \sum_{u,v=-\infty}^{\infty} q^{12u^2-8u+12v^2-2v} + q \sum_{u,v=-\infty}^{\infty} q^{12u^2+4u+12v^2+10v} \\
 &= \prod_{n \geq 1} (1 + q^{24n-20})(1 + q^{24n-14})(1 + q^{24n-10})(1 + q^{24n-4})(1 - q^{24n})^2 \\
 &\quad + q \prod_{n \geq 1} (1 + q^{24n-22})(1 + q^{24n-16})(1 + q^{24n-8})(1 + q^{24n-2})(1 - q^{24n})^2.
 \end{aligned}$$

We can write (8) as follows.

$$\begin{aligned}
 &\left( q^2, q^6, q^{12}, q^{12}, q^{18}, q^{22}, q^{24}, q^{24} \right. \\
 &\quad \left. q, q^3, q^9, q^{11}, q^{13}, q^{15}, q^{21}, q^{23}; q^{24} \right)_{\infty} \\
 &= \left( q^8, q^{24}, q^{24}, q^{40}, q^{48}, q^{48} \right. \\
 &\quad \left. q^4, q^{10}, q^{14}, q^{34}, q^{38}, q^{44}; q^{48} \right)_{\infty} + q \left( q^4, q^{24}, q^{24}, q^{44}, q^{48}, q^{48} \right. \\
 &\quad \left. q^2, q^8, q^{22}, q^{26}, q^{40}, q^{46}; q^{48} \right)_{\infty}
 \end{aligned}$$

If we multiply by  $\left( q^{10}, q^{14} \right. \left. q^{12}, q^{24}; q^{24} \right)_{\infty}$  we obtain

$$\begin{aligned}
 (9) \quad &\left( q^2, q^6, q^{10}, q^{12} \right. \\
 &\quad \left. q, q^3, q^9, q^{11}; q^{12} \right)_{\infty} = \left( q^8, q^{24}, q^{40}, q^{48} \right. \\
 &\quad \left. q^4, q^{12}, q^{36}, q^{44}; q^{48} \right)_{\infty} \\
 &\quad + q \left( q^4, q^{10}, q^{14}, q^{24}, q^{34}, q^{38}, q^{44}, q^{48} \right. \\
 &\quad \left. q^2, q^8, q^{12}, q^{22}, q^{26}, q^{36}, q^{40}, q^{46}; q^{48} \right)_{\infty}
 \end{aligned}$$

Combining Theorem 3 and (9), we find

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{3n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{12n^2} + 2q \left( q^4, q^{10}, q^{14}, q^{24}, q^{34}, q^{38}, q^{44}, q^{48} \right. \left. q^2, q^8, q^{12}, q^{22}, q^{26}, q^{36}, q^{40}, q^{46}; q^{48} \right)_{\infty}.$$

If we extract odd powers, divide by  $q$  and replace  $q^2$  by  $q$  we obtain

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} + \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left( q^2, q^5, q^7, q^{12}, q^{17}, q^{19}, q^{22}, q^{24} \right. \left. q, q^4, q^6, q^{11}, q^{13}, q^{18}, q^{20}, q^{23}; q^{24} \right)_{\infty}$$

If we put  $-q$  for  $q$  we find

$$\sum_{-\infty}^{\infty} q^{2n(n+1)} - \sum_{-\infty}^{\infty} q^{6n(n+1)+1} = 2 \left( q, q^{10}, q^{11}, q^{12}, q^{13}, q^{14}, q^{23}, q^{24} \right. \left. q^4, q^5, q^6, q^7, q^{17}, q^{18}, q^{19}, q^{20}; q^{24} \right)_{\infty} \quad \square$$

PROOF OF THEOREM 6: We shall start by proving

$$(10) \quad \prod_{n \geq 1} (1 + q^{10n-9})(1 + q^{10n-1})(1 - q^{10n})$$

$$= \prod_{n \geq 1} (1 + q^{40n-28})(1 + q^{40n-12})(1 - q^{40n})$$

$$+ q \prod_{n \geq 1} (1 + q^{40n-32})(1 + q^{40n-8})(1 - q^{40n}).$$

We have

$$\prod_{n \geq 1} (1 + q^{10n-9})(1 + q^{10n-1})(1 - q^{10n})$$

$$= \sum_{-\infty}^{\infty} q^{5n^2-4n} = \sum_{-\infty}^{\infty} q^{20n^2-8n} + q \sum_{-\infty}^{\infty} q^{20n^2+12n}$$

$$= \prod_{n \geq 1} (1 + q^{40n-28})(1 + q^{40n-12})(1 - q^{40n})$$

$$+ q \prod_{n \geq 1} (1 + q^{40n-32})(1 + q^{40n-8})(1 - q^{40n}).$$

We can write (10) as follows.

$$\left( \begin{matrix} q^2, q^{10}, q^{18}, q^{20} \\ q, q^9, q^{11}, q^{19}; q^{20} \end{matrix} \right)_{\infty} = \left( \begin{matrix} q^{24}, q^{40}, q^{56}, q^{80} \\ q^{12}, q^{28}, q^{52}, q^{68}; q^{80} \end{matrix} \right)_{\infty} + q \left( \begin{matrix} q^{16}, q^{40}, q^{64}, q^{80} \\ q^8, q^{32}, q^{48}, q^{72}; q^{80} \end{matrix} \right)_{\infty}.$$

If we multiply by  $\left( \begin{matrix} q^8, q^{12} \\ q^4, q^{16}; q^{20} \end{matrix} \right)_{\infty}$  we obtain

$$(11) \quad \left( \begin{matrix} q^2, q^8, q^{10}, q^{12}, q^{18}, q^{20} \\ q, q^4, q^9, q^{11}, q^{16}, q^{19}; q^{20} \end{matrix} \right)_{\infty}$$

$$= \left( \begin{matrix} q^8, q^{32}, q^{40}, q^{48}, q^{72}, q^{80} \\ q^4, q^{16}, q^{36}, q^{44}, q^{64}, q^{76}; q^{80} \end{matrix} \right)_{\infty} + q \left( \begin{matrix} q^{12}, q^{28}, q^{40}, q^{52}, q^{68}, q^{80} \\ q^4, q^{24}, q^{36}, q^{44}, q^{56}, q^{76}; q^{80} \end{matrix} \right)_{\infty}.$$

Combining Theorem 4 and (11) we obtain

$$\sum_{-\infty}^{\infty} q^{n^2} + \sum_{-\infty}^{\infty} q^{5n^2} = \sum_{-\infty}^{\infty} q^{4n^2} + \sum_{-\infty}^{\infty} q^{20n^2} + 2q \left( \begin{matrix} q^{12}, q^{28}, q^{40}, q^{52}, q^{68}, q^{80} \\ q^4, q^{24}, q^{36}, q^{44}, q^{56}, q^{76}; q^{80} \end{matrix} \right)_{\infty}.$$

If we extract odd powers, divide by  $q$  and replace  $q^4$  by  $q$  we obtain

$$\sum_{-\infty}^{\infty} q^{n(n+1)} + \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2 \left( \begin{matrix} q^3, q^7, q^{10}, q^{13}, q^{17}, q^{20} \\ q, q^6, q^9, q^{11}, q^{14}, q^{19}; q^{20} \end{matrix} \right)_{\infty}.$$

If we put  $-q$  for  $q$  we find

$$\sum_{-\infty}^{\infty} q^{n(n+1)} - \sum_{-\infty}^{\infty} q^{5n(n+1)+1} = 2 \left( \begin{matrix} q, q^9, q^{10}, q^{11}, q^{19}, q^{20} \\ q^2, q^3, q^7, q^{13}, q^{17}, q^{18}; q^{20} \end{matrix} \right)_{\infty}.$$

□

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