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Computing the effectively computable bound in Baker's inequality for linear forms in logarithms, and: Multiplicative relations in number fields: Corrigenda and addenda

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We indicate a number of qualifications and amendments that are necessary so as to correct the statements and proofs of the theorems in our papers "Computing the effectively computable bound in Baker's inequality for linear forms in logarithms", 15 (1976), 33-57, and its sequel "Multiplicative relations in number fields", 16 (1977), 83-98, and remark on recent observations that would yield yet sharper results.

1. Corrigenda

Throughout, in both [3] and [4], log is ill-defined; it is necessary to fix the $\log \alpha_j$ once for all (for example, as principal values), and to require that the A_j be chosen so as to satisfy $|\log \alpha_j| < A_j$ $(1 \le j \le n)$. It is not made clear in [3] that we suppose that $\log \log A_j \ge 1$ $(1 \le j \le n)$; this should again be remarked on in [4]. Similarly, in [3] it should be emphasised that we take $B \ge e^2$.

In [4], we could define $s(\alpha) = \max\{\text{logden}\, \alpha, \log|\alpha^{(j)}|\}$ and

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$S(\alpha) = \max\{s(\alpha), s(\alpha^{-1})\}$ for an algebraic number α. Then the conclusion of Theorem 1 can be

$$|t_k| \le C_1\left(\prod_{j=1}^{m} S(\alpha_j)\right)/S(\alpha_k) \ , \quad (1 \le k \le m) \ ,$$

where

$$C_1 = 4D \log\log 6D\left(\frac{\log 2}{C_2}\right)^{\min\{m-1,D\}}\left(\frac{m-1}{\log 2}\right)^{m-1} \ ,$$

and $C_2 = C_2(D)$ is the constant of Lemma 3.

The following corrections are needed in the proof: in Lemma 1,

$$\left(\sum_{j=0}^{d} a_j^2\right)^{\frac{1}{2}} \le (d+1)^{\frac{1}{2}} \max_{0\le j\le d} |a_j| \ .$$

In the proof of Theorem 1 (p. 88, line 3) replace factorisation of $N_{K/Q}\alpha_j$ by ideal factorisation

$$[\alpha_j] = \prod_{\mathfrak{p}} \mathfrak{p}^{e_{\mathfrak{p}j}} \ ,$$

where $\mathfrak{p}$ runs nominally through all prime ideals of K; wherever we refer to the indices p we now refer to the indices $\mathfrak{p}$. Finally replace $\log H(\alpha_j)$ by $S(\alpha_j)$ and delete the spurious factors D in the numerators and denominators on the right sides of the inequalities on page 89.

There is a serious error on page 91 of [4], in that, in view of the multivaluedness of log, we cannot take logarithms of (14) and write $\log 1 = 0$; thus (13) and (14) need not yield a counterexample with $n - 1$ instead of n logarithms. The solution to this real difficulty is to suppose one has the results of [3] for the linear form

$$\Lambda = b_0 \log \alpha_0 + b_1 \log \alpha_1 + \dots + b_n \log \alpha_n$$

with $\alpha_0 = e^{\pi i/q}$. The argument then goes through and one obtains the asserted results with $n + 2$ replacing $n + 1$ and $D_0 \le (q-1)D$ replacing D. In fact a slight change in the argument on page 94 (redefine γ so that it is γ^{q^l} which is given in terms of $\alpha_0, \ldots, \alpha_n$, with $16qD_0 > q^l \geq 16D_0$) allows one to choose q minimally as $q = 13$, and in that case a shortening of the extrapolation, Lemma 4 of [3], is possible and would allow us to recover the bounds we asserted in [4]. However, avoiding substantial change but reselecting the parameters (defined [3], page 35) as $\mu = 1$, $k = \frac{1}{2}$, allows us to assert that $C = \left(150(n+1)D\right)^{10(n+1)}$ suffices in the theorems of [4]; here the shape of C is quite artificially selected so as to minimally change the bound we originally claimed.

2. Addenda

We are indebted to remarks of Baker, Stewart, and Waldschmidt which alerted us to the principal omission in [4]. For details of a correct final argument see Baker [1]; it is appropriate to comment that the proof of [1], Theorem 2, is quite similar to ours and that the bound obtained in [1] is equivalent (our exponent 10 is obtained by virtue of rather tiresome computations not detailed in the proof; it easily becomes 200).

Since the papers [3], [4] were written it has become apparent that considerable economies and further refinements are possible in the argument. Thus a different 'size', say $s\left(\alpha_j\right)$ in place of $\log A_j$, would lead to economies in the argument of [4]. In place of the box principle lemma applied in Lemma 2 of [3] one should use the refinement, Lemme 1.3.1 of Waldschmidt [7], p. 10, which would permit a better dependence on D; one could do better yet by replacing the integers $p(\lambda)$ by algebraic integers $\sum_{\mu=1}^{D} p(\lambda, \mu)\xi^{\mu}$ where ξ is a generator of the field K over $\mathbb{Q}$. Our complicated choice of parameters can be eliminated if one follows an idea of Waldschmidt in the extrapolation argument, Lemma 4 of [3]; namely at each step in the induction one uses the same number, $\frac{1}{2n} S_0$ of the derivatives and one extrapolates for $n - 1$ steps (rather than using the sampe proportion of the remaining derivatives as we do, and extrapolating for more steps). Then one can choose $q = 2$ for the

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remainder of the proof, which leads to smaller constants than can be achieved by our methods. Finally, in Lemma 3 of [3] our estimate for $|f(z, m)|$ is unnecessarily pessimistic in that one has a considerably sharper upper bound whenever for all j, $1 \le j \le n$, $|\log \alpha_j|$ is much less than $\log A_j$. The sharper estimate then permits one to choose a large radius for the contour Γ of equation (9) of [3], and finally allows one to place a factor $(\log E)^{-(n-1)}$, with, say, $E = en\left\{\sum |\log \alpha_j| \log A_j\right\}^{-1}$ in all the bounds obtained; *cf.* Shorey [5]. In many applications E is large, so the sharpening effected can be considerable.

Incidentally a Schinzel-Zassenhaus result (as in [4], Lemma 3) can be recovered from our bound on the exponents in multiplicative relations for algebraic numbers. By specialising Theorem 1 of [4] to the case $n = 2$, we obtain the following assertion: if α, β are multiplicatively dependent algebraic numbers in a field K of degree D and not roots of unity, then there is a non-trivial relation $\alpha^s \beta^t = \zeta$, a root of unity, with $|s| \le S(\beta)/C_2$, $|t| \le S(\alpha)/C_2$. In particular, $S(\alpha) \ge C_2$. Now suppose α is a non-zero algebraic integer in K with $\log \overline{|\alpha|} \le C_2$. By the results of Cassels [2] or Smyth [6], α must be reciprocal and so $S(\alpha) = \log \overline{|\alpha|}$. Consequently from the previous remarks, α must be a root of unity, which is what we wished to show. Our remark is based on an idea of C.L. Stewart.

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