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On a problem of Noether–Lefschetz type

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Abstract. In this paper we study the following question: Is it true that a generic hypersurface X of degree d in $\mathbb{P}^{n+1}$, where $(d, n) \neq (3, 1), (2, 2), (3, 2)$, does not admit a non-trivial, non-isomorphic surjective map to another smooth variety Y, except of course $\mathbb{P}^n$? It is easy to see that it is true for $n = 1, 2$. We try to prove this for $n = 3$ and can exclude all possibilities for Y except $Y = G(1, 4) \cap \mathbb{P}^6$ and $Y = V_{22}^s$, a special Fano threefold of type V_{22} found by Mukai and Umemura in [MU].

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1. Introduction

In [L], Lazarsfeld proves (in characteristic zero) that the only smooth variety which can be an image of $\mathbb{P}^n$ under a non-constant morphism is $\mathbb{P}^n$ itself. Moreover, there is an analogous result for smooth quadrics ([PS]): if $n \geqslant 3$, a smooth n-dimensional quadric does not admit a finite surjective morphism to another smooth variety except $\mathbb{P}^n$ and itself, and any surjective endomorphism of a quadric is an isomorphism. Also, there is evidence that a statement of this kind holds for certain homogeneous varieties ([PS]).

QUESTION. *Which other smooth projective varieties satisfy this property?*

A sufficiently general hypersurface in $\mathbb{P}^n$, $n \geqslant 2$, seems to be a good candidate for this (though *any* hypersurface certainly would not do: there are obvious maps between Fermat hypersurfaces). In fact, for $n = 3$ it is not difficult to prove the following.

THEOREM 1.1. *If X is a general hypersurface in $\mathbb{P}^3$ of degree at least four, Y a smooth projective surface and $f: X \to Y$ a surjective* (*finite*) *morphism, then Y is either $\mathbb{P}^2$ or isomorphic to X, and in the last case f is an isomorphism.*

'Finite' stands in brackets here because a general hypersurface in $\mathbb{P}^3$ of degree at least four has Picard group isomorphic to $\mathbb{Z}$, and so the morphism has to be finite if Y is not a point.

The key observation for Theorem 1.1 is Proposition 2.1 below, which is not difficult to deduce from several results by Deligne and which asserts that the

Hodge structure on the middle cohomologies of a general hypersurface in $\mathbb{P}^n$ does not have Hodge substructures (if the hypersurface is not a quadric or a 2-dimensional cubic), except, of course, the obvious ones, i.e. Hodge substructures generated by a multiple of the linear section class (and orthogonal to these). It immediately implies the analogue of Theorem 1.1 for curves (one must assume that the degree of the plane curve is at least four) and together with some general facts, this proposition easily yields Theorem 1.1.

The case $n = 4$ requires however a more detailed analysis. The problem is to exclude the possibility of maps to three (types of) Fano threefolds which have the same Hodge numbers as $\mathbb{P}^3$, namely, the 3-dimensional quadric, a linear section of $G(1,4)$ in the Plücker embedding (denoted in the sequel as V_5) and varieties of type $V_{22} \subset \mathbb{P}^{13}$ (these are Fano threefolds of index one and sectional genus 12; see for example [I], [M] for their construction and descriptions). So far, I do not have a complete proof. Namely, it remains to exclude the linear section of $G(1,4)$ and a special variety of type V_{22}, which has been constructed in the paper [MU] as the projective closure of the $\mathrm{SL}_2(\mathbb{C})$-orbit of a certain binary form of degree 12. We will call it *the Mukai–Umemura variety* and denote it as V_{22}^s. V_{22}^s has non-reduced Hilbert scheme of lines. As it was shown by Prokhorov in [P], V_{22}^s is characterized by this property: the Hilbert scheme of lines on any V_{22} different from the Mukai–Umemura variety, has only finitely many singular points. So the main result of this paper is as follows:

THEOREM 1.2. *A general hypersurface X in $\mathbb{P}^4$ does not admit a non-trivial, non-isomorphic map onto a smooth variety $Y \not\cong \mathbb{P}^3$, except possibly for $Y = V_5$ or Y the Mukai–Umemura variety.*

However, some discussion will be given for these remaining cases.

The paper is organized as follows: in paragraph 2, we prove Proposition 2.1 mentioned above, and we deduce Theorem 1.1 from this. In paragraph 3, we reduce the problem in the 3-dimensional case to the study of maps to the Fano threefolds with vanishing b_3 – the quadric, V_5 and varieties of type V_{22}. Some generalities on Fano threefolds are also recalled there. In paragraph 4, the Infinitesimal Noether–Lefschetz theorem is applied to prove the absence of maps from a general hypersurface in $\mathbb{P}^4$ onto a quadric. Finally, in 5, we apply results of C. Voisin on curves on general hypersurfaces to prove that the latter do not admit maps to a V_{22} with reduced scheme of lines.

We work over the field of complex numbers. The word ‘general’ applied to a hypersurface in $\mathbb{P}^n$ is used in the sense ‘outside of a countable union of proper subvarieties in the parametrizing space’. One notation is frequently used in the paper: for $X \subset \mathbb{P}^N$, H_X denotes a hyperplane section of X.

2. Hodge structures

In what follows we make the following convention: for a smooth projective n-dimensional variety X we say that a Hodge substructure in $H^n(X)$ is *trivial* if it is either the whole $H^n(X)$, or (in the case of n even) it is a 1-dimensional Hodge substructure generated by the multiple of the linear section class. We say that the Hodge numbers of X are trivial, if they coincide with that of $\mathbb{P}^n$.

We put together several results mostly due to Deligne to prove the following.

PROPOSITION 2.1. *Any Hodge substructure in $H^n(X)$ where X is a general hypersurface of degree d in $\mathbb{P}^{n+1}$ is either trivial or orthogonal to a trivial one, except when $d = 2$ or $(d, n) = (3, 2)$.*

Proof. Recall the definition of the Mumford–Tate group $MT(V)$ of a rational Hodge structure V of weight n.

Consider a natural homomorphism

$$\phi: T = \mathbb{C}^* \times \mathbb{C}^* \to \mathrm{GL}(V_{\mathbb{C}})$$

given by

$$\phi(a, b)v = a^p b^q v$$

for $v \in V^{p,q}$.

The Mumford–Tate group is then the minimal algebraic subgroup of $\mathrm{GL}(V)$ defined over $\mathbb{Q}$ such that its group of points over $\mathbb{C}$, $MT(\mathbb{C})$, contains $\phi(T)$.

It is easy to see that if the Hodge structure V has a substructure W, then W is globally invariant under $MT(V)$; conversely, subspaces of V invariant under MT give rational Hodge substructures in V.

We will recall the relation between Mumford–Tate groups and the monodromy as explained in [D1], [Z].

Let now $f: Y \to S$ be a smooth projective morphism of algebraic varieties over $\mathbb{C}$, and denote by Y_s the fiber over $s \in S$. Let n be the dimension of Y_s. Denote by G the Zariski-closure in $\mathrm{Aut}(H^n(Y_s, \mathbb{C}))$ of the monodromy group $\Gamma_s = \mathrm{im}\, \pi_1(S, s) \subset \mathrm{Aut}(H^n(Y_s, \mathbb{Q}))$. G is defined over $\mathbb{Q}$, so $G = M(\mathbb{C})$ for an algebraic subgroup M of $\mathrm{GL}(H^n(Y_s, \mathbb{Q}))$. Let M^0 be a connected component of M. Then the theorem of Deligne says that for s outside a countable union of proper subvarieties of S, M^0 is a normal subgroup of $MT(H^n(Y_s))$.

Now let $f: Y \to S$ be a general Lefschetz pencil of hypersurfaces in $\mathbb{P}^{n+1}$, i.e., Y_s for a general s is our hypersurface X. Let G_{prim} be the Zariski-closure of the monodromy group in $\mathrm{GL}(H^n_{\mathrm{prim}}(X, \mathbb{C}))$ (this makes sense because the primitive part of the cohomologies is of course globally invariant under monodromy). As Deligne proves in [D2], G_{prim} is either as big as possible or finite, and for odd n it is always as big as possible: concretely, for ψ the intersection form,

$$G_{\text{prim}} = \mathrm{Sp}(H^n_{\text{prim}}(X), \psi) \quad \text{if } n \text{ is odd;}$$
$$G_{\text{prim}} = \mathrm{O}(H^n_{\text{prim}}(X), \psi) \quad \text{or } G_{\text{prim}} \text{ is finite if } n \text{ is even.}$$

Using the irreducibity of the monodromy action on primitive cohomologies, it is easy to see ([SGA]) that G_{prim} can be finite only if X is a quadric or a 2-dimensional cubic. In other cases, we get that the action of the Mumford–Tate group is irreducible on $H^n_{\text{prim}}(X)$. From this we conclude that the only possible invariant subspaces of the Mumford–Tate group, and so the only Hodge substructures in $H^n(X)$ are the trivial ones and orthogonal to the trivial ones.

COROLLARY 2.2. *If X is a general hypersurface in $\mathbb{P}^{n+1}$ which is not a quadric or a 2-dimensional cubic, Y a smooth projective variety (not a point) and $f: X \to Y$ a surjective morphism, then the Hodge numbers of Y coincide either with that of X or with that of $\mathbb{P}^n$.*

Indeed, f must be finite (we have that either $\mathrm{Pic}(X) \cong \mathbb{Z}$, or X is a curve); therefore the inverse image map f^* in cohomologies becomes an injection after tensoring with $\mathbb{Q}$. The map f^* is a morphism of Hodge structures, and so we are done.

Remark 2.3. In fact the corollary can also be proved without the use of Mumford–Tate groups. The argument would be a modification of the monodromy argument which is used to prove the Noether–Lefschetz theorem (see e.g. lecture 4 by C. Voisin in [CIME]). One still needs, however, that G_{prim} is big except in a few cases.

Remark 2.4. If the Hodge numbers of our general hypersurface X and its smooth image Y coincide, K_X is an effective divisor and $\chi(X) \neq 0$, then f must be an isomorphism by Hurwitz's formula

$$K_X = f^*(K_Y) + R,$$

where R is the ramification divisor. Indeed, it is easy to see that $H^0(K_X) \neq 0$ and $H^{p,q}(X) = H^{p,q}(Y)$ imply $R = 0$ and so either $\chi(X) = 0$, or f is an isomorphism.

Also, straightforward computation shows that $\chi(X) = 0$ if and only if X is a plane cubic.

In dimension 2, the proof of Theorem 1.1 is now ready. Indeed, as we assume that $\deg(X) \geqslant 4$, by the previous remark we only have to deal with the case when Y has trivial Hodge numbers. But it is well-known (see for example [BPV], p. 230) that such Y either is $\mathbb{P}^2$ or has the unit ball as its universal covering. So if there is a morphism f from a smooth hypersurface X onto such Y, then, if Y is not $\mathbb{P}^2$, there should also exist a holomorphic map g from X to the unit ball such that $f = \pi \cdot g$, where π is the universal covering map. This is clearly impossible.

3. Fano threefolds

In dimension three, we have by Proposition 2.1 that the Hodge structure of a general hypersurface of any degree does not have non-trivial Hodge substructures, but Remark 2.4 does not apply to a hypersurface of degree less then 5. So, for sufficiently general cubics and quartics in $\mathbb{P}^4$ we also have to show the non-existence of maps to varieties with the same Hodge numbers. From Hurwitz's formula it is immediate that any possible smooth image of a cubic or a quartic under a finite morphism is also a Fano variety.

Also (and this is a much more serious problem), we must exclude the possibility of maps from a generic hypersurface in $\mathbb{P}^4$ to threefolds with Hodge numbers of $\mathbb{P}^3$. Clearly, varieties with trivial Hodge numbers must be either Fano or of general type. Contrary to the 2-dimensional case, where we had surfaces of general type (which were all quotients of the unit ball), in the 3-dimensional case we get Fano varieties:

LEMMA 3.0. *A smooth threefold with Hodge numbers of* $\mathbb{P}^3$ *is Fano.*

Proof. The general type case is impossible by the Bogomolov inequality

$$(c_1^2(Y) - 3c_2(Y)) \cdot H_Y \leqslant 0,$$

which applies to threefolds of general type with Picard group $\mathbb{Z}$ ([B]). Indeed, the Riemann–Roch formula yields

$$c_1(Y)c_2(Y) = 24\chi(\mathcal{O}_Y) = 24\chi(\mathcal{O}_{\mathbb{P}^3}) = 24,$$

so $c_2(Y) \cdot H_Y < 0$, and this contradicts the above inequality.

So let us recall some general facts on Fano threefolds ([I]) which we will frequently use.

The index of a Fano variety V is the maximal number k such that $-K_V = kL$ with L ample. The index of a Fano threefold is at most 4, the only Fano threefolds of index 4 resp. 3 are $\mathbb{P}^3$ resp. a quadric. If L is very ample, then on a Fano threefold of index two embedded by L there is a 2-dimensional family of lines. A general line has trivial normal bundle, and there is a 1-dimensional subfamily of lines with the normal bundle $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$ (in what follows, such lines on a Fano 3-fold of index 2 are called $(-1, 1)$-lines). So the Hilbert scheme of lines on such a 3-fold V is smooth, and if $\mathcal{X}$ is the universal family over this scheme, then the ramification locus of the natural map $p: \mathcal{X} \to V$ consists exactly of $(-1, 1)$-lines.

A Fano threefold of index one V with $-K_V$ very ample has a 1-dimensional family of lines. The normal bundle of a line is either $\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$, or $\mathcal{O}_{\mathbb{P}^1}(1) \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$. $(1, -2)$-lines must, of course, form a closed subfamily, as these correspond exactly to the singular points of the Hilbert scheme. However, this subfamily need not be proper: for example, on the Fermat quartic each line is $(1, -2)$. If every line on a Fano 3-fold V of index one is $(1, -2)$, then the divisor of lines on V is

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either a cone, or a tangent surface to a curve (if the genus of V is at least 4, then V does not contain cones, so this divisor is always a tangent surface). If a general line on V is $(0,-1)$, then this is not the case and $p: \mathcal{X} \to V$ (with $\mathcal{X}$ the universal family of lines) is an immersion along a general line. These last statements are easy to see comparing normal bundles of a line in $\mathcal{X}$ and V.

From the classification of Fano threefolds with Picard number one ([I]) we have that there is one family of Fano threefolds with the Hodge numbers of a cubic (besides the family of cubics), namely, the family of varieties V_{14} – linear sections of the Grassmannian $G(1,5) \subset \mathbb{P}^{14}$, and one family of Fano threefolds with the Hodge numbers of a quartic (besides the family of quartics): these threefolds are double covers of a quadric in $\mathbb{P}^4$, ramified along a quartic section of this quadric. For this last family, we refer to the Theorem 4.1.1 below: it will be proven there that a general hypersurface in $\mathbb{P}^4$ does not admit non-trivial morphisms to the quadric. Let us consider the three remaining cases (note that the generator of the Picard group is very ample in these cases).

PROPOSITION 3.1. (i) *A general quartic in $\mathbb{P}^4$ cannot be mapped onto another quartic. Any endomorphism of a general quartic in $\mathbb{P}^4$ is an isomorphism.*

(ii) *There are no finite maps from a general cubic to V_{14}.*

(iii) *Any finite map between smooth cubics in $\mathbb{P}^4$ is an isomorphism.*

Proof. (i) A standard computation with Chern classes (see e.g. [T]) yields that if a quartic X is sufficiently general, then the surface S_X formed by lines on X is of degree 320.

By a Torelli-type theorem ([Don]), a general quartic threefold X is determined by its polarized intermediate Jacobian $J(X)$. A morphism $f: X \to X'$ of quartics induces an isogeny of $J(X)$ and $J(X')$. This implies that the image of a general quartic is also a general quartic, i.e., that if a general quartic admits morphisms of certain degree onto other quartics, then among the images there will be quartics with a surface of lines of degree 320 (of course to make this observation one must first remark that morphisms of fixed degree from quartics to quartics form an algebraic family, but this is more or less standard.)

Now if X' is a quartic with $\deg S_{X'} = 320$ and if $f: X \to X'$ is a morphism such that $f^*(\mathcal{O}_{X'}(1)) = \mathcal{O}_X(m)$, then by Hurwitz's formula

$$-H_X = -mH_X + R$$

we get that the ramification divisor R is $(m-1)H_X$. As $f^{-1}(S_{X'}) \equiv 80mH_X$ (counting with multiplicities), this means that some component of the inverse image of $S_{X'}$ does not lie in the ramification.

Let C be an irreducible component of the inverse image of a line l which is not contained in the ramification, and let D be the full preimage of l. We have

$$(\mathcal{I}_D/\mathcal{I}_D^2)^* = \mathcal{O}_D \oplus \mathcal{O}_D(-m).$$

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There is a natural morphism

$$\phi\colon (\mathcal{I}_C/\mathcal{I}_C^2)^* \to (\mathcal{I}_D/\mathcal{I}_D^2)^*|_C$$

which must be an isomorphism at a smooth point of D. Also, as C is reduced, the map

$$\psi\colon T_X|_C \to (\mathcal{I}_C/\mathcal{I}_C^2)^*$$

must be a surjection at a general point of C. It is easy to see that the bundle $T_X(2)$ is globally generated. Therefore we must have $m \leqslant 2$. But this is already a contradiction: we saw that, under an assumption that a general quartic admits non-isomorphic maps onto other quartics, a general quartic must also occur as an image of such a map. Composing these maps, we can get maps between quartics of arbitrarily high degree. So we always have $m = 1$, i.e. any surjective map from a general quartic to a quartic is an isomorphism.

(iii) Let X be a cubic. Denote by U_X the surface formed by $(-1,1)$-lines on X.

Again, a standard computation with Chern classes gives that the Plücker embedding of the Fano surface into $\mathbb{P}^9$ is canonical. Using this, one computes the degree and genus of the following curve A on the Fano surface:

$$A = \{\text{lines intersecting a given (sufficiently general) line } l \\ \text{and not coinciding with } l\},$$

(note that A is complete if the normal bundle of l is trivial) and gets $g(A) = 11$. For a general l, the curve A is smooth ([CG]). Now there is a finite map $\alpha\colon A \to l$, $\alpha(l') = l' \cap l$, deg$(\alpha) = 5$ (there are 6 lines through a general point of a cubic). The ramification locus consists of points corresponding to $(-1,1)$-lines, so the branch locus consists of intersection points of l and U_X. We conclude by Hurwitz that the degree of the ramification divisor is 30. The branch locus must then consist of at least 8 different points, so the divisor U_X on our cubic is at least $8H_X$. As before, one sees that if $f\colon X \to X'$ is a finite map between cubics, then some component of the inverse image of a general $(-1,1)$-line is not in the ramification.

Denote by C a reduced irreducible component of the inverse image of some $(-1,1)$-line, and by D the full inverse image of a $(-1,1)$-line. Then

$$(\mathcal{I}_D/\mathcal{I}_D^2)^* = \mathcal{O}_D(m) \oplus \mathcal{O}_D(-m).$$

Again we get a generic surjection

$$\chi\colon T_X|_C \to (\mathcal{I}_D/\mathcal{I}_D^2)^*|_C.$$

But this time already $T_X(1)$ is globally generated, and thus $m = 1$ and f is an isomorphism.

(ii) From the arguments above it follows that it is enough to prove that a $V_{14} \subset \mathbb{P}^9$ such that lines on it cover only a hyperplane section divisor, can vary only in a family of dimension less than 10 (as 10 is the dimension of the family of cubic threefolds modulo projective equivalence). This is not difficult: in fact in this case the scheme of lines must be non-reduced (Iskovskih shows in [I] that the reduced one has bigger degree in the Grassmannian) and all the lines must be tangent to some curve. Moreover, one checks easily that this curve is a rational normal octic. Namely, denote as S the surface of lines; a general hyperplane section of S is a curve of arithmetic genus 8, geometric genus $p_g(A)$ and with at least deg(A) singularities. But A generates $\mathbb{P}^8$, so deg$(A) \geqslant 8$; therefore, equality holds and $p_g(A) = p_a(A) = 0$ as stated.

But all the rational normal octics lying on $G(1,5)$ together with all their tangents, are in the same orbit of Aut$(G(1,5)) \cong$ Aut$(\mathbb{P}^5)$. Indeed, such a curve A must be the image of the Gauss map γ for some curve B in $\mathbb{P}^5$ ([Pi], Satz 11.2: the union of lines in $\mathbb{P}^n$, corresponding to points of a smooth curve which lies in $G(1,n)$ with all its tangents, is either a cone, or a tangent surface to a curve), and it is not difficult to conclude from the Plücker formulae for degrees of Gauss images ([GH], p. 270) that B is a rational normal quintic.

Therefore one easily sees that the family of smooth V_{14}'s containing a tangent surface to some A (up to isomorphism) is either empty or 5-dimensional, so we are done.

Remark 3.2. In fact one even can make an explicit computation and show that this family actually *is* empty, in other words, all 3-dimensional linear sections of $G(1,5)$ containing the tangent surface to A are singular. This shows that there cannot exist a map from *any* smooth cubic to a smooth V_{14}.

Again, according to the classification of Fano threefolds ([I]), apart from $\mathbb{P}^3$ there are three types of Fano threefolds with trivial Hodge numbers

(1) the 3-dimensional quadric Q_3;
(2) $V_5 \subset \mathbb{P}^6$ of index two; V_5 is $G(1,4) \cap \mathbb{P}^6$, where the Grassmann variety of lines in $\mathbb{P}^4$ is embedded in $\mathbb{P}^9$ by Plücker coordinates;
(3) $V_{22} \subset \mathbb{P}^{13}$; V_{22} is a Fano threefold of index one and sectional genus 12, with $-K_V$ very ample.

Varieties of type V_{22} form a six-dimensional family and they admit several descriptions ([M]). Recall that all V_{22}'s except the Mukai–Umemura variety V_{22}^s constructed in [MU] have reduced Hilbert scheme of lines, that is, a general line on such a V_{22} is $(0,-1)$. On V_{22}^s, any line is $(1,-2)$ ([MU]). It was shown in [I] that the surface S formed by lines on a $V_{22} \neq V_{22}^s$ is linearly equivalent to $-2K_{V_{22}}$ ([I]) (on V_{22}^s it is then, of course, the anticanonical divisor, as the canonical class is the generator of the Picard group of V_{22}.)

The geometry of V_{22}^s thus differs from the geometry of other varieties V_{22}, and this is important for the sequel. As we remarked already, on V_{22}^s all the lines must be tangent to some curve, and on a $V_{22} \neq V_{22}^s$ this it not the case: the natural map $p: \mathcal{X} \to V_{22}$, where $\mathcal{X}$ is the universal family of lines on the V_{22}, must be an immersion along a general line.

4. An application of infinitesimal Noether–Lefschetz theorem

4.1. The quadric

Let f be a (finite) map between a hypersurface X of degree $d \geqslant 3$ in $\mathbb{P}^4$ and a 3-dimensional quadric Q.

THEOREM 4.1.1. (a) *Let* $f^*(\mathcal{O}_Q(1)) = \mathcal{O}_X(m)$. *Then* $m \leqslant 3d$.

(b) *If* X *is generic, then there are no maps* $f: X \to Q$.

Before starting the proof of the theorem, let us briefly recall the concept of Infinitesimal Noether–Lefschetz theorem from [CGGH]

Let Z be a smooth complete intersection $D_1 \cap \cdots \cap D_k$ in a smooth projective variety Y. Assume (for simplicity) that Z is a surface, i.e. $\dim Y = k + 2$ (this is the only case we will need here). Define the subspace $H^{1,1}_{i.f.}(Z)$ of *infinitesimally fixed* classes in $H^{1,1}(Z)$ as a subspace of classes which stay infinitesimally in all directions of type $(1, 1)$, in other words (cf. [CGGH]).

$\lambda \in H^{1,1}_{i.f.}(Z)$ if and only if it is in the right kernel of the multiplication map $T \otimes H^{1,1}(Z) \to H^{0,2}(Z)$, where $T \subset H^1(Z, T_Z)$ is the Kodaira–Spencer image of the tangent space at Z to the parametrizing space $\mathbb{P}(H^0(Y, \bigoplus_{i=1}^k \mathcal{O}(D_k)))$ (this multiplication map is induced by the derivative of the period map).

We say that the Infinitesimal Noether–Lefschetz theorem holds for complete intersections of type $(D_1, \ldots, D_k)$ (i.e. complete intersections of divisors linearly equivalent to $D_1, \ldots, D_k$) in Y, if for any smooth Z of type $(D_1, \ldots, D_k)$, $H^{1,1}_{i.f.}(Z)$ consists exactly of those classes which are restrictions of $(1, 1)$-classes on Y. The Infinitesimal Noether–Lefschetz theorem implies the

Noether–Lefschetz theorem: For a generic Z *of type* $(D_1, \ldots, D_k)$, $\mathrm{Pic}(Z) \cong \mathrm{Pic}(Y)$.

The locus of smooth Z with $\mathrm{Pic}(Z) \neq \mathrm{Pic}(Y)$ is called the Noether–Lefschetz locus; for Z in the Noether–Lefschetz locus and λ in $H^{1,1}(Z)$ which is not a restriction of a class on Y, the vector subspace of T which annihilates λ is the Kodaira–Spencer image of the tangent space to the corresponding component of the Noether–Lefschetz locus.

Proof. (a) Let l be a line on Q and H a smooth hyperplane section of Q which contains it. As Q is a homogeneous variety, for a general choice of l and H the inverse images $C = f^{-1}(l)$ and $M = f^{-1}(H)$ will be smooth ([H], Ch. 3, Thm. 10.8). We have an exact sequence

$$0 \to N_{l,H} \to N_{l,Q} \to \mathcal{O}_l(1) \to 0.$$

As $N_{l,H} = \mathcal{O}_l$, this sequence splits and $N_{l,Q} = \mathcal{O}_l \oplus \mathcal{O}_l(1)$.

It is easy to see that $N_{C,X} = f^*(N_{l,Q}) = \mathcal{O}_C \oplus \mathcal{O}_C(m)$. So the exact sequence

$$0 \to N_{C,M} \to N_{C,X} \to N_{M,X}|_C \to 0$$

is just

$$0 \to \mathcal{O}_C \to \mathcal{O}_C \oplus \mathcal{O}_C(m) \to \mathcal{O}_C(m) \to 0$$

and therefore splits. In particular, the map

$$\alpha: H^0(N_{C,X}) \to H^0(N_{M,X}|_C)$$

is surjective, so for $\beta: H^0(N_{M,X}) \to H^0(N_{M,X}|_C)$ we obviously have $\operatorname{Im}\beta \subset \operatorname{Im}\alpha$. But this means that every infinitesimal deformation of M in X contains an infinitesimal deformation of C. C is not linearly equivalent to a multiple of a hyperplane section on M, and this means that the Infinitesimal Noether–Lefschetz theorem does not hold for divisors from $|\mathcal{O}(m)|$ on X. But as Ein and Lazarsfeld prove in [EL], Proposition 3.4, this can only happen if $m \leqslant 3d$.

(b) Now if the hypersurface X is generic, we even have that every infinitesimal deformation of M in $\mathbb{P}^4$ contains an infinitesimal deformation of C. M is a complete intersection of type (d, m); it is known that the Infinitesimal Noether–Lefschetz fails for such complete intersections only if $(d, m) = (2, 1), (3, 1)$ or $(2, 2)$ (see e.g. [E2] for a much more general result).

Remark 4.1.2. The proof of part (a) works for X any smooth threefold with Picard group $\mathbb{Z}$. In fact, the result from [EL] is as follows

Let X be a smooth projective threefold, and let A be a very ample and B a nef line bundle on X. If Y is a smooth divisor from the linear system $|3K_X + 16A + B|$, then the Infinitesimal Noether–Lefschetz holds for Y.

So this means that we can easily bound the possible degree of a map from a smooth threefold X with Picard (in fact, even Neron-Severi) group $\mathbb{Z}$ to a 3-dimensional quadric in terms of $c_1(X)$ and the numerical index of X.

A result of this type had been first obtained by C. Schuhmann ([S]) by different methods. However, our bound for m seems to be better in some cases (e.g. for hypersurfaces) and it uses less invariants of X (in [S], the bound also depends on $c_2(X)$). Also, the method given here admits a simple generalisation for maps from n-folds to n-quadrics (however, the bound grows very fast with n, as it becomes more difficult to obtain all the vanishing results needed to prove Infinitesimal Noether–Lefschetz). I hope to return to bounding degrees of maps to certain Fano varieties in a forthcoming note.

Remark 4.1.3. Infinitesimal Noether–Lefschetz theorem implies the ‘usual’ one, but the converse does not have to be true, as it is clear from the discussion preceeding the proof of Theorem 4.1.1. Indeed, if a smooth complete intersection Z is a very singular point of a component of the Noether–Lefschetz locus, then the Infinitesimal Noether–Lefschetz can fail at Z. The obvious map from the Fermat hypersurface of degree $2n$ in $\mathbb{P}^4$ to the quadric provides us with an explicit example. Indeed, the Noether-Lefschetz theorem holds for divisors from $|nH_{F_{2n}}|$ on F_{2n}: any curve on a generic intersection of the Fermat hypersurface F_{2n} with a hypersurface of degree n is a complete intersection ([Mois]). However, as the proof of Theorem 4.1.1 shows, the infinitesimal Noether–Lefschetz cannot be true at a smooth divisor from $|nH_{F_{2n}}|$ if this divisor is the inverse image of a hyperplane section of the quadric.

4.2. A DISCUSSION

Trying to apply the same method to the V_5 and V_{22}'s, one must produce a curve D in a surface S on each of these Fano's such that $C = f^{-1}D$ and $M = f^{-1}S$ are smooth (so D should vary in a large family) and the sequence of normal bundles for $D \subset S \subset V_5$ (or V_{22}) splits (by the 'sequence of normal bundles' for $X \subset Y \subset Z$ we mean, of course

$$0 \to N_{X,Y} \to N_{X,Z} \to N_{Y,Z}|_X \to 0).$$

This seems difficult on a V_{22} which is not Mukai–Umemura (in the next section, another method will be applied to deal with this class of varieties). On the Mukai–Umemura variety V_{22}^s and on the V_5, there are 1-dimensional families of lines such that the normal bundle sequence for $l \subset H \subset V_5(V_{22}^s)$ splits. Moreover, it is not difficult to show that for a curve linked $(1, 1)$ to such a line the sequence of the normal bundles also splits. However, there is a problem with the smoothness of the inverse image of a hyperplane section passing through a line l with the normal bundle $\mathcal{O}(-1) \oplus \mathcal{O}(1)$ on V_5 or a line with the normal bundle $\mathcal{O}(-2) \oplus \mathcal{O}(1)$ on V_{22}^s.

Let us consider the case of V_5, for example. It is easy to see that the $(-1, 1)$-lines on V_5 are all tangent to some curve Σ. It can be shown that Σ is a rational normal sextic, so it is exactly the singular locus of the surface T formed by $(-1, 1)$-lines and every $(-1, 1)$-line intersects Σ at one point. Observe that we have the following:

If $f: X \to Y \subset \mathbb{P}^n$ is a finite morphism of non-singular varieties and H a hyperplane in $\mathbb{P}^n$ which does not contain Y, then $f^{-1}(Y \cap H)$ is non-singular at a point x if and only if $f_*(T_xX)$ is not contained in H. $(*)$

Now it is clear what the problem is: even for a general choice of l, it can happen that for the point $y = l \cap \Sigma$ there is some point x in $f^{-1}(y)$ such that $f_*(T_xX) = T_yl = T_y\Sigma$, in other words, $rk(f) = 1$ along some component of $f^{-1}\Sigma$. This will make x a singular point of $f^{-1}H$ if H contains l.

If, however, Σ does not lie in $\Gamma = \{f(x): rk_x f = 1\}$, then for a general choice of l we can find a smooth inverse image of a hyperplane section through l: clearly, l does not lie in Γ; if $z \in l \cap \Gamma$, $z \neq y$, then for any $t \in f^{-1}(z)$ $f_*(T_t X) \neq T_z l$. If now T is not contained in the branch locus of f, then obviously we can choose H through a general line on T such that $f^{-1}H$ is smooth; if T is in the branch locus, we just make the elementary observation that the tangent space to T, and therefore the image of the tangent space to X, stays constant along the line l.

Having a smooth $M = f^{-1}H$, we argue as in Theorem 4.1.1. The result will be as follows.

PROPOSITION 4.2. *If $f: X \to V_5$ where X is a general hypersurface in $\mathbb{P}^4$ is a finite morphism, then the curve Σ tangent to all the $(-1,1)$-lines on V_5 must be contained in the locus $\{f(x): rk_x f = 1\}$.*

The case of a V_{22}^s is completely analogous.

5. Curves of low genus and a general V_{22}

The other method to rule out finite morphisms from a *general* hypersurface to V_5 and V_{22} is to notice that if they exist, then there must be curves of low genus on a general hypersurface in $\mathbb{P}^4$ (obtained for example as inverse images of lines or conics), and thus try to get a contradiction with the results of C. Voisin ([V]) or L. Ein ([E]). More concretely, the following results are due to Voisin:

(V1) *Let X be the universal family of hypersurfaces of degree d in $\mathbb{P}^n$, $d \geqslant n+2$, and X_t a fiber. Then the bundle $T\mathcal{X}(1)|_{X_t}$ is generated by global sections.*

(V2) *Let X be a general hypersurface of degree $d \geqslant n+2$ in $\mathbb{P}^n$. Then, for a divisor D on X, any desingularization $\sigma: \widetilde{D} \to D$ satisfies*

The map given by the linear system $|K_{\tilde{D}} + \sigma^*(n+2-d)H_X|$ is generically finite onto its image.

OBSERVATION 5.0. It follows easily from (V1) that if $f: X \to V_5$ is a finite map (X is, as before, a general hypersurface in $\mathbb{P}^4$) and $\deg X \geqslant 6$, then the inverse image of T, the surface formed by $(-1,1)$-lines on V_5, is contained in the ramification. Indeed, if some component of $f^{-1}T$ is not in the ramification, then there exist a reduced irreducible component C of $f^{-1}l = D$, where l is a $(-1,1)$-line. As our hypersurface is general, this gives rise to a family $\mathcal{C} \subset \mathcal{X}$. We have the natural morphism

$$\psi: T\mathcal{X}|_C \to (\mathcal{I}_{\mathcal{C}}/\mathcal{I}_{\mathcal{C}}^2)^*|_C = (\mathcal{I}_C/\mathcal{I}_C^2)^*,$$

and this is surjective at a smooth point of C. But there is also a morphism

$$\phi: (\mathcal{I}_C/\mathcal{I}_C^2)^* \to (\mathcal{I}_D/\mathcal{I}_D^2)^*|_C = \mathcal{O}_C(m) \oplus \mathcal{O}_C(-m),$$

which is a generic isomorphism. As $T\mathcal{X}(1)|_X$ is globally generated, this is impossible if $m \neq 1$. The case $m = 1$ obviously cannot occur.

The same argument shows that if $f\colon X \to V_{22}$ is a finite map, then the inverse image of the surface of lines $S \subset V_{22}$ is in the ramification.

Using the results (V2), (V3), we are able to rule out the case of a non-Mukai–Umemura V_{22}.

PROPOSITION 5.1. *There are no surjective morphisms from a general hypersurface in $\mathbb{P}^4$ to a $V_{22} \neq V_{22}^s$.*

Proof. Let $f\colon X \to V_{22}$ be a morphism. As in the paragraph 4.2, we consider the surface $M \subset X$, which is the inverse image of a hyperplane section H of V_{22}, such that H contains a line l. The point is that if a V_{22} is not the Mukai–Umemura variety, then for a general choice of H and l the surface M has very simple singularities.

LEMMA 5.2. *One can choose l and H so that M is smooth but for a finite number of A_{k-1}-singularities* (*i.e. of singularities locally given by* $x^2+y^2+z^k=0$), *where for each singular point of M the number k is the number of sheets of the covering $f\colon X \to V_{22}$ coming together in this point. Moreover, all the singular points of M are mapped by f to the same point on l.*

Proof. By Bertini, only points of $f^{-1}(l)$ can be singular on the inverse image of a general $H \supset l$. We notice that on our V_{22} there does not exist a curve such that all lines are tangent to it (see the end of Paragraph 3). Therefore, if a general line l passes through the points p_i such that at some point $q_{i,j} \in f^{-1}(p_i)$ we have $rk(f)_{q_{i,j}} = 1$, then the image space $v_{i,j} = f_*(T_{q_{i,j}}X) \subset T_{p_i}V_{22}$ cannot be the tangent space to l. We can suppose that l lies in the branch locus of f. Then at all the other points of l we have a finite number of 2-dimensional images of the tangent spaces to X, say planes $P_{t,j}$ at a point $t \in l$. Notice that at a smooth point of the branch locus there is only one plane P_t, which coincides with the tangent plane to the surface S formed by lines.

Now we recall the observation $(*)$ which says that as soon as our hyperplane H does not pass through $P_{t,j}$ or $v_{i,j}$, then the inverse image of H will be non-singular at the corresponding point. Of course a general hyperplane will pass through some $P_{t,j}$'s, but we can choose a hyperplane H such that it passes only through those of the planes P_t for which t a smooth point of the branch locus, and, moreover, t satisfies the following property.

> *Near each point $q \in f^{-1}t$, we can write the map f as*
>
> $$u = x, \;\; v = y, \;\; w = z^k, \tag{$**$}$$
>
> *where (x, y, z) resp. (u, v, w) are local coordinates near q resp. t, and $k = k(q)$ is a positive integer.*

Obviously, we can also assume that in these local coordinates our line l is given by $w = 0$, $u = 0$.

In fact there will be exactly one point t_0 such that H contains P_{t_0}, i.e. is tangent to S, and at this point the intersection of l with the other component A of $H \cap S_X$

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will be transversal. (This is because we can choose H such that H does not contain other lines except l; then A will induce a section in the universal family of lines on V_{22}, i.e. will intersect l in the singular points of S plus at one non-singular point, in which the intersection will be transversal.) This means that in local coordinates near this point we have H given as

$$w + f_2(u,v) + awu + bwv + cw^2 + \textit{higher order terms}$$

with f_2 a non-degenerate quadratic form (the coefficient at w is non-zero because H is smooth). In other words, the inverse image of H will be locally given as

$$z^k \cdot (\textit{invertible power series}) + f_2(x,y) + g(x,y),$$

where $g(x,y)$ starts with cubic terms, and this is obviously an A_{k-1}-singularity.

It is well-known that an A_{k-1}-singularity is resolved by a chain of rational curves $E_1, \dots, E_{k-1}$, $E_i^2 = -2$, $E_i \cdot E_j = 1$ if $|i-j| = 1$ and 0 otherwise. Let us assume the following notations.

- $\pi\colon \widetilde{M} \to M$ is the resolution of singularities;
- $g = f \cdot \pi\colon \widetilde{M} \to H$;
- C is a reduction of some irreducible component of $f^{-1}(l)$; at a general point of C there are k sheets of the covering coming together, so C passes through A_{k-1}-singularities of M; C is smooth at these singular points of M because of $(**)$;
- $\widetilde{C}$ is the proper transform of C on $\widetilde{M}$; $\widetilde{C} \cong C$;
- C maps to l, say, generically $n\colon 1$, so on C we have n singular points of M, say $p_1, \dots, p_n$, and a point p_i is resolved by a chain $E_{i,1}, \dots, E_{i,k-1}$.

An elementary computation shows that we can assume $\widetilde{C} \cdot E_{i,1} = 1$, and $\widetilde{C} \cdot E_{i,l} = 0$ for $l \neq 1$.

LEMMA 5.3.

$$g^*(l) = k\widetilde{C} + \sum_{i=1}^{n} \sum_{j=1}^{k-1} (k-j) E_{i,j} + F,$$

where $F \cdot E_{i,j} = 0$ *for* $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant k-1$, *and* $F \cdot \widetilde{C} \geqslant 0$ (*in other words,* F *comes from other components of the inverse image of* l).

Proof. Obviously

$$g^*(l) = k\widetilde{C} + \sum_{i=1}^{n} \sum_{j=1}^{k-1} a_{i,j} E_{i,j} + F,$$

and $a_{i,j}$ are easy to compute from the equalities $g^*(l) \cdot E_{i,j} = 0$ for any i, j.

Now we have

$$g^*(l) \cdot k\widetilde{C} = k(l \cdot g_*(\widetilde{C})) = -2kn,$$

(as on a smooth K3-surface H we have $l^2 = -2$),

$$\begin{aligned}(k\widetilde{C})^2 &= (g^*(l) - \sum_{i=1}^{n}\sum_{j=1}^{k-1}(k-j)E_{i,j} - F) \cdot k\tilde{C} \\ &\leqslant -2kn - (k-1)kn = -(k+1)kn,\end{aligned}$$

so $(\widetilde{C})^2 \leqslant -((k+1)/k) \cdot n$. Notice that

$$\frac{n}{\deg(f)} = \frac{\deg(C)}{\deg(f^{-1}(l))},$$

so $n = m \cdot \deg(C)$. Now $K_{\widetilde{M}} = \pi^*(\mathcal{O}_M(d+m-5))$, so we can estimate from above the arithmetic genus of $\widetilde{C}$ and therefore of C. We get

$$2p_a(C) - 2 \leqslant \left(d - \frac{m}{k} - 5\right) \deg C.$$

To finish the proof, suppose now that k is as small as possible, i.e. that all components of the inverse image of S have multiplicity at least k. As S is linearly equivalent to $2H_{V_{22}}$, we have that $k \leqslant 2m$ with equality if and only if the set-theoretic inverse image of S is a hyperplane section of X. In other cases, $k \leqslant m$.

If X is a quintic, we therefore get that C is a rational curve of degree at most four. But this is a contradiction, because it is well-known that on a general quintic there is only a finite number of such curves, and as our line $l \subset V_{22}$ varies in a 1-dimensional family,we also have that C must vary.

If $\deg(X) \geqslant 6$, we can apply the theorems of Voisin mentioned above. Namely, if $m/k \geqslant 1$, this is a contradiction with (V2); if the set-theoretic inverse image of S is a hyperplane section A of X, or contains such a hyperplane section, then by Zak's theorem on tangencies A has only isolated singularities. On A, we have a 1-dimensional family of curves which are reductions of irreducible components of the inverse images of lines on V_{22}. A general one of these curves will not pass through the singularities of A. This and the computation above imply that on the desingularization of A we will have a 1-dimensional family of curves with negative squares, which is impossible.

If $m = k$, we get a contradiction with (V2).

Finally, if X is a quartic or a cubic, the simplest argument is to remark that, as in the Proposition 3.1, some component of the inverse image of $S \subset V_{22}$ does not lie in the ramification (recall that $S \equiv 2H_{V_{22}}$) and so we can come to our conclusion as in that proposition.

Remark 5.4. It is difficult to argue in the same way for V_5 and V_{22}^s , since the singularities we get on the inverse image of a hyperplane section containing a $(-1,1)$ or $(-2,1)$-line can become uncontrollable. There is a theorem by L. Ein ([E]) which reads as follows for generic hypersurfaces in $\mathbb{P}^4$.

If $\mathcal{Y}$ is a family of smooth curves on a generic hypersurface X of degree d in $\mathbb{P}^4$ such that the members of this family cover a subscheme of X of dimension k, then for a curve $Y \in \mathcal{Y}$ we have that $K_Y + (8-k-d)H_Y$ is effective.

If we take preimages of lines on V_5 (resp. conics on V_{22}) as members Y of a family $\mathcal{Y}$, we get $K_Y = (d-5)H_Y$, i.e. we have the smallest canonical class which agrees with Ein's theorem. The results of Voisin cited above improve these of Ein (V3) is an improvement for $k = 1$ and (V2) for $k = 2$. However, I do not know how to obtain an improvement for $k = 3$.

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References

[B] Bogomolov, F. A.: Holomorphic tensors and vector bundles on projective varieties, *Math. USSR Izv*. 13 (1979) 499–555.
[BPV] Barth, W., Peters, C. and Van de Ven, A.: *Compact Complex Surfaces*, Springer-Verlag, 1984.
[CIME] Green, M. et al.: *Algebraic Cycles and Hodge Theory*, Lecture Notes in Math. 1594, Springer-Verlag, 1994.
[CG] Clemens, H. and Griffiths, Ph.: The intermediate Jacobian of a cubic threefold, *Ann. Math*. 95 (1972) 281–356.
[CGGH] Carlson, J., Green, M., Griffiths, Ph. and Harris, J.: Infinitesimal variations of Hodge structures, *Comp. Math.* 50 (1983).
[D1] Deligne, P.: La conjecture de Weil pour les surfaces $K3$, *Inv. Math.* 15 (1972) 206–226.
[D2] Deligne, P.: La conjecture de Weil II, *Publ. Math. IHES* 52 (1980) 137–252.
[Don] Donagi, R.: Generic Torelli for projective hypersurfaces, *Comp. Math*. 50 (1983) 325–353.
[E] Ein, L.: Subvarieties of generic complete intersections, *Inv. Math*. 94 (1988) 163–169.
[E2] Ein, L.: An analogue of Max Noether's theorem, *Duke Math. J*. 52 (1985) 689–706.
[EL] Ein, L. and Lazarsfeld, R.: Syzygies and Koszul cohomologies of smooth projective varieties, *Inv. Math*. 111 (1993) 53–67.
[GH] Griffiths, Ph. and Harris, J.: *Principles of Algebraic Geometry*, Wiley, 1978.
[H] Hartshorne, R.: *Algebraic Geometry*, Springer-Verlag, 1977.
[I] Iskovskih, V.: Fano 3-folds I, II, *Math. USSR Izv*. 11 (1977) 485–527, and 12 (1978), 469–506.
[L] Lazarsfeld, R.: *Some Applications of the Theory of Positive Vector Bundles*, in Lecture Notes in Math. 1092, pp. 29–61.

[M] Mukai, S.: Fano 3-folds, in: *Complex Projective Geometry*, ed. by G. Ellingsrud et al., London Math. Soc. Lecture Notes 179, 255–263.

[Mois] Moishezon, B.: Algebraic homology classes on algebraic varieties, *Math. USSR Izv*. 1 (1967) 209–251.

[MU] Mukai, S. and Umemura, H.: Minimal rational 3-folds, in: *Algebraic Geometry (Proceedings, Tokyo/Kyoto 1982)*, Lecture Notes in Math. 1016, Springer, 1983, pp. 490–518.

[PS] Paranjape, K. and Srinivas, V.: Self-maps of homogeneous spaces, *Inv. Math*. 98 (1989) 425–444.

[Pi] Piontkowski, J.: *Abwickelbare Varietäten*, thesis, Düsseldorf, 1995.

[P] Prokhorov, Yu.: On exotic Fano varieties, *Moscow Univ. Math. Bull*. 45(3) (1990) 36–38.

[S] Schuhmann, C.: Mapping 3-folds onto 3-quadrics, *Math. Ann*. 306 (1996) 571–581.

[SGA] Deligne, P. and Katz, N.: SGA 7.II *Groupes de Monodromie en Geometrie Algebrique*, Lecture Notes in Math. 340, Springer-Verlag, 1973.

[T] Tennison, B.: On the quartic threefold, *Proc. London Math. Soc*. 29 (1974) 714–734.

[V] Voisin, C.: On a conjecture of Clemens on rational curves on hypersurfaces, *J. Diff. Geom*. 44 (1996) 200–214.

[Z] Zarkhin, Yu.: Weights of simple Lie algebras in the cohomology of algebraic varieties, *Math. USSR Izv.* 24 (1985) 245–281.