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Skew products and minimal dynamical systems on separable Hilbert manifolds

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Abstract. We prove that any locally compact, non-compact, second countable group acts minimally on any metrizable connected manifold modelled on the separable Hilbert space.

0. *Introduction*

The purpose of this article is to prove the following theorem.

THEOREM. *If M is a connected separable Hilbert manifold and G is a locally compact, non compact, second countable topological group, we can find a continuous and minimal action of G on M.*

This theorem is a generalization of the one we obtained in [**5**]. Let us point out that, as in [**5**], we do not construct smooth actions in the case where G is a Lie group. To our knowledge, it is still not known that there exists a minimal diffeomorphism of l^2.

The main ingredients for this paper are our previous work [**5**] and the paper of Glasner and Weiss [**6**] about existence of minimal skew product extension. In fact, it is quite natural, once one knows the existence of a minimal action on Hilbert space, to try to apply the ideas of Glasner and Weiss and to lift this action on l^2 to a minimal skew product extension on $l^2\times M$ which is homeomorphic to M. Unfortunately, this is impossible due to the non-compactness of M. However it is easy to construct skew product extensions which have dense orbits, and as we showed in [**5**] it is pretty natural in the context of infinite dimensional topology to try to construct extensions such that the subset of points of $l^2\times M$ with a dense orbit is homeomorphic to $l^2\times M$.

It is not necessary to be familiar with [**5**] or [**6**] to read this paper. In fact we obtain, as a by product, a generalization of the theorem of Glasner and Weiss to a general group and a general base space (see theorem 4.3).

1. *Topology on the space of skew products*

We consider a topological space X endowed with a continuous action α of a locally compact group G. Since this action is fixed in the sequel we will denote the effect of $g\in G$ on $x\in X$ by gx instead of $\alpha(g,x)$. We suppose that Y is a metric space, whose metric is denoted by d. Without loss of generality, we can assume that the metric d is bounded by 1 (replace $d(x,y)$ by $\min[1, d(x,y)]$).

We will consider actions A of G on $X \times Y$ which are skew products over α. Such an action $A: G \times X \times Y \to X \times Y$ can be written as $A(g, x, y) = (gx, a_{g,x}(y))$. We will call $a_{g,x}$ the cocycle associated to A. It verifies the cocycle equations:

$$a_{g',gx} \circ a_{g,x} = a_{g'g,x}, \qquad \forall x \in X, \forall g \in G, \forall g' \in G,$$
$$a_{e,x} = \mathrm{Id}_Y, \qquad \forall x \in X,$$

(e is the neutral element in G).

We denote by $\mathcal{S}(\alpha)$ the set of skew products on $X \times Y$ over α. For each compact subset C of G, we define a semi-metric D_C on $\mathcal{S}(\alpha)$ by:

$$D_C(A, A') = \sup \{d(a_{g,x}(y), a'_{g,x}(y)) | g \in C, x \in X, y \in Y\},$$

where $(a_{g,x})$ and $(a'_{g,x})$ are the cocycles associated with A and A'.

The set of semi-metrics $\{D_C | C$ a compact subset of $G\}$ defines a uniform structure on $\mathcal{S}(\alpha)$ which is easily seen to be Hausdorff. Moreover if G is σ-compact, it is easy to define a metric on $\mathcal{S}(\alpha)$ which gives the same uniform structure.

LEMMA 1.1. *If Y is complete for d, then $\mathcal{S}(\alpha)$ is complete. In particular if G is σ-compact, $\mathcal{S}(\alpha)$ is metric complete.*

Proof. Suppose $(A^i)_{i \in I}$ is a Cauchy net in $\mathcal{S}(\alpha)$ with associated cocycles $(a^i_{g,x})$. It is easy to show that there exists a continuous map $G \times X \times Y \to Y$, $(g, x, y) \mapsto a_{g,x}(y)$, such that if we put $A(g, x, y) = (gx, a_{g,x}(y))$, we have, for each compact subset C of G, $D_C(A^i, A) \to 0$ as i goes to infinity in I. We still must check that A is an *action* of G on $X \times Y$. This means that we have to prove that $a_{g,x}$ verifies the cocycle conditions.

Suppose $g, g' \in G$, $x \in X$ and $y \in Y$; we have for each $i \in I$:

$$\begin{aligned} d[a_{g',gx}(a_{g,x}(y)), a_{g'g,x}(y)] \le\ & d[a_{g',gx}(a_{g,x}(y)), a_{g',gx}(a^i_{g,x}(y))] \\ & + d[a_{g',gx}(a^i_{g,x}(y)), a^i_{g',gx}(a^i_{g,x}(y))] \\ & + d[a^i_{g',gx}(a^i_{g,x}(y)), a^i_{g'g,x}(y)] + d[a^i_{g'g,x}(y), a_{g'g,x}(y)]. \end{aligned}$$

Now, as i goes to infinity, it is clear that the second and fourth terms go to zero because they are bounded respectively by $D_{\{g'\}}(A, A^i)$ and $D_{\{g'g\}}(A, A^i)$. The first term goes to zero as i goes to infinity because $a^i_{g,x}(y) \to a_{g,x}(y)$ and $a_{g',gx}$ is a continuous map. The third term is zero because of the cocycle conditions on $(a^i_{g,x})$. Of course we have $a_{e,x} = \mathrm{Id}_Y$, for all $x \in X$, because this is true for each $a^i_{e,x}$. □

LEMMA 1.2. *The topology defined on $\mathcal{S}(\alpha)$ is finer than the compact open topology.*

Proof. Let p_1 and p_2 be the canonical projections of $X \times Y$ on the two factors. The topology on $\mathcal{S}(\alpha)$ is the topology of uniform convergence of the maps $p_2 \circ A$: $G \times X \times Y \to Y$ on each set of the form $C \times X \times Y$, where C is an arbitrary compact subset of G. Certainly this topology is finer than the compact open topology on the maps from $G \times X \times Y$ to Y.

This proves the lemma since $p_1 \circ A$ is independent of $A \in \mathcal{S}(\alpha)$. □

We now introduce a subset $\tilde{\mathcal{S}}(\mathrm{Id}_X)$ of $\mathcal{S}(\mathrm{Id}_X)$ the set of skew products on $X \times Y$ over the identity of X. The set $\tilde{\mathcal{S}}(\mathrm{Id}_X)$ is defined as the set of homeomorphisms H of $X \times Y$ of the form $H(x, y) = (x, h_x(y))$ such that the two sets $(h_x)_{x \in X}$, $(h_x^{-1})_{x \in X}$

of self maps of Y are *uniformly equicontinuous* (with respect to d). It is easy to verify that $\tilde{\mathcal{S}}(\mathrm{Id}_X)$ is a group (no topology involved here).

Given A in $\mathcal{S}(\alpha)$ and H in $\tilde{\mathcal{S}}(\mathrm{Id}_X)$, we can define HAH^{-1} by

$$HAH^{-1}(g, x, y) = H(A(g, H^{-1}(x, y))).$$

It is clear that HAH^{-1} is in $\mathcal{S}(\alpha)$.

LEMMA 1.3. *If $H \in \tilde{\mathcal{S}}(\mathrm{Id}_X)$, the map $\mathcal{S}(\alpha) \to \mathcal{S}(\alpha)$, $A \to HAH^{-1}$ is a homeomorphism.*

Proof. We have only to check that it is continuous. Since H is in $\tilde{\mathcal{S}}(\mathrm{Id}_X)$, the function $\theta: \mathbb{R}_+ \to \mathbb{R}_+$ defined by

$$\theta(\delta) = \sup \{d(h_x(y), h_x(y')) | x \in X, y, y' \in Y, \quad d(y, y') \leq \delta\}$$

satisfies $\lim_{\delta \to 0} \theta(\delta) = 0$. The lemma follows then from the easily verified inequality:

$$D_C(HAH^{-1}, HA'H^{-1}) \leq \theta(D_C(A, A')). \qquad \square$$

2. *Construction of some real valued functions*

Let G be a locally compact group, C a compact subset of G and n a positive integer. We define a function $\varphi_{n,C}: G \to [0, 1]$ by:

$$\varphi_{n,C}(C) = 1;$$
$$\varphi_{n,C}(C^{m+1} - C^m) = \max((n-m)/n, 0), \qquad \forall m \in \mathbb{N}^*;$$
$$\varphi_{n,C} = 0 \qquad \text{outside } \bigcup_{m \in \mathbb{N}^*} C^m.$$

We state some properties of $\varphi_{n,C}$ in the following lemma.

LEMMA 2.1. *$\varphi_{n,C}$ is (Borel) measurable; $\varphi_{n,C}(C) = 1$, $\varphi_{n,C}(G) \subset [0, 1]$; $\varphi_{n,C} = 0$ outside $\bigcup_{i=1}^n C^i$. Moreover if C is symmetric (i.e. $C = C^{-1} = \{g^{-1} | g \in C\}$), we have for each $g \in C$ and $g' \in G$;*

$$|\varphi(g'g) - \varphi(g')| \leq \frac{1}{n} \quad \textit{and} \quad |\varphi(gg') - \varphi(g')| \leq \frac{1}{n}.$$

Proof. The first four properties are trivial. The last one follows from the fact that if C is symmetric, we have: $g \in C$, $g' \in C^{m+1} \setminus C^m$ implies gg' and $g'g$ belong to

$$C^{m+2} \setminus C^{m-1} \subset C^{m+2} \setminus C^{m+1} \cup C^{m+1} \setminus C^m \cup C^m \setminus C^{m-1}. \qquad \square$$

We will need the following well-known technical lemma. We will provide the proof, since it is very short.

LEMMA 2.2. *Suppose $\alpha: G \times X \to X$ is a continuous action of the group G on X. If $K \subset G$ is compact and $A \subset X$ is closed, then the set $\alpha(K \times A)$ is closed in X.*

Proof. Since α can be written as the composition of the homeomorphism $G \times X \to G \times X$, $(g, x) \mapsto (g, \alpha(g, x))$ followed by the projection $G \times X \to X$, it suffices to prove that if K is a compact space, the projection p: $K \times X \to X$ is closed. Let B be a closed subset of $K \times X$, we have:

$$X \setminus p(B) = \{x \in X | K \times \{x\} \subset K \times X \setminus B\}.$$

Using the compactness of K, it is very easy to show that this last set is open in X. $\square$

We can now state and prove the main lemma of this work.

LEMMA 2.3. *For each compact subset C of G and each $\varepsilon>0$, we can find a compact subset C' of G which satisfies the following property:*

Given a continuous action α of G on a normal space X, a closed subset F of X, with $gF\cap F=\emptyset$ for all $g\in G$, $g\neq e$, and a neighbourhood V of $C'F=\{gx|g\in C', x\in F\}$, there exists a continuous map $\theta: X\to[0,1]$ such that:

(i) $\theta(F)=1$;

(ii) $\theta=0$ *outside* V;

(iii) $\forall g\in C, \forall x\in X, \quad |\theta(gx)-\theta(x)|<\varepsilon$.

Proof. We can suppose that C is a symmetric neighbourhood of e in G. Let μ be a right invariant Haar measure on G. Choose C_0 a compact symmetric neighbourhood of e such that $C_0^2\subset \mathring{C}$, the interior of C. We have $\mu(C_0)>0$. Fix n such that $2\mu(C)/n\mu(C_0)<\varepsilon$. Put $C'=C^{n+1}$. Suppose now α, X, F and V are given as stated above. Using the inclusion $C_0^2\subset\mathring{C}$ and the fact that $gF\cap F=\emptyset$, for all $g\neq e$, we have

$$[(C^{2n}-\mathring{C})C_0F]\cap C_0F=\emptyset.$$

Since our space X is normal, we can separate the two closed (see lemma 2.2) sets $(C^{2n}-\mathring{C})C_0F$ and C_0F by two open sets W and W'. Using the compactness of $(C^{2n}-\mathring{C})$ and C^n we can find an open neighbourhood U of C_0F such that $U\subset W'$, $(C^{2n}-\mathring{C})U\subset W$, $C^nU\subset V$ (recall that V is a neighbourhood of $C^{n+1}F=C'F$). So up to this point we have found a neighbourhood U of the closed set C_0F such that:

$$[(C^{2n}-\mathring{C})U]\cap U=\emptyset \quad\text{and}\quad C^nU\subset V.$$

We now choose a continuous map $\rho: X\to[0,1]$ such that $\rho(C_0F)=1$, $\rho=0$ outside U. We consider the map $\varphi_{n,C}$ defined above and we define $\theta_1: X\to[0,\infty[$ by:

$$\theta_1(x)=\int_G\varphi_{n,C}(k)\rho(kx)\,d\mu(k)=\int_{C^n}\varphi_{n,C}(k)\rho(kx)\,d\mu(k).$$

The last equality follows from the fact (lemma 2.1) that $\varphi_{n,C}$ is 0 outside $\bigcup_{i=1}^n C^i=C^n$, $(e\in C!)$.

Using the Lebesgue dominated convergence theorem, it is easy to show that θ_1 is continuous. We remark also that $\theta_1(x)\neq 0$ implies that there exists $k\in C^n$ such that $kx\in U$, in particular θ_1 is zero outside $C^nU\subset V$. Moreover if $x\in X$, k and $k'\in G$ are such that $\varphi_{n,C}(k)\rho(kx)$ and $\varphi_{n,C}(k')\rho(k'x)$ are non-zero, we have $k, k'\in C^n$, $kx, k'x\in U$. In particular, we obtain $kk'^{-1}\in C^{2n}$ and $kk'^{-1}(k'x)\in U$, hence $kk'^{-1}\in C$ by the choice of U. Thus we have shown that for each $x\in X$, there exists $k'\in G$ such that the function $k\mapsto\varphi_{n,C}(k)\rho(kx)$ is zero outside Ck'. The same property is also true for any function of the form $k\mapsto\varphi_{n,C}(kh^{-1})\rho(kx)$ since it is a right translated function obtained from $k\mapsto\varphi_{n,C}(k)\rho[k(hx)]$. Let us now remark that:

$$\theta_1(hx)=\int_G\varphi_{n,C}(k)\rho(khx)\,d\mu(k)=\int_G\varphi_{n,C}(kh^{-1})\rho(kx)\,d\mu(k),$$

by right invariance of μ. Now using lemma 2.1, what has just been said above and the right invariance of μ, we obtain that if $h\in C$ and $x\in X$:

$$|\theta_1(hx)-\theta_1(x)|\leq 2\mu(C)/n.$$

Since $\rho(C_0F)=1$ and $\varphi_{n,C}(C)=1$, we have $\theta_1(x)\geq\mu(C_0)$ for each $x\in F$. If we define $\tilde{\theta}_1$ by $(1/\mu(C_0))\theta_1$, it satisfies properties (ii) and (iii), but it takes values ≥ 1 and it is only ≥ 1 on F. It is easy to rectify this by composing θ_1 with the retraction $r:[0,\infty[\to[0,1]$, $r(x)=\inf(x,1)$. Property (iii) is true for $\theta=r\theta_1$ because we have

$$|r(t)-r(t')|\leq|t-t'|,\qquad \forall t,t'\in[0,\infty[.\qquad\square$$

Remarks. (1) The function θ in lemma 2.3 is, in the case where $G=\mathbb{Z}$, very similar to functions constructed in [7, proof of theorem 3] and [**8**, top of p. 482].

(2) In fact, we need lemma 2.3 in the case where F is compact. Of course, in this context, lemma 2.2 is obvious.

3. *Construction of some skew products*

We consider a continuous action α of the locally compact group G on the normal (Hausdorff) space X. We suppose also that our metric space Y satisfies the following condition:

($\mathcal{H}$) The metric space Y is locally compact, and the group of homeomorphisms with compact support, which are isotopic to the identity through an isotopy with compact support, acts minimally on Y.

Another formulation of ($\mathcal{H}$) is the following:

($\mathcal{H}$) The metric space Y is locally compact, and for every non-void open set $U\subset Y$ and any compact set $K\subset Y$, there exists a finite number of isotopies $(k_{i,t})_{t\in[0,1]}$, $i=1,\dots,n$, with compact support, such that $k_{i,0}=\mathrm{Id}_Y$, $i=1,\dots,n$, and $K\subset\bigcup_{i=1}^{n}(k_{i,1})^{-1}(U)$.

Recall that '$(k_t)_{t\in[0,1]}$ is an isotopy with compact support on Y' means that the self-map of $Y\times[0,1]$, $(y,t)\mapsto(k_t(y),t)$ is a homeomorphism with compact support. This implies that the set of maps $(k_t)_{t\in[0,1]}$ and $(k_t^{-1})_{t\in[0,1]}$ are uniformly equicontinuous with respect to any metric on Y compatible with its topology.

We remark that the condition ($\mathcal{H}$) is satisfied if Y is locally compact and given any two points $y,y'\in Y$, there exists an isotopy with compact support $(k_t)_{t\in[0,1]}$ such that $k_1(y)=y'$. This last condition is satisfied in the case where Y is a connected finite dimensional manifold without boundary. It is also satisfied if Y is a connected Hilbert cube manifold (see [**3**] for the properties of Hilbert cube manifolds).

The condition ($\mathcal{H}$) is also satisfied if Y is a compact space on which a path connected topological group acts minimally.

We now use the results of § 2 to obtain the following approximation lemma.

LEMMA 3.1. *Let $\delta>0$, C a compact subset of G, K and V respectively a compact and an open non-void subset of Y, U an open non-void subset of X, be given. We suppose that $F\subset X$ is a closed subset satisfying $gF\cap F=\varnothing$, for all $g\in G$, $g\neq e$, and $\{g\in G|gF\subset U\}$ is not relatively compact in G (in particular it is non-void). Then we can find $H\in\tilde{\mathcal{G}}(\mathrm{Id}_X)$ such that*:

(i) $D_C(\alpha\times\mathrm{Id}_Y, H(\alpha\times\mathrm{Id}_Y)H^{-1})<\delta$;

(ii) *the set $F\times K$ is contained in the orbit of $U\times V$ under $H(\alpha\times\mathrm{Id}_Y)H^{-1}$.*

Proof. Without loss of generality, we can assume that C is a symmetric set containing $e\in G$. We can find a finite number of isotopies of Y, $(k_{i,t})_{t\in[0,1]}$, $i=1,\dots,n$, with compact supports such that $k_{i,0} = \mathrm{Id}_Y$ and $K \subset \bigcup_{i=1}^{i=n} (k_{i,1})^{-1}(V)$. Using the fact that the isotopies have compact support, we can find an $\varepsilon > 0$ such that for all $t, t' \in [0, 1]$, $|t - t'| < \varepsilon$, and for all $i \in \{1, \ldots, n\}$, and all $y \in Y$, $d(k_{i,t}(y), k_{i,t'}(y)) < \delta$.

Given this ε and the compact subset C of G, we can obtain, using lemma 2.3, a compact subset C' of G which will allow us to apply lemma 2.3.

Since $\{g \in G \mid gF \subset U\}$ is not relatively compact, we can construct by induction $g_1, \ldots, g_n \in G$ (n is the same as the number of isotopies) such that

$$g_iF \subset U, \qquad i = 1, \ldots, n,$$
$$e \notin C'g_i, \qquad i = 1, \ldots, n,$$
$$CC'g_i \cap CC'g_j = \varnothing, \qquad 1 \le i < j \le n.$$

Applying lemma 2.3, we can find continuous functions $\varphi_1, \ldots, \varphi_n : X \to [0, 1]$ such that:

(a) $\varphi_i(g_iF) = 1$;

(b) $\forall g \in C, \forall x \in X, \forall i \in \{1, \ldots, n\}, |\varphi_i(gx) - \varphi_i(x)| < \varepsilon$;

(c) supp φ_i, the (closed) support of φ_i, is so close to $C'g_iF$ that $\varphi_i|F = 0$, $i = 1, \ldots, n$, and $C \operatorname{supp} \varphi_i \cap C \operatorname{supp} \varphi_j = \varnothing$, $1 \le i < j \le n$.

We now define H: $X \times Y \to X \times Y$, $(x, y) \mapsto (x, h_x(y))$ by:

$$h_x(y) = \begin{cases} k_{i,\varphi_i(x)}(y), & \text{if } x \in \operatorname{supp} \varphi_i, \quad i = 1, \ldots, n \\ y, & \text{if } x \notin \bigcup_{i=1}^n \operatorname{supp} \varphi_i. \end{cases}$$

The condition (c) above shows that H is a well defined homeomorphism. Moreover, this same condition (c) shows that for any $x \in X$ we can find $i \in \{1, \ldots, n\}$ such that for any $g \in C$ (in particular $g = e$) and any $y \in Y$,

$$h_{gx}(y) = k_{i,\varphi_i(gx)}(y).$$

It follows then from condition (b) and the definition of ε that for all $x \in X$, $g \in C$, and $y \in Y$, $d(h_{gx}h_x^{-1}(y), y) < \delta$. Since

$$H(\alpha \times \mathrm{Id}_Y)H^{-1}(g, x, y) = (gx, h_{gx}h_x^{-1}(y)),$$

we obtain condition (i) of the lemma. The fact that H is in $\tilde{\mathscr{S}}(\mathrm{Id}_X)$ follows easily from the compactness of the supports of the isotopies $(k_{i,t})_{t \in [0,1]}$.

Let us verify condition (ii). Given $y \in K$, we can find $i \in \{1, \ldots, n\}$ such that $k_{i,1}(y) \in V$. If $x \in F$, we have

$$HAH^{-1}(g_i, x, y) = (g_ix, h_{g_ix}h_x^{-1}(y)).$$

By construction of g_i, $g_ix \in U$. Since $x \in F$, properties (a) and (c) on the $(\varphi_j)_{j=1,\ldots,n}$ show that $h_{g_ix} = k_{i,1}$ and $h_x = \mathrm{Id}_Y$, in particular

$$h_{g_ix}h_x^{-1}(y) = k_{i,1}(y) \in V. \qquad \square$$

4. *Topologically transitive and minimal skew products.*

In this section, we will suppose that G is a locally compact σ-compact topological group acting continuously on the normal space X. We will suppose that X is second countable (i.e. there exists a countable basis of open sets), and that Y satisfies the condition $(\mathscr{H})$ of § 3. Note that since Y is locally compact and metric, we can suppose that the metric d we have on Y is complete. Since G is σ-compact $\mathcal{S}(\alpha)$ is metrizable and complete, in particular each closed subspace of $\mathcal{S}(\alpha)$ is a Baire space. Since Y satisfies condition $(\mathcal{H})$ of § 3, it is easy to check that the path components of Y are dense, in particular Y is connected. It follows easily from [**2**, theorem 5, p. 109], that a metric locally compact connected space is σ-compact, and hence also second countable. We conclude from this that $X\times Y$ is second countable.

Following [**6**], we introduce $\vartheta(\alpha)\subset\mathcal{S}(\alpha)$, the orbit of $\alpha\times \mathrm{Id}_Y$ under the action of the group $\tilde{\mathcal{S}}(\mathrm{Id}_X)$:

$$\vartheta(\alpha)=\{H(\alpha\times\mathrm{Id}_Y)H^{-1}|H\in\tilde{\mathcal{S}}(\mathrm{Id}_X)\}.$$

By what we have seen $\overline{\vartheta(\alpha)}$, the closure of $\vartheta(\alpha)$ in $\mathcal{S}(\alpha)$, is a Baire space.

Let K be compact subset of X, such that $gK\cap K=\emptyset$, for all $g\in G$, $g\neq e$, and for each non-void open subset U of X, the set $\{g|gK\subset U\}$ is not relatively compact in G.

LEMMA 4.1. *Under the above hypothesis, there exists a dense G_δ subset $\mathcal{G}$ of the Baire space $\overline{\vartheta(\alpha)}$ such that for each A in $\mathcal{G}$ and each open non-void subset W of $X\times Y$, the orbit of W under A contains the set $K\times Y$.*

Proof. Since $\overline{\vartheta(\alpha)}$ is a Baire space, Y is σ-compact, and $X\times Y$ second countable, it suffices to prove that for each open non-void subset W of $X\times Y$, and each compact subset $\tilde{K}$ of Y the set

$$\mathcal{E}(\tilde{K},W)=\left\{A\in\overline{\vartheta(\alpha)}\,\middle|\,K\times\tilde{K}\subset\bigcup_{g\in G}A(g)W\right\}$$

is open and dense in $\overline{\vartheta(\alpha)}$. We have used $A(g)W$ to denote the image of W under the homeomorphism $(x,y)\mapsto A(g,x,y)$. The fact that $\mathcal{E}(\tilde{K},W)$ is open, is an immediate consequence of lemma 1.2. The fact that $\alpha\times\mathrm{Id}_Y\in\overline{\mathcal{E}(\tilde{K},W)}$ results from lemma 3.1. By the definition of $\vartheta(\alpha)$, all we have to show is that $H(\alpha\times\mathrm{Id}_Y)H^{-1}$ is in $\overline{\mathcal{E}(\tilde{K},W)}$ for each H in $\tilde{\mathcal{S}}(\mathrm{Id}_X)$. Now we have by lemma 1.3:

$$H(\alpha\times\mathrm{Id}_Y)H^{-1}\in\overline{\mathcal{E}(\tilde{K},W)}\Leftrightarrow\alpha\times\mathrm{Id}_Y\in\overline{H^{-1}\mathcal{E}(\tilde{K},W)H}.$$

But

$$H^{-1}\mathcal{E}(\tilde{K},W)H=\left\{A\in\overline{\vartheta(\alpha)}\,\middle|\,K\times\tilde{K}\subset\bigcup_{g\in G}HAH^{-1}(W)\right\}$$
$$=\left\{A\in\overline{\vartheta(\alpha)}\,\middle|\,H^{-1}(K\times\tilde{K})\subset\bigcup_{g\in G}A(g)(H^{-1}(W))\right\}.$$

Since H is a skew product over the identity of X, we can find a compact subset $\tilde{\tilde{K}}$ of Y such that $H^{-1}(K\times\tilde{K})\subset K\times\tilde{\tilde{K}}$. In particular we obtain:

$$\mathcal{E}(\tilde{\tilde{K}},H^{-1}(W))\subset H^{-1}\mathcal{E}(\tilde{K},W)H.$$

Since $\alpha\times\mathrm{Id}_Y\in\overline{\mathcal{E}(\tilde{K},W)}$ holds for any compact set $\tilde{K}$ in Y, and any non-void open set W in $X\times Y$, we obtain:

$$\alpha\times\mathrm{Id}_Y\in\overline{\mathcal{E}(\tilde{\tilde{K}},H^{-1}(W))}\subset\overline{H^{-1}\mathcal{E}(\tilde{K},W)H}.$$

This completes the proof of the lemma. □

We now easily obtain from this lemma two theorems.

THEOREM 4.2. *Suppose Y satisfies* ($\mathcal{H}$), *and that the continuous action* α *of the* σ*-compact locally compact group G on the second countable normal space X is such that there exists a point* $x_0 \in X$ *with* Gx_0 *dense in X,* $gx_0 \neq x_0$, *for all* $g \in G$, $g \neq e$, *and for every compact subset C of G the set* Cx_0 *has no interior in X. Then there exists a skew-product extension A of* α *to* $X \times Y$ *which is topologically transitive. In fact, A can be chosen such that every point of the form* $(x_0, y) \in X \times Y$ *has a dense orbit under the action of A.*

Proof. Since Gx_0 is dense and for each compact subset C of G, the set Cx_0 has no interior in X, it is easy to show that $\{g \in G | gx_0 \in U\}$ is not relatively compact in G for each non-void open subset of X. We can now apply lemma 4.1 to find $A \in \mathcal{S}(\alpha)$ such that $\{x_0\} \times Y \subset \bigcup_{g \in G} A(g) W$ for each non-void open set W of $X \times Y$. In particular if $y \in Y$, there exists $g \in G$ such that $(x_0, y) \in A(g) W$, which implies $A(g^{-1})(x_0, y) \in W$. □

THEOREM 4.3. *Suppose there exists a minimal action of a path-connected group on the compact metric space Y. Suppose that the locally compact* σ*-compact group G acts continuously and minimally on the second countable normal space X, in such a way that there exists* $x_0 \in X$, *on which the action of G is free, and* Cx_0 *has no interior in X for each compact subset C of G. Then there exists a skew product extension of G to* $X \times Y$ *which is minimal.*

Proof. By theorem 4.2 (or lemma 4.1), we can find a skew product extension A of α, such that for every non-void open set W of $X \times Y$, we have $\{x_0\} \times Y$ with $\{x_0\} \times Y \subset \bigcup_{g \in G} A(g) W$. Since Y is compact and $\bigcup_{g \in G} A(g) W$ open, it is easy to find an open neighbourhood U of x_0 in X such that $U \times Y \subset \bigcup_{g \in G} A(g) W$. The invariance of $\bigcup_{g \in G} A(g) W$ under G implies that

$$\bigcup_{g \in G} A(g) W \supset \left(\bigcup_{g \in G} gU \right) \times Y = X \times Y,$$

where the last equality follows from the minimality of the action of G on X. □

Remarks. (1) In fact in theorems 4.2 and 4.3 (and also in lemma 4.1 if $K \neq \emptyset$), the group G has to be second countable (hence metric). This follows from the fact that G is σ-compact, and the fact that each compact subset C of G is second countable because it is homeomorphic to Cx_0 which is contained in the second countable space X.

(2) In the case X compact metric and $G = \mathbb{Z}$, theorem 4.3 is in [**6**]. In the case X compact, Y a compact connected topological group and $G = \mathbb{Z}$ or $\mathbb{R}$, theorem 4.3 is in [**4**] and [**7**].

(3) Theorem 4.2 is well known in the cases $G = \mathbb{Z}$ or $\mathbb{R}$.

(4) J. C. Yoccoz has shown me that theorem 4.3 is false if we do not assume that the metric space Y is compact. In fact, if we look at the action α of $\mathbb{T}^1$ on the first factor of $\mathbb{T}^1 \times \mathbb{R}$, there is no minimal homeomorphism in $\overline{\vartheta(\alpha)}$.

(5) The fact that we must know that Cx_0 has empty interior in X, for each compact set in G, is necessary for theorems 4.2 and 4.3. For example if this hypothesis is not

verified in the context of a minimal action of G on X, it is easy to show that the dynamical system α is homeomorphic to the one given by left translation on G. However this dynamical system has no minimal extension to $G \times Y$ (as soon as Y has more than one point!), because in a skew product on $G \times X$ the orbits are in fact graphs of (continuous) maps $G \to Y$.

5. *Existence of minimal actions on Hilbert manifolds*

The purpose of this section is to establish the following theorem.

THEOREM 5.1. *Any locally compact, non-compact, second countable topological group acts continuously and minimally on any connected, second countable, infinite dimensional Hilbert manifold.*

Fix a topological group G which is locally compact, non-compact and second countable, hence metrizable. We will let M denote a second countable, infinite dimensional Hilbert manifold. We will have to use some results from the theory of infinite dimensional manifolds, most of them can be found either in [**1**] or in [**3**]. The first result we will use is the fact that we can find a (connected) Hilbert cube manifold Y such that M is homeomorphic to $l^2 \times Y$, where l^2 is the separable infinite dimensional Hilbert space. Then we will try to obtain the minimal action as a skew product of a minimal action on l^2. Of course by remark 4 at the end of § 4 our methods do not allow us to prove that. What we will prove is that we can find a skew product on $l^2 \times Y$ whose set of points with a dense orbit is homeomorphic to $l^2 \times Y$. To prove the existence of such a homeomorphism we will need the notion of Z-set, which we recall now.

Definition. (See [**1**, p. 151]). A closed subset F of a Hilbert manifold M is called a *Z-subset* (or *Z-set*) if the set of continuous maps $C^0(Q, M \setminus F)$ is dense in $C^0(Q, M)$ for the uniform topology. We have used Q to denote the product $[0, 1]^{\mathbb{N}}$ (i.e. the Hilbert cube).

We have the following theorem ([**1**, corollary 7.3, p. 316]):

THEOREM 5.2. *If a set A, in the Hilbert manifold M, is a countable union of (closed) Z-sets, then $M \setminus A$ is homeomorphic to M.*

After these preliminaries of infinite dimensional topology, we now exhibit an action of G on l^2 with some good properties (see [**5**]).

We consider the space $C^0(G)$ of real valued continuous functions on G, endowed with the topology of uniform convergence on compact subsets of G. Since G is locally compact and second countable, the space $C^0(G)$ is in fact a separable Fréchet space. The group G acts continuously on $C^0(G)$ by $(gf)(x) = f(g^{-1}x)$ where $f \in C^0(G)$, $g \in G$, $x \in G$. It is easy to see that G acts effectively on $C^0(G)$; this means given any $g \in G$, $g \neq e$, there exists $f \in C^0(G)$ such that $gf \neq f$. By considering functions with compact support, and the fact that G is non-compact, it is easy to show that this action has the properties (P) stated below.

Definition. A continuous action of G on a topological vector space E, $(g, x) \to gx$, *satisfies properties* (P) if:

(i) for each $g \in G$ the map $x \to gx$ is a (continuous) linear map;

(ii) the vector subspace $\{x \in E \mid \lim_{g\to\infty} gx = 0\}$ is dense in E.

Properties (P) are important because of the following facts.

LEMMA 5.3. *If an action of G on E has properties* (P), *then the action of G on $E^{\mathbb{N}}$ also has properties* (P).

The proof of lemma 5.3 is easy.

LEMMA 5.4. *If an action of G on E has properties* (P), *and if E is locally convex, then for any compact space K the induced action of G on $C^0(K, E)$ (endowed with the uniform or compact open topology) has properties* (P).

Proof. Given a convex neighbourhood V of 0 in E and $f: K \to E$, it is easy to construct a cover $(U_i)_{i=1,\dots,n}$ of K such that if we choose $k_i \in U_i$, $i = 1, \dots, n$, then for all $k \in U_i$, $f(k) - f(k_i) \in V$. Let $(\varphi_i)_{i=1,\dots,n}$ be a partition of unity on K with supp $\varphi_i \subset U$. Using the fact that V is convex, we obtain:

$$f(k) - \sum_{i=1}^{n} \varphi_i(k) f(k_i) \in V, \qquad \forall k \in K.$$

We can, by the density of $\{x \in E \mid \lim_{g\to\infty} gx = 0\}$, choose $x_1, \dots, x_n$ in this set such that $f(k_i) - x_i \in V$, $i = 1, \dots, n$. Again convexity of V shows that

$$\sum_{i=1}^{n} \varphi_i(k)(f(k_i) - x_i) \in V.$$

In particular the function defined by

$$f'(k) = \sum_{i=1}^{n} \varphi_i(k) x_i$$

is $V + V$ close to f. Moreover $\lim_{g\to\infty} gf' = 0$, since f' takes values in a finite dimensional vector subspace of $\{x \mid \lim_{g\to\infty} gx = 0\}$ and the action of G is linear. □

LEMMA 5.5 (Rolewicz [**9**]). *If the action of G on E has properties* (P), *then it is topologically transitive.*

Proof. Let U and V be open non-void subsets of E; we have to show that $gU \cap V$ is non-void for some g in G. By properties (P), we can find $a \in U$ and $b \in V$ such that

$$\lim_{g\to\infty} ga = \lim_{g\to\infty} gb = 0.$$

In particular, if g is big enough, $a + g^{-1}b \in U$ and $ga + b \in V$. But, by the linearity of the action of g, we have $g(a + g^{-1}b) = ga + b$. □

Let us go back now to our action of G on $C^0(G)$. We will in fact consider the action of G on the separable Fréchet space $X = C^0(G)^{\mathbb{N}}$. We have:

LEMMA 5.6 (West [**10**] or [**1**, p. 168]). *The set of continuous maps $F: Q \to X$ such that for all $g \in G$, $g \neq e \Rightarrow gf(Q) \cap f(Q) = \varnothing$ is a dense G_δ in $C^0(Q, X)$.*

Proof. Since G is second countable and locally compact we can write $G\setminus\{e\}=\bigcup_{n\in\mathbb{N}} C_n$ a countable union of compact subsets of G. Each of the sets

$$\mathscr{E}_n=\{f\in C^0(Q,X)|\forall g\in C_n,\quad gf(Q)\cap f(Q)=\varnothing\}$$

is easily seen to be open. This implies that the set defined in the lemma, which is $\bigcap_{n\in\mathbb{N}}\mathscr{E}_n$, is a G_δ subset of $C^0(Q,X)$. We have still to show the density of this set. Let us use $(a_i)_{i\in\mathbb{N}}$ to denote a dense countable subset of the separable space $C^0(G)$. If $f: Q\to X=C^0(G)^{\mathbb{N}}$ is a continuous map, we can define for each $m\in\mathbb{N}$ a continuous map $f_m: Q\to C^0(G)^{\mathbb{N}}$, by

$$p_nf_m=p_nf \qquad \text{if } n<m,$$

$$p_nf_m=a_{n-m} \qquad \text{if } n\geq m,$$

where p_n is the nth projection of $C^0(G)^{\mathbb{N}}$ on $C^0(G)$. It is easy to verify that $\lim_{m\to\infty}f_m=f$. Moreover, since $(a_n)_{n\in\mathbb{N}}$ is dense in $C^0(G)$ and the action of G is effective, given any $g\in G$, $g\neq e$, there exists a_n with $ga_n\neq a_n$, it follows easily by looking at the $(m+n)$th component of f_m that

$$gf_m(Q)\cap f_m(Q)=\varnothing. \qquad \square$$

We can now state the properties we need for the action of G on X.

LEMMA 5.7. *There exists a continuous action α of G on a separable Fréchet space X, for which there exists a countable dense subset $(f_i)_{i\in\mathbb{N}}$ of $C^0(Q,X)$ such that*:

(a) $\forall i\in\mathbb{N}, \forall g\in G, g\neq e\Rightarrow gf_i(Q)\cap f_i(Q)=\varnothing$;

(b) $\forall i\in\mathbb{N}, \forall U$ *open non-void subset of X, the set $\{g\in G|gf_i(Q)\subset U\}$ is not relatively compact in G.*

Proof. We take the action of G on $X=C^0(G)^{\mathbb{N}}$ given above, we know that (lemma 5.3) it has properties (P). By lemma 5.6, the set $\mathscr{G}$ of functions $f: Q\to X$ such that for all $g\neq e$, $gf(Q)\cap f(Q)=\varnothing$ is a dense G_δ. Using lemma 5.4 and lemma 5.5, we obtain that if U is an open non-void subset of U, the open subset of $C^0(Q,X)$

$$\mathscr{U}(U)=\{f\in C^0(Q,X)|\exists g\in G,\quad gf(Q)\subset U\}\text{ is dense.}$$

Since $C^0(Q,X)$ is a complete metrizable space, the set $\mathscr{G}\cap\bigcup_{g\in\mathbb{N}}\mathscr{U}(U_j)$ is a dense G_δ subset of $C^0(Q,X)$, where $(U_j)_{j\in\mathbb{N}}$ denotes a basis of open non-void subsets of X. It is easy to check, using the fact that a compact subset of the infinite dimensional vector space has empty interior, that for every $f\in\bigcap_{i\in\mathbb{N}}\mathscr{U}(U_i)$ and every non-void open subset U of X, the set $\{g\in G|gf(Q)\subset U\}$ is not relatively compact in G. Since $C^0(Q,X)$ is separable, we can obtain the set $(f_i)_{i\in\mathbb{N}}$ by selecting a countable dense subset of $\mathscr{G}\cap\bigcap_{j\in\mathbb{N}}\mathscr{U}(U_j)$. $\square$

Proof of theorem 5.1. We will find a skew product action A of G on $X\times Y$ where the action α of G on X is given by lemma 5.7. The fact that X is only a separable Fréchet space and not l^2 is not a restriction, since all separable infinite dimensional Fréchet spaces are homeomorphic (see [**1**, theorem 5.2, p. 189]). Remark that, since Y is a connected Q-manifold, it satisfies hypothesis ($\mathscr{H}$) via the fact, already mentioned, that isotopies with compact support operate transitively on Y. Moreover, if f_i is one of the maps $Q\to X$ given by lemma 5.7, we can apply lemma 4.1 with $f_i(Q)$ as a compact subset of X. Using the fact that $\overline{\vartheta(\alpha)}$ is a Baire space, we can

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in fact find a G_δ dense subset $\mathcal{M}$ of $\overline{\vartheta(\alpha)}$, such that, for each $A \in \mathcal{M}$, each open non-void subset U of $X \times Y$ and each $i \in \mathbb{N}$, we have:

$$f_i(Q) \times Y \subset \bigcup_{g \in G} A(g) U.$$

It follows easily from the fact that the $(f_i)_{i \in \mathbb{N}}$ are dense in $C^0(Q, X)$, that for each A in $\mathcal{M}$ and each open non-void subset U of $X \times Y$, the complement in $X \times Y$ of $\bigcup_{g \in G} A(g) U$ is a Z-set in $X \times Y$. In particular if $A \in \mathcal{M}$, and $(U_i)_{i \in \mathbb{N}}$ is a basis of open non-void subsets of $X \times Y$, we obtain by theorem 5.2 that $\bigcap_{i \in \mathbb{N}} (\bigcup_{g \in G} A(g) U_i)$ is homeomorphic to $X \times Y$. The action A restricts to a minimal action on $\bigcap_{i \in \mathbb{N}} (\bigcup_{g \in G} A(g) U_i)$, since this set is precisely the set of points in $X \times Y$ whose orbit under A is dense in $X \times Y$. □

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