A CHARACTERIZATION OF A CLASS OF BARRELLED SEQUENCE SPACES

by J. SWETITS

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1. Introduction. In a recent paper [**4**] Bennett and Kalton characterized dense, barrelled subspaces of an arbitrary *FK* space, E. In this note, it is shown that if E is assumed to be an *AK* space, then the characterization assumes a simpler and more explicit form.

2. Definition and preliminaries. ω denotes the vector space of sequences of complex numbers. A subspace E of ω is a *K space* if it is endowed with a locally convex topology τ such that the linear functionals

$$x \to x_j \qquad (j=0,1,2,\ldots.)$$

are continuous. In addition, if τ is complete and metrizable, then (E, τ) is an *FK* space.

If $x=\{x_k\}$, let $P_n x=\{x_0, x_1, \ldots, x_n, 0, \ldots\}$. If a K space (E, τ) has the property that $P_n x \to x$ in τ for each $x \in E$, then (E, τ) is called an *AK* space.

If E is an *FK–AK* space then the dual of E may be identified with

$$E^\beta = \left\{ y \in \omega : \sum_{j=0}^{\infty} x_j y_j \text{ converges } \forall x \in E \right\}.$$

If F is a subspace of E^β containing the space ϕ of sequences with only finitely many non-zero terms then E, F form a separated pair under the bilinear form

$$\langle x, y \rangle = \sum_{j=0}^{\infty} x_j y_j.$$

$\sigma(E, F)$, $\tau(E, F)$ and $\beta(E, F)$ denote the weak, Mackey and strong topologies, respectively, on E by F (see, e.g., [**7**]).

If $A=(a_{nk})$ is an infinite matrix of complex numbers, the sequence $Ax=\{(Ax)_n\}$ is defined by

$$(Ax)_n = \sum_{k=0}^{\infty} a_{nk} x_k \qquad (n=0,1,2,\ldots).$$

$E_A=\{x : Ax \in E\}$, where E is a given sequence space. A' denotes the transpose of A.

The following theorem is established in [**8**].

THEOREM 2.1. *Let E and F be sequence spaces, each containing ϕ, such that $(E^\beta, \sigma(E^\beta, E))$ and $(F, \sigma(F, F^\beta))$ are sequentially complete. If $A=(a_{nk})$ is an infinite*

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matrix, then the following are equivalent:

(i) F_A *contains* E;

(ii) $E^{\beta}_{A'}$ *contains* F^{β};

(iii) $F_{A'}$ *contains* $(E^{\beta})^{\beta}$.

Proof. (i)⇒(ii). Let $\{t_k\} \in F^{\beta}$ and $\{x_k\} \in E$. Define the matrix $B = (b_{nk})$ by

$$b_{nk} = \begin{cases} t_k & (0 \le k \le n), \\ 0 & (k > n). \end{cases}$$

Then

$$\sum_{n=0}^{\infty} t_n \sum_{k=0}^{\infty} a_{nk} x_k = \lim_{j\to\infty} \sum_{n=0}^{j} t_n \sum_{k=0}^{\infty} a_{nk} x_k$$

$$= \lim_{j\to\infty} \sum_{k=0}^{\infty} x_k \sum_{n=0}^{j} t_n a_{nk}$$

$$= \lim_{j\to\infty} [(BA)x]j.$$

The hypotheses on E insure that

$$\lim_{j\to\infty} [(BA)x]_j = \sum_{k=0}^{\infty} x_k \lim_{j\to\infty} [(BA)e^k]_j$$

$$= \sum_{k=0}^{\infty} x_k \sum_{n=0}^{\infty} t_n a_{nk},$$

where e^k denotes the sequence with a one in the kth coordinate and zeros elsewhere.

Since $\{t_k\} \in F^{\beta}$, $\{x_k\} \in E$ are arbitrary, it follows that A' maps F^{β} to E^{β}.

(ii)⇒(iii) follows from (i)⇒(ii) and the fact that $F = (F^{\beta})^{\beta}$ if $(F, \sigma(F, F^{\beta}))$ is sequentially complete [**10**, p. 974].

(iii)⇒(i) is trivial.

3. A class of barrelled spaces.

THEOREM 3.1. *Let* E *be an FK–AK space and* E_0 *a subspace of* E *containing* ϕ. E_0 *is barrelled in* E *if and only if*

(i) $E_0^{\beta} = E^{\beta}$, *and*

(ii) $(E^{\beta}, \sigma(E^{\beta}, E_0))$ *is sequentially complete.*

Proof. (*Necessity*) Let $\{t_k\} \in E_0^{\beta}$, and define $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} t_k & (0 \le k \le n), \\ 0 & (k > n). \end{cases}$$

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If c denotes the space of convergent sequences, then c_A includes E_0. Since c_A is an FK space [**9**, ch. 12], it follows from [**4**, Theorem 1] that c_A includes E. Thus, for any $x \in E$, $\sum_{k=0}^{\infty} t_k x_k$ converges. Consequently E^β includes E_0^β. Since the reverse inclusion is satisfied, we have $E_0^\beta = E^\beta$.

Let $\{a^{(n)}\}$ be a sequence in E^β that is $\sigma(E^\beta, E_0)$ Cauchy. If $A = (a_{nk})$ is defined by $a_{nk} = a_k^{(n)}$, then c_A includes E_0. Consequently, c_A includes E, [**4**, Theorem 1]. Condition (ii) now follows from the the fact that E^β is $\sigma(E^\beta, E)$ sequentially complete.

(*Sufficiency*). Let $\{a^{(n)}\}$ be a sequence in E^β that is $\sigma(E^\beta, E_0)$ bounded. Let m denote the space of bounded sequences, and define $A = (a_{nk})$ by $a_{nk} = a_k^{(n)}$. Then m_A includes E_0. Conditions (i) and (ii) and Theorem 2.1 imply that m_A includes E since $(m, \sigma(m, \ell))$ is sequentially complete (ℓ = space of absolutely convergent series). Thus, $\sigma(E^\beta, E_0)$ and $\sigma(E^\beta, E)$ define the same bounded sequences and, hence, the same bounded sets. Thus, the topology $\beta(E_0, E^\beta)$ is the restriction of $\beta(E, E^\beta) = \tau(E, E^\beta) = FK$ topology of E to E_0. It follows that E_0 is barrelled in E.

Remarks. If E_0 is monotone (i.e., the coordinatewise product $xy \in E_0$ if $x \in E_0$ and y is a sequence of zeros and ones) then condition (ii) of Theorem 3.1 can be omitted [**3**, p. 55].

Let $\{r_n\}$ denote a non-decreasing unbounded sequence of positive integers with $r_0 = 1$ and $r_n = o(n)$. For each $x \in \omega$ and each $n = 0, 1, 2, \ldots$, let $c_n(x)$ denote the number of non-zero elements in $\{x_0, x_1, \ldots, x_n\}$. If E is a sequence space, a scarce copy of E is the linear span of

$$\{x \in E : c_n(x) \le r_n, n = 0, 1, 2, \ldots\}.$$

As corollaries to Theorem 3.1, we obtain Theorems 7, 8 and 10 of [**2**]. In each case the spaces are monotone and the verification of condition (i) of Theorem 3.1 is straightforward.

ω has the topology of coordinatewise convergence, and, for $p > 0$, $\ell^p = \left\{x : \sum_{j=0}^{\infty} |x_j|^p < \infty\right\}$.

Corollary 3.2. *Every scarce copy of ω is barrelled.*

Corollary 3.3. *Every scarce copy of $\bigcap_{p>0} \ell^p$ is barrelled as a subspace of ℓ.*

Corollary 3.4. *Let E be a monotone FK–AK space. The union of all the scarce copies of E is a barrelled subspace of E.*

It is noted that Corollary 3,4 strengthens Theorem 10 of [**2**], which is stated for solid spaces.

Another consequence of Theorem 3.1 is the following result.

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Corollary 3.5. *Let E be an FK–AK space and E_0 a subspace of E containing ϕ. The following are equivalent:*

(i) *E_0 is barrelled;*

(ii) *If G is a separable FK space containing E_0, then G contains E.*

Proof. (i) $\Rightarrow$ (ii). This is a consequence of [**4**, Theorem 1]. (ii) $\Rightarrow$ (i). Let $\{t_k\} \in E_0^\beta$, and define $A = (a_{nk})$ by

$$a_{nk} = \begin{cases} t_k & (0 \le k \le n), \\ 0 & (k > n). \end{cases}$$

Then c_A includes E_0. Since c_A is a separable FK space [**1**, p. 199], c_A includes E. Thus, $\{t_k\} \in E^\beta$, and condition (i) of Theorem 3.1 is satisfied.

Let $\{a^{(n)}\}$ be a sequence in E^β that is $\sigma(E^\beta, E_0)$ Cauchy. If $A = (a_{nk})$ is the matrix defined by $a_{nk} = a_k^{(n)}$, then c_A includes E_0. It follows that c_A includes E. Since E^β is $\sigma(E^\beta, E)$ sequentially complete, condition (ii) of Theorem 3.1 is satisfied. Thus, E_0 is barrelled.

Remark. For FK–AK spaces, (ii) $\Rightarrow$ (i) of Corollary 3.5 improves (ii) $\Rightarrow$ (i) of [**4**, Theorem 1].

In Theorem 3.1, if it is not assumed that E is an AK space, then (i) and (ii) are not sufficient to insure that E_0 is barrelled in E.

Let E be ac_0, the space of sequences that are almost convergent to 0, (see [**6**]). For $x \in ac_0$, let

$$\|x\| = \sup_n |x_n|.$$

Let $E_0 = bs + c_0$, where

$$c_0 = \left\{ x \in \omega : \lim_{n \to \infty} x_n = 0 \right\},$$

$$bs = \left\{ x \in \omega : \sup_n \left| \sum_{j=0}^{n} x_j \right| < \infty \right\}.$$

Then $E_0^\beta = E^\beta = \ell$, and E_0 is dense in E [**5**, p. 29]. Furthermore, ℓ is $\sigma(\ell, E_0)$ sequentially complete. However, E_0 is a normed FK space when topologized by

$$\|x\| = \inf\left\{ \sup_n |y_n| + \sup_n \left| \sum_{j=0}^{n} z_j \right| : x = y + z,\ y \in c_0,\ z \in bs \right\}.$$

It follows from [**4**, Theorem 1] that E_0 is not barrelled in E.

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Department of Mathematical and Computing Sciences
Old Dominion University
Norfolk, Virginia 23508