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SHRINKING TARGETS FOR NONAUTONOMOUS DYNAMICAL SYSTEMS CORRESPONDING TO CANTOR SERIES EXPANSIONS

LIOR FISHMAN, BILL MANCE, DAVID SIMMONS✉ and MARIUSZ URBAŃSKI

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Abstract

We provide a closed formula of Bowen type for the Hausdorff dimension of a very general shrinking target scheme generated by the nonautonomous dynamical system on the interval $[0, 1)$, viewed as $\mathbb{R}/\mathbb{Z}$, corresponding to a given method of Cantor series expansion. We also examine a wide class of examples utilising our theorem. In particular, we give a Diophantine approximation interpretation of our scheme.

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1. Introduction

Recall that in the framework of autonomous dynamical systems, the evolution process of the system is determined by continuously iterating a fixed map. In contrast, the system is said to be nonautonomous if at each stage of the iteration process we allow the action of a possibly different map. In this paper, we will be concerned with a shrinking target problem in the context of the nonautonomous dynamical system generated by a sequence $Q = (q_n) \in \mathbb{N}_{\geq 2}^{\mathbb{N}}$. Given such a sequence, the maps $T_{Q,n} : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and $T_Q^n : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ are defined for any $n \in \mathbb{N}$ as follows:

$$\begin{aligned} T_{Q,n}(x) &= q_n x \pmod 1, \\ T_Q^n(x) &= T_{Q,n} \circ \cdots \circ T_{Q,1}(x) = q_n \cdot q_{n-1} \cdots q_1 x \pmod 1. \end{aligned} \tag{1.1}$$

This system was first introduced and investigated in an implicit manner by Cantor in [6], where he considered what is now called the *Q-Cantor series expansion* of a real

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number x, that is, the (unique) expansion of the form

$$x = \omega_0 + \sum_{j=1}^{\infty} \frac{\omega_j}{q_1 q_2 \cdots q_j},$$

where $\omega_0 = \lfloor x \rfloor$, ω_j is in $\{0, 1, \ldots, q_j - 1\}$ for $j \geq 1$ and $\omega_j \neq q_j - 1$ for infinitely many j. The relation between this definition and the nonautonomous dynamical system (1.1) is that for every $n \geq 1$, $\omega_n = \lfloor q_n T_Q^{n-1}(x) \rfloor$, where $T_Q^{n-1}(x)$ denotes the representative of $T_Q^{n-1}(x)$ in $[0, 1)$.

The study of normal numbers and various statistical properties of real numbers with respect to large classes of Cantor series expansions dates back to Erdős and Rényi [7, 8] and was continued by Rényi [16–18] and Turán [23]. Later, certain aspects of this area were extensively studied by many authors, notably by Galambos [10], Šalát (see for example [19]) and Schweiger [20]. Most recently, the second author and his collaborators continued to develop this area of research in [1, 2, 14], where the primary results concern various concepts of normality, relations between them and the Hausdorff dimensions of appropriate significant sets.

In this paper, we follow a different approach. We are prompted by the (relatively) recent activity focused on determining the Hausdorff dimension of the set of points 'hitting' some shrinking target infinitely often. As with research regarding Cantor expansions, one may trace this approach to the pioneering work of Besicovič [3] and Jarník [13] with respect to continued fraction expansions. There are also connections to conformal dynamics, Kleinian groups and conformal iterated function systems; we list here a (by no means exhaustive) selection for the reader's convenience: [4, 5, 11, 12, 15, 21, 22, 24]. We emphasise that unlike all the papers mentioned above, we work in the context of a nonautonomous dynamical system, namely the one defined in (1.1).

The shrinking target scheme considered in the present paper is quite general, at least in the context of sequences. Let $\alpha = (\alpha_i)_{i=1}^{\infty}$ be a sequence of nonnegative real numbers and, for each $n \geq 1$, let

$$\alpha(n) := \sum_{i=1}^{n} \alpha_i.$$

Let

$$\mathcal{D}_\infty^Q(\alpha) = \{x \in X := \mathbb{R}/\mathbb{Z} : \|T_Q^n(x)\| \leq e^{-\alpha(n)} \text{ for infinitely many } n\},$$

where $\|\cdot\|$ denotes distance to the nearest integer. We would like to bring the reader's attention to the fact that the set $\mathcal{D}_\infty^Q(\alpha)$ has a precise Diophantine approximation interpretation. A general scheme of Diophantine analysis has been laid down by three of the authors in [9]. In the setting considered here, for every integer $n \geq 1$, denote by Q_n the set of all Q-adic rationals of order n in X, that is, the set of all numbers of the form

$$\sum_{j=1}^{n} \frac{\omega_j}{q_1 \cdots q_j},$$

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where $\omega_j \in \{0, 1, \ldots, q_j - 1\}$. For every Q-adic rational number $w \in \mathcal{Q} := \bigcup_{n\geq 1} \mathcal{Q}_n$, let $H_Q(w)$ denote the least integer $n \geq 1$ such that $w \in \mathcal{Q}_n$. We can interpret $H_Q(w)$ as the *height* (induced by the sequence Q) of the number w. The triple $(X, \mathcal{Q}, H_Q)$ then forms a *Diophantine space*, that is, a complete metric space, a dense set and a function measuring the 'complexity' of elements of this dense set. Define the (approximation) function $\psi : \mathbb{N} \to (0, \infty)$ as follows:

$$\psi(n) = \frac{\exp(-\alpha(n))}{q_1 \cdots q_n}.$$

In Diophantine approximation terminology, a point $x \in X$ is called *ψ-approximable* (relative to the Diophantine space $(X, \mathcal{Q}, H_Q)$) if there exists a sequence $(w_k)_1^\infty$ in $\mathcal{Q}$ converging to x with the property that $|x - w_k| \leq \psi(H_Q(w_k))$ for all $k \geq 1$. We observe then that $\mathcal{D}_\infty^Q(\alpha)$ is precisely the set of ψ-approximable numbers, a set which is a standard object of study in Diophantine analysis.

Using the definitions above and prompted by thermodynamic formalism, given $s \geq 0$, we define the *(upper) pressure function*

$$P_{Q,\alpha}(s) := \limsup_{n\to\infty} \frac{1}{n}[(1 - s)\log(q_1 \cdots q_n) - s\alpha(n)]. \tag{1.2}$$

Note that since $q_i \geq 2$ and $\alpha_i \geq 0$, the function $s \mapsto P_{Q,\alpha}(s)$ is strictly decreasing in its domain of finiteness. We further define

$$b_Q(\alpha) := \sup\{t \geq 0 : P_{Q,\alpha}(t) > 0\} = \inf\{t \geq 0 : P_{Q,\alpha}(t) < 0\}.$$

THEOREM 1.1. *For any sequence $\alpha = (\alpha_i)_{i=1}^\infty$ of nonnegative real numbers,*

$$\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) = b_Q(\alpha) = \limsup_{n\to\infty} \frac{\log(q_1 \cdots q_n)}{\log(q_1 \cdots q_n) + \alpha(n)}.$$

Here and in what follows, HD denotes Hausdorff dimension.

The general form of this theorem does not differ much from the ones obtained in the papers mentioned above. What is perhaps surprising is that it holds in such generality in the realm of a nonautonomous system. We note that it also covers the autonomous case of q-ary sequences, where $q \geq 2$ is an integer; simply consider the constant sequence $Q = (q)_1^\infty$, every term of which is equal to q. It also captures the cases of periodic and eventually periodic sequences Q, that is, the ones that can also be approached with the methods of autonomous dynamical systems.

We prove Theorem 1.1 in the next section and in Section 3 we describe a number of classes of examples which illustrate its content and scope.

2. Proof of Theorem 1.1

Fix an arbitrary $t > b_Q(\alpha)$, so that $P_{Q,\alpha}(t) < 0$. For each $n \geq 1$, let $Q_n = q_1 \ldots q_n$ and, for each $j = 0, \ldots, Q_n - 1$, let

$$\Delta_{n,j}(\alpha) := \left\{x \in X : \left\|x - \frac{j}{Q_n}\right\| \leq \frac{e^{-\alpha(n)}}{Q_n}\right\},$$

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so that

$$\mathcal{D}_\infty^Q(\alpha) = \limsup_{n\to\infty} \bigcup_{j=0}^{Q_n-1} \Delta_{n,j}(\alpha) = \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{j=0}^{Q_n-1} \Delta_{n,j}(\alpha).$$

We have

$$|\Delta_{n,j}(\alpha)| = 2(Q_n)^{-1} e^{-\alpha(n)}$$

and thus, for all sufficiently large $n \geq 1$,

$$\sum_{j=0}^{Q_n-1} |\Delta_{n,j}(\alpha)|^t = Q_n(2(Q_n)^{-1} e^{-\alpha(n)})^t = 2^t Q_n^{1-t} e^{-t\alpha(n)} \leq 2^t \exp\Big(\frac{1}{2} P_{Q,\alpha}(t) \cdot n\Big).$$

Thus, for all sufficiently large $N \geq 1$,

$$\sum_{n=N}^{\infty} \sum_{j=0}^{Q_n-1} |\Delta_{n,j}(\alpha)|^t \leq 2^t \sum_{n=N}^{\infty} \exp\Big(\frac{1}{2} P_{Q,\alpha}(t) \cdot n\Big)$$

and so

$$H^t(\mathcal{D}_\infty^Q(\alpha)) \leq 2^t \lim_{N\to\infty} \sum_{n=N}^{\infty} \exp\Big(\frac{1}{2} P_{Q,\alpha}(t) \cdot n\Big) = 0,$$

where H^t denotes Hausdorff t-dimensional measure. Hence, $\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) \leq t$ and, since $t > b_Q(\alpha)$ was arbitrary, $\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) \leq b_Q(\alpha)$.

We now prove that $\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) \geq b_Q(\alpha)$. Fix $0 \leq s < b_Q(\alpha)$ and fix a sufficiently fast growing sequence $(n_i)_{i=1}^\infty$, to be determined later in the proof, along which the lim sup in (1.2) is achieved, that is, for which

$$\lim_{l\to\infty} \frac{1}{n_l}[(1-s)\log(Q_{n_l}) - s\cdot\alpha(n_l)] = P_{Q,\alpha}(s) > 0. \tag{2.3}$$

For each $l \geq 1$, let

$$S_l = \{\Delta_{n_l,j}(\alpha) : j = 0, \ldots, Q_{n_l} - 1\}.$$

We assume that n_1 is chosen large enough so that $\alpha(n_1) > \log(2)$ (ignoring the case $\alpha(n) \leq \log(2)$ for all n as trivial), so that for each $l \geq 1$, S_l is a disjoint collection. Now we construct inductively a sequence of sets $(R_l)_{l=1}^\infty$ as follows. We start by letting $R_1 = S_1$. For the inductive step, suppose that the set $R_l \subseteq S_l$ has been defined. Then we let R_{l+1} be the set consisting of all elements $\Delta \in S_{l+1}$ contained in some interval from the family R_l. We define the following nonempty compact set:

$$K := \bigcap_{l=1}^{\infty} \bigcup (S_l) = \bigcap_{l=1}^{\infty} \bigcup_{\Delta\in S_l} \Delta.$$

For every $\Delta \in S_l$, let

$$R_{l+1}(\Delta) := \{\Gamma \in R_{l+1} : \Gamma \subseteq \Delta\}.$$

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Then

$$R_{l+1} = \bigcup_{\Delta \in S_l} R_{l+1}(\Delta)$$

and, for all $\Delta \in S_l$,

$$\begin{aligned}\#(R_{l+1}(\Delta)) &\geq \frac{|\Delta|}{(Q_{n_{l+1}})^{-1}} - 2 = \frac{(Q_{n_l})^{-1}e^{-\alpha(n_l)}}{(Q_{n_l}q_{n_l+1}\cdots q_{n_{l+1}})^{-1}} - 2 \\ &= q_{n_l+1}\cdots q_{n_{l+1}}e^{-\alpha(n_l)} - 2 \geq \frac{1}{2}q_{n_l+1}\cdots q_{n_{l+1}}e^{-\alpha(n_l)}, \end{aligned} \tag{2.4}$$

where the last inequality holds provided that $n_{l+1} \geq 1$ is large enough. We shall now recursively define maps $m_l : R_l \to [0, 1]$, $l \geq 1$, as follows. Let $m_1(\Delta) = 1/\#(R_1)$ for all $\Delta \in R_1$. Proceeding inductively, fix $\Delta \in R_l$ and, for every $\Gamma \in R_{l+1}$, set

$$m_{l+1}(\Gamma) := \frac{m_l(\Delta)}{\#(R_{l+1}(\Delta))}.$$

Then it is easy to see (for example, by choosing arbitrary measures extending the functions m_l and then taking a weak limit) that there exists a Borel probability measure m supported on K such that $m(\Delta) = m_l(\Delta)$ for all $l \geq 1$ and $\Delta \in S_l$. Equation (2.4) shows that

$$m_{l+1}(\Gamma) \leq 2m_l(\Delta)e^{\alpha(n_l)} \cdot (q_{n_l+1}\cdots q_{n_{l+1}})^{-1}$$

and iterating this estimate gives

$$m(\Delta) \leq 2^{l-1}\exp(\alpha(n_1) + \alpha(n_2) + \cdots + \alpha(n_{l-1})) \cdot (Q_{n_l})^{-1}. \tag{2.5}$$

Let $x \in K$ be arbitrary. We want to show that there exists some $C > 0$ independent of x such that for every $r > 0$

$$m(B(x, r)) \leq C \cdot r^s. \tag{2.6}$$

Since m is a probability measure, it is of course enough to show this for all small enough $r > 0$. Fix $r \in (0, e^{-\alpha(n_1)}(Q_{n_1})^{-1})$ and then let l be the largest integer such that

$$e^{-\alpha(n_l)} \cdot (Q_{n_l})^{-1} \geq r. \tag{2.7}$$

By our choice of r, we have that $l \geq 1$. Now, cover $B(x, r)$ by a union of intervals of the form $[(j - 1/2)/Q_n, (j + 1/2)/Q_n]$, $j = 0, \ldots, Q_n - 1$. We can do it by taking no more than

$$\frac{r}{(Q_{n_{l+1}})^{-1}} + 2 \leq 2rQ_{n_{l+1}}$$

such intervals. But then we can also cover $K \cap B(x, r)$ by at most $2rQ_{n_{l+1}}$ intervals of R_{l+1}. By invoking (2.5) and the fact that the measure m is supported on K,

$$\begin{aligned} m(B(x, r)) &\leq 2rQ_{n_{l+1}}2^l\exp(\alpha(n_1) + \alpha(n_2) + \cdots + \alpha(n_l)) \cdot Q_{n_{l+1}}^{-1} \\ &= 2^{l+1}r\exp(\alpha(n_1) + \alpha(n_2) + \cdots + \alpha(n_l)). \end{aligned}$$

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Hence, in order to show that (2.6) holds, it is enough to check that

$$2^{l+1}\exp(\alpha(n_1)+\alpha(n_2)+\cdots+\alpha(n_l))\le C\cdot r^{s-1}.$$

Because of (2.7), and since $s<b_Q(\alpha)\le 1$, it is enough to show that

$$2^{l+1}\exp(\alpha(n_1)+\alpha(n_2)+\cdots+\alpha(n_l))\le C(Q_{n_l})^{1-s}\exp((1-s)\alpha(n_l)).$$

Equivalently,

$$(l+1)\log 2+\alpha(n_2)+\cdots+\alpha(n_{l-1})\le \log C+(1-s)\log(Q_{n_l})-s\cdot\alpha(n_l).$$

But, because of our choice of s, we have $P_{Q,\alpha}(s)>0$, so, by (2.3),

$$(1-s)\log(Q_{n_l})-s\cdot\alpha(n_l)\ge \tfrac{1}{2}P_{Q,\alpha}(s)\cdot n_l$$

for all large enough l. Thus, it suffices to verify that

$$\tfrac{1}{2}P_{Q,\alpha}(s)\cdot n_l\ge (l+1)\log 2+\alpha(n_1)+\alpha(n_2)+\cdots+\alpha(n_{l-1})-\log C.$$

This can be done by defining n_l inductively to be large enough depending on $n_1,\ldots,n_{l-1}$ and on l. Note that this choice does not conflict with the requirement (2.3). This completes the proof that $\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha))=b_Q(\alpha)$.

To finish the proof, we show that $b_Q(\alpha)=\delta:=\limsup_{n\to\infty}\log(Q_n)/(\log(Q_n)+\alpha(n))$. If $s>t>\delta$, then, for all n sufficiently large,

$$\frac{\log(Q_n)}{\log(Q_n)+\alpha(n)}\le t,\quad \alpha(n)\ge\frac{1-t}{t}\log(Q_n)$$

and thus

$$\begin{aligned}P_{Q,\alpha}(s)&\le\limsup_{n\to\infty}\frac{1}{n}\Big[(1-s)\log(Q_n)-s\frac{1-t}{t}\log(Q_n)\Big]\\&=s\Big(\frac{1-s}{s}-\frac{1-t}{t}\Big)\liminf_{n\to\infty}\frac{\log(Q_n)}{n}\le s\Big(\frac{1-s}{s}-\frac{1-t}{t}\Big)\log 2<0\end{aligned}$$

and so $s>b_Q(\alpha)$. Conversely, if $s<t<\delta$, then, for infinitely many n,

$$\frac{\log(Q_n)}{\log(Q_n)+\alpha(n)}\ge t,\quad \alpha(n)\le\frac{1-t}{t}\log(Q_n)$$

and thus

$$\begin{aligned}P_{Q,\alpha}(s)&\ge\limsup_{n\to\infty}\frac{1}{n}\Big[(1-s)\log(Q_n)-s\frac{1-t}{t}\log(Q_n)\Big]\\&=s\Big(\frac{1-s}{s}-\frac{1-t}{t}\Big)\limsup_{n\to\infty}\frac{\log(Q_n)}{n}\ge s\Big(\frac{1-s}{s}-\frac{1-t}{t}\Big)\log 2>0\end{aligned}$$

and so $s<b_Q(\alpha)$.

3. Examples

In this section we consider a few classes of examples. Having proved Theorem 1.1, our task reduces to examining sequences $(q_n)_{n=1}^{\infty}$ with appropriate arithmetical properties. Our examples show how to get a very explicit closed formula for the Hausdorff dimension of $\mathcal{D}_{\infty}^{Q}(\alpha)$ for many classes of sequences $(q_n)_{n=1}^{\infty}$.

Our first example follows directly from Theorem 1.1.

Example 3.1. Let $G(a_1, \ldots, a_n)$ denote the geometric mean of the positive real numbers $a_1, \ldots, a_n$. Suppose that Q is eventually periodic. That is, we can write Q in the form

$$(d_1, d_2, \ldots, d_k, \overline{p_1, p_2, \ldots, p_m}).$$

Let $\alpha_n = c > 0$ for all n. Then $\alpha(n) = cn$ for all n.

A short calculation shows that

$$\mathrm{HD}(\mathcal{D}_{\infty}^{Q}(\alpha)) = \frac{\log G(p_1, \ldots, p_m)}{\log G(p_1, \ldots, p_m) + c}.$$

In particular, if $Q = (2, 3, 2, 3, \ldots)$, then $\mathrm{HD}(\mathcal{D}_{\infty}^{Q}(\alpha)) = \log \sqrt{6}/(\log \sqrt{6} + c)$. As mentioned in the introduction, one could obtain this result by the methods of autonomous dynamics. This is particularly transparent in the case when $q_n = b$ for all n.

We observe that if

$$L = \liminf_{n\to\infty} \frac{\alpha(n)}{\log(Q_n)},$$

then Theorem 1.1 says that

$$\mathrm{HD}(\mathcal{D}_{\infty}^{Q}(\alpha)) = \frac{1}{1+L}, \tag{3.8}$$

with the convention that the right-hand side is 0 if $L = \infty$. On the other hand, we have the following simple observation.

Observation 3.2. Let $L \in \mathbb{R} \cup \{\pm\infty\}$ and let $(a_n)_{n=1}^{\infty}$ and $(b_n)_{n=1}^{\infty}$ be two sequences of positive real numbers such that

$$\sum_{n=1}^{\infty} b_n = \infty \quad \text{and} \quad \lim_{n\to\infty} \frac{a_n}{b_n} = L.$$

Then

$$\lim_{n\to\infty} \frac{a_1 + a_2 + \cdots + a_n}{b_1 + b_2 + \cdots + b_n} = L.$$

Combining Observation 3.2 with (3.8) yields the following result.

Corollary 3.3. *Suppose that the limit*

$$L := \lim_{n\to\infty} \frac{\alpha_n}{\log(q_n)}$$

exists in $[0, \infty]$. *Then*

$$\mathrm{HD}(\mathcal{D}_{\infty}^{Q}(\alpha)) = \frac{1}{1+L},$$

with the convention that the right-hand side is 0 *if* $L = \infty$.

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The next two examples follow directly from Corollary 3.3.

EXAMPLE 3.4. Suppose that Q is a sequence such that $\lim_{n\to\infty} q_n/n^k \in (0,\infty)$ for $k > 0$ and $\alpha_n = c\log n$ for $c > 0$. Then

$$\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) = \frac{k}{k+c}.$$

For example, if $q_n = n+1$, then $\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) = 1/(1+c)$. As another example, if

$$q_n = 2 + \left\lfloor \frac{\sqrt{n+\sqrt{n}\cos n}}{\sqrt[3]{n}} \right\rfloor,$$

then $\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) = (1/6)/(1/6+c)$.

EXAMPLE 3.5. Suppose that Q is a sequence such that $\lim_{n\to\infty} q_n/b^n \in (0,\infty)$ for $b > 1$ and $\alpha_n = cn$ for $c > 0$. Then

$$\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) = \frac{\log b}{\log b + c}.$$

For example, if $q_n = 2^n$, then $\mathrm{HD}(\mathcal{D}_\infty^Q(\alpha)) = \log 2/(\log 2 + c)$.

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LIOR FISHMAN, University of North Texas,
Department of Mathematics, 1155 Union Circle #311430,
Denton, TX 76203-5017, USA
e-mail: lior.fishman@unt.edu

BILL MANCE, University of North Texas,
Department of Mathematics, 1155 Union Circle #311430,
Denton, TX 76203-5017, USA
e-mail: mance@unt.edu

DAVID SIMMONS, Ohio State University,
Department of Mathematics, 231 W. 18th Avenue,
Columbus, OH 43210-1174, USA
e-mail: simmons.465@osu.edu

MARIUSZ URBAŃSKI, University of North Texas,
Department of Mathematics, 1155 Union Circle #311430,
Denton, TX 76203-5017, USA
e-mail: urbanski@unt.edu