PROBLEMS ON MEASURE ALGEBRAS

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Suppose that G is a locally compact abelian group, u an element of infinite order, and w a complex number of modulus 1. It is a familiar fact that there is a complex homomorphism Ψ of the measure algebra M of G, which maps ϵ_u (the unit mass concentrated at u) to w. Beyond this, one may specify an element μ of M, and require a homomorphism Ψ which does not annihilate μ. The resolution of this problem leads to an abstract lemma on measurable transformations, derived in some generality in the first section. Next, following the examples of Hewitt and Kakutani **(1)**, we construct a family of perfect compact subsets P of real numbers, such that for each measure μ supported by P, we can have $|\Psi(\mu)| = \lim_{n\to\infty} ||\mu^n||^{1/n}$. This is based on a probabilistic construction, which allows us to derive some miscellaneous facts about measures; for example, P supports measures μ with $\hat{\mu}(n) = o(1)$. The paper of Salem **(4)** seems to be the ultimate source for this method. We prove also that the closed ideal in M generated by $\epsilon_u - we$ ($e = \epsilon_0$, the identity of M) is not the intersection of maximal ideals when G is the circle group; the last section gives another example.

1. A lemma on measurable transformations. Let S be a set, $\mathfrak{S}$ a σ-field of subsets of S, and T a measurable transformation of S into itself: $T^{-1}E \in \mathfrak{S}$ for each E in $\mathfrak{S}$. For a complex number w of modulus 1, a measurable function h on S is (T, w)*-invariant* if $h(Ts) = wh(s)$, $s \in S$. We write C.A. for the complex Banach space of finite countably additive measures in $(S, \mathfrak{S})$ and define a linear contraction U in C.A.:

$$(U\mu)(E) = \mu(T^{-1}E), \qquad E \in \mathfrak{S},\ \mu \in \text{C.A.}$$

Set, for a measure μ fixed throughout this section,

$$d = \inf ||U\lambda - w\lambda - \mu||, \qquad \lambda \in \text{C.A.}$$

LEMMA 1. $d = \sup |\int g(s)\mu(ds)|$, $|g(s)| \leqslant 1$, $g(T, w)$*-invariant.*

Proof. For a bounded measurable function h, and $\lambda \in$ C.A.,

$$\int h(s)(U\mu)(ds) = \int h(Ts)\mu(ds).$$

For a function g as in the lemma, this shows that g is orthogonal to $U\lambda - w\lambda$, proving the easy half of the statement. Define now a positive measure λ on S by the formula

$$\lambda(E) = \sum_0^\infty \frac{|\mu|(T^{-n}E)}{(n+1)^2}, \qquad E \in \mathfrak{S};$$

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then $\mu \ll \lambda$ and $\lambda(E) = 0$ implies that $\lambda(T^{-1}E) = 0$. Let $Y = L^1(S, \mathfrak{S}, \lambda)$, a subspace left invariant by U. By the definition of d there is a linear functional x^* of norm $\leqslant 1$ on Y such that $x^*(\mu) = d$, $x^*(U\sigma) \equiv wx^*(\sigma)$. This means, of course, that there is a measurable function h on S such that $|h| \leqslant 1$ and

$$\int h(s)\mu(ds) = d, \qquad \int h(s)(U\sigma)(ds) \equiv w\int h(s)\sigma(ds).$$

From the identity at the beginning of the proof it is clear that $h(s) = wh(Ts)$ except on a set of λ-measure 0. Now put

$$h_*(s) = \lim_{N\to\infty} \frac{1}{N} \sum_1^N w^{-t}h(T^t s)$$

whenever the limit exists, and $h^*(s) = 0$ otherwise. Then $h^* = h$ λ-almost everywhere and h^* is (T, w)-invariant on all of S, as required.

2. Applications to measures in groups. Measurability in the group G is in reference to the σ-field determined by all continuous functions on G, which we call the Baire σ-field. We let J be the closed principal ideal of M generated by $\epsilon_u - we$. To apply the lemma on measurable transformations, we note that if E is a Baire set and $\lambda \in M$, then $[(\epsilon_u - w)^*\lambda](E) = \lambda(E - u) - w\lambda(E)$. Taking $T(g) = g + u$, $g \in G$, we conclude that the distance of a measure μ from G is d, described in the lemma.

THEOREM 1. *These statements are equivalent for $c > 0$:*

(i) *There is a complex homomorphism Ψ of M for which $\Psi(J) = 0$, $|\Psi(\mu)| \geqslant c$.*

(ii) *For every number d in $(0, c)$ there is a bounded Baire function h such that $h(g + u) = wh(g)$, $g \in G$ and*

$$\limsup_{n\to\infty} \left| \int h(x)\mu^n(dx) \right|^{1/n} \geqslant d.$$

Proof. If (ii) holds for some $d > 0$ and some h, it holds also for $h/||h||_\infty$, so that by Lemma 1,

$$\limsup_{n\to\infty} ||\mu^n + J||^{1/n} \geqslant d.$$

The implication (ii) $\to$ (i) follows now from the formula of Beurling and Gelfand (**3**, I.6, III.1), applied to the Banach algebra M/J (**3**, pp. 43–44). To prove the converse, let Z be the Banach space of Baire functions h as described in the theorem. Suppose that for some $d > 0$ and each $h \in Z$

$$\limsup_{n\to\infty} \left| \int h(x)\mu^n(dx) \right|^{1/n} \leqslant d,$$

or equivalently

$$\lim_{n\to\infty} r^{-n} \left| \int h(x)\mu^n(dx) \right| = 0, \qquad \text{whenever } r > d.$$

By the principle of uniform boundedness, for each $r > d$ there exists a constant $B(r)$ such that

$$\left| \int h(x)\mu^n(dx) \right| \leqslant r^n B(r)||h||_\infty \qquad (h \in Z, 1 \leqslant n < \infty).$$

By Lemma 1, $||\mu^n + J|| \leqslant B(r)r^n$, $1 \leqslant n < \infty$. Because each complex homomorphism of M is norm-decreasing, $\Psi(J) = 0$ forces $|\Psi(\mu)| \leqslant r$, and so $|\Psi(\mu)| \leqslant d$, as required.

The sequence of lemmas below is intended mainly to elucidate the examples which conclude this article.

LEMMA 2. *If $\mu \in L^1$, $\mu \notin J$, there is a continuous character χ of G such that $\chi(u) = w$ and $\int \chi\, d\mu \neq 0$.*

Proof. Suppose first that u generates a discrete subgroup H, so there is certainly a continuous character χ of G, with $\chi(u) = w$. The automorphism $f \to \chi \cdot f$ of L^1 onto itself maps μ to an element not in the closure of the ideal $L^1(\epsilon_u - e) \subseteq L^1$. The quotient algebra of L^1 by the closure of this ideal is $L^1(G/H)$, so that for some character γ of G with $\gamma(u) = 1$, $\int \gamma\chi\, d\mu \neq 0$, as required.

In the remaining case H may be supposed compact with Haar measure m_H. By the hypothesis we know there is a bounded Baire function h such that $h(x + u) = wh(x)$, $x \in G$, and $\int h(x)\mu(dx) = 1$. From the absolute continuity of μ we conclude that the function F, defined by

$$F(y) = \int h(x + y)\mu(dx), \qquad y \in G,$$

is continuous. Since $F(nu) = w^n$ for each integer n, there is a continuous character γ on H fulfilling the condition $\gamma(u) = \bar{w}$. Writing $\nu = (\gamma \cdot m_H) * \mu$,

$$\begin{aligned}\int h(x)\nu(dx) &= \int_G \int_H h(x + y)\gamma(y) m_H(dy)\mu(dx)\\ &= \int_H \gamma(y)F(y)m_H(dy) = 1.\end{aligned}$$

Choose a character χ for which $\int \chi\, d\nu \neq 0$—it is clear that $\chi(u) = w$ and $\int \chi\, d\mu \neq 0$.

LEMMA 3. *Suppose that μ is supported in the compact set E. If for a certain integer $m \geqslant 1$*

$$nu \notin \pm E \pm \ldots \pm E \text{ ($2m$ summands)}, \qquad \textit{all } n \geqslant 1,$$

then $||\mu^m + J|| = ||\mu^m||$.

Proof. Since μ^m is supported in the m-fold sum of E with itself, the measures $\mu^m * \epsilon_{nu}$ $(-\infty < n < \infty)$ are mutually singular. Writing $\lambda = \bar{w}\epsilon_u$,

$$\left\|\frac{1}{N}\sum_1^N \lambda^k * \mu^m\right\| = ||\mu^m|| \qquad \text{for every } N \geqslant 1.$$

But $||(1/N)\sum \lambda^k * \nu||$ converges to zero for every ν in J, whence

$$||\mu^m + J|| = ||\mu^m||.$$

Remarks. It may be observed that, in the event E is not a Baire set, it is contained in a compact G_δ, E_1, which meets the requirement of the lemma.

Further, if F_1 is a compact G_δ and F_2 a compact set, $F_1 + F_2$ is a G_δ. For if

$$F_1 = \bigcap_{k=1}^{\infty} (U_k),$$

then

$$F_1 + F_2 = \bigcap_{k=1}^{\infty} (U_k + F_2),$$

where $U_1 \supseteq U_2 \supseteq \ldots$ is a sequence of open sets, and each U_k has compact closure contained in U_{k-1}.

The construction of examples for the theory presented is based on the well-known fact (Kolmogoroff *et al.*) that if $r_1, r_2, r_3, \ldots$ are positive numbers and

$$\sum_1^{\infty} r_i^2 < \infty,$$

then there is a probability measure μ on $(-\infty, \infty)$ such that

$$\hat{\mu}(u) = \int e^{iut}\mu(dt) = \prod_{i=1}^{\infty} \cos(ur_i), \qquad -\infty < u < \infty$$

(Pitt **2**, pp. 44, 97). We consider, though, sequences subject to the stronger condition

$$\sum_{i=1}^{\infty} r_i < \infty.$$

In this case μ is supported in the "symmetric set"

$$\sum (r) = \left\{ \sum_{i=1}^{\infty} \epsilon_i r_i : \epsilon_i = \pm 1, 1 \leqslant i < \infty \right\}.$$

Following Salem **(4)** we introduce the product space of sequences

$$\xi = \{\xi_1, \xi_2, \xi_3, \ldots\},$$

$0 \leqslant \xi_i \leqslant 1, 1 \leqslant i < \infty$, and let P be the product of the Lebesgue measures $d\xi_i$ on the intervals $0 \leqslant \xi_i \leqslant 1$. According to our convention we write

$$\sum (\xi \cdot r) = \left\{ \sum_{i=1}^{\infty} \epsilon_i \xi_i r_i : \epsilon_i = \pm 1, 1 \leqslant i < \infty \right\}.$$

Let us say that the multiplier sequence $(r) = \{r_1, r_2, r_3, \ldots\}$ is "m-sparse" $(m = 1, 2, 3, \ldots)$ if for each number $b \neq 0$, it is almost certain that the m-fold difference set $\pm \sum (\xi \cdot r) \pm \ldots \pm (\xi \cdot r)$ does not contain b. A technical detail is the fact that the subset of C whose measure is to be 1 is open in C.

THEOREM 2. *(r) is m-sparse if*

$$\liminf_{N \to \infty} (2m + 1)^N \sum_{N+1}^{\infty} r_i = 0.$$

Proof. An element of the difference set has the form

$$\sum_1^{\infty} \delta_i \xi_i r_i, \qquad -m \leqslant \delta_i \leqslant m, 1 \leqslant i < \infty.$$

It is enough to prove that, almost certainly, this form does not represent $b \neq 0$, provided also that one fixed δ_M has a specified value $\neq 0$; for example, we may arrange the sequences of δ's according to the first number $\neq 0$.

This restriction in effect, whenever

$$\sum_1^\infty \delta_i \xi_i r_i = b \quad \text{and} \quad N > M,$$

$$\left| \sum_1^N \delta_i \xi_i r_i - b \right| \leqslant (2m+1) \sum_{N+1}^\infty r_i.$$

There are at most $(2m+1)^N$ N-tuples $(\delta_1, \ldots, \delta_N)$ which contribute to the probability; for each choice, and any Borel set A of Lebesgue measure $|A|$,

$$P\left\{\xi: \sum_1^N \delta_i \xi_i r_i \in A\right\} \leqslant \frac{1}{\delta_M r_M} |A|.$$

Therefore the probability that the above inequality hold is at most

$$2(\delta_M r_M)^{-1}(2m+1)^{N+1} \sum_{N+1}^\infty r_i.$$

As $N \to \infty$, the probability is seen to be 0.

An immediate corollary is that if $(1/i) \log r_i \to -\infty$, (r) is m-sparse for every m, so that the symmetric set $\sum(\xi \cdot r)$ almost certainly satisfies all the conditions under Lemma 3 (i.e. for all n and m). The set $\sum(\xi \cdot r)$ is then an example of a set P mentioned in the introduction. Observe, however, that if $r_i = e^{-i\alpha}$ where $\alpha > 5$, then for almost all $\xi \in C$, the conditions in Lemma 3 with $m = 1$ are fulfilled.

LEMMA 4 **(4)**. (a) *If* $r_i = e^{-i\alpha}$ *with* $\alpha > 1$, *then for some* $j = j(\alpha)$ *and almost all* ξ,

$$\sum_{n=-\infty}^\infty \left| \prod_{i=1}^\infty \cos n r_i \xi_i \right|^j = \sum_{-\infty}^\infty |\hat{\mu}(n)|^j < \infty.$$

(b) *If* $r_i = e^{-i \log \log i}$, $i > 3$, *then for almost all* ξ, $\hat{\mu}(n) = o(1)$ *as* $n \to \infty$.

Proof. For any number $s > 0$ and integer $n \geqslant 1$,

$$\int |\hat{\mu}(n)|^s P(d\xi) = \prod_{i=1}^\infty \int_0^1 |\cos n r_i \xi_i|^s d\xi_i.$$

The typical multiplicand is equal to

$$(nr_i)^{-1} \int_0^{nr_i} |\cos y|^s dy \leqslant \left(\frac{1}{\pi} + (nr_i)^{-1}\right) \int_0^\pi |\cos y|^s dy.$$

Under (a), $nr_i \geqslant 1$ for $[\log n/\log \alpha]$ indices i, so

$$\int |\hat{\mu}(n)|^s P(d\xi) \leqslant \left\{\frac{\pi+1}{\pi} \int_0^\pi |\cos y|^s dy\right\}^{[\log n/\log \alpha]}.$$

As soon as

$$\frac{\pi+1}{\pi} \int_0^\pi |\cos y|^s dy < \alpha^{-1}$$

we have

$$\sum_{-\infty}^\infty \int |\mu(n)|^s P(d\xi) < \infty$$

and (a) follows from this.

In (b), $nr_i \geqslant 1$ for large n and

$$i \leqslant 4 \frac{\log n}{\log \log n}.$$

Now

$$\int_0^\pi |\cos y|^s dy \leqslant Cs^{-1/2} \qquad \text{as } s \to \infty.$$

Taking

$$s = s(n) = \frac{\log n}{\log \log n},$$

we find that for large n, $\log s(n) \geqslant \frac{3}{4} \log \log n$,

$$\int |\hat{\mu}(n)|^s P(d\xi) \leqslant \exp\left(-\frac{3}{2} \log n + C \frac{\log n}{\log \log n}\right).$$

For each integer $M > 1$,

$$M^s \int |\hat{\mu}(n)|^s P(d\xi) \leqslant \exp\left(-\frac{3}{2} \log n + C_1 \frac{\log n}{\log \log n}\right)$$

so that

$$\sum_{-\infty}^{\infty} M^{s(|n|)} |\hat{\mu}(n)|^{s(|n|)} < \infty$$

almost certainly. Therefore $\hat{\mu}(n) = o(1)$ almost certainly.

THEOREM 3. *If G is the circle group T, the ideal J is not the intersection of maximal ideals of M.*

Proof. We consider T as the (additive) group of reals R, mod 2π. We choose $r_i = e^{-bi}$ according to the remarks after Theorem 2. Let H be the subgroup of R which is mapped onto the infinite subgroup of T generated by u. It is almost certain that $H \cap (\pm\sum \pm \sum) = \{0\}$, because H is countable. Writing $\sum'$ for the image of $\sum(\text{mod } 2\pi)$, it is almost certain that $nu \notin \pm\sum'(\xi \cdot r) \pm \sum'(\xi \cdot r)$ so that the random measure μ constructed in Lemma 4, and now projected from R to T, is almost certainly not in J. Call this new measure λ; its jth power is almost certainly in $L^1(T)$. Thus the present theorem follows from Lemma 2 *unless* $\chi(u) = w$ for some continuous character χ of T.

In the last event, χ is unique, so we can choose a positive measure σ with finite support F, so that $\int \chi \, d\sigma = 0$, whence $\sigma * \lambda^j$ is almost certainly in J. If we also arrange that $nu \notin \pm F \pm F$ for $n \neq 0$, then, almost certainly,

$$nu \notin \pm\sum' \pm \sum' \pm F \pm F.$$

This is sufficient, because $\sigma * \lambda$ is concentrated in $\sum' + F$, to yield $\sigma * \lambda \notin J$.

3. Another example. In the following example we construct a compact metric abelian group G, a Borel probability μ in G, a Borel set $E \subseteq G$, and an element u of G such that

(i) $(E + mu) \cap E = \emptyset$, for $m \neq 0$,

(ii) $\mu(E) = 1$,

(iii) $\mu^*\mu^*\mu$ is equivalent to the Haar measure m in G, with derivative bounded away from 0 and ∞.

The construction uses the following notations, p being always an *odd prime*. Z_p is the group Z/pZ; $\mathbf{m}_p$ is the invariant probability in Z_p. $F_p = Z_p - \{0\}$ and μ_p is the probability uniformly distributed in F_p. $G = \prod_p Z_p$, $\mathbf{m} = \prod_p \mathbf{m}_p$, $\mu = \prod_p \mu_p$, $F = \prod_p F_p$, $u = (1, 1, 1, \ldots)$, and $E = F \sim \bigcup_{m \neq 0} (F + mu)$.

Of assertions (i)–(iii), (i) holds by construction and (ii) will follow as soon as it is proved that $\mu(F + mu) = 0$ if $m \neq 0$. However,

$$\mu(F + mu) = \mu(\textstyle\prod_p (F_p + m \cdot 1)) = \prod_{p \geqslant 3} \mu_p(F_p + m \cdot 1).$$

Now $\mu_p(F_p + m \cdot 1) \leqslant 1 - p^{-1}$ if $m \not\equiv 0 \pmod p$, so $\mu(F + mu) = 0$ if $m \neq 0$.

To prove (iii) we denote by e_p the unit mass at 0 in Z_p. Then

$$\mu_p = p(p-1)^{-1}(\mathbf{m}_p - p^{-1}e_p).$$

Thus

$$\mu_p{}^*\mu_p{}^*\mu_p = p^3(p-1)^{-3}[\mathbf{m}_p - 3p^{-1}\mathbf{m}_p + 3p^{-2}\mathbf{m}_p - p^{-3}e_p].$$

Here we used the facts that $\mathbf{m}_p$ is invariant and that $\mathbf{m}_p$ and e_p are idempotent. Continuing,

$$\mu_p{}^*\mu_p{}^*\mu_p = [1 + (p-1)^{-3}]\mathbf{m}_p - (p-1)^{-3}e_p.$$

If we write $\mu_p = f_p \cdot \mathbf{m}_p$, then

$$f_p \geqslant 1 + (p-1)^{-3} - p(p-1)^{-3} = 1 - (p-1)^{-2} \geqslant \tfrac{3}{4}.$$

Moreover, $|f_p - 1| = O(p^{-2})$ so the product $\prod_p f_p$ converges uniformly to a function f bounded away from 0 and ∞; of course, $\mu = f \cdot \mathbf{m}$. This is (iii).

We note that if w is a complex number of modulus 1, but not a root of unity, there is no continuous character mapping u to w. Thus for such a w, the closed principal ideal J generated by $\mathfrak{E}_u - we$ does not contain μ (cf. Lemma 4) but contains $\mu^*\mu^*\mu$ (Lemma 2). Thus J is not the intersection of the maximal ideals in M which contain it.

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