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FINITE DIMENSIONAL PERTURBATIONS OF DIFFERENTIAL EXPRESSIONS

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Operators in L_2, or more generally, L_p spaces, which are generated by differential expressions, have had extensive study. More recently some authors, in particular Krall [**3**; **4**; **5**; **6**; **7**], Kim [**2**], and Krall and Brown [**8**], have studied operators which are generated by a differential expression plus an additional term. This additional term is of the nature of a perturbation of the differential expression by an operator with finite dimensional range. However even if the basic operator is specifically of the form of a finite dimensional perturbation of a differential operator, this is not true of the adjoint, since the boundary conditions which arise on the adjoint are not appropriate to the adjoint of the differential operator alone. This has led to a consideration of such operators subjected to more general boundary conditions than the ones appropriate to differential operators.

It is the object of this paper to show that these more general conditions arise naturally from a consideration of a class of finite dimensional perturbations of differential expressions. We shall also show that the theory of the operators arising from such expressions is very closely analogous to that for those arising from differential expressions.

1. The basic expression. Let $\tau y = \sum_{j=0}^{n} p_j y^{(n-j)}$ be a differential expression with $p_j \in C^{n-j}$ on an interval I.

$$(1.1) \qquad \mathscr{L} y = \tau y + \sum_{j=1}^{m} \chi_j F_j(y),$$

where $\chi_j \in L_p(I)$ and F_j depends linearly on y.

In order that this be tractable it is only reasonable to assume that F_j has some continuity properties. Associated with the differential expression τ on $L_p(I)$ there is a maximal domain $\mathscr{D}_1(\tau, p, I)$, a minimal domain $\mathscr{D}_0(\tau, p, I)$, and closed operators $T_1(\tau, p, I)$ and $T_0(\tau, p, I)$ with domains $\mathscr{D}_1$ and $\mathscr{D}_0$ respectively (see Kemp [**1**]). The domains $\mathscr{D}_1$ and $\mathscr{D}_0$ are Banach spaces in the τ-norm $||f||_\tau = ||\tau f||_p + ||f||_p$, and a natural assumption to impose on the F_j is that each F_j is a continuous linear functional on the Banach Space $\mathscr{D}_1(\tau, p, I)$.

LEMMA 1.1. *If F is a continuous linear functional on $\mathscr{D}_1(\tau, p, I)$ $(1 \leqq p < \infty)$,*

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there exist $g, h \in L_q(I)(1/p + 1/q = 1)$ *such that*

$$F(f) = \int_I [(\tau f)\bar{g} + f\bar{h}]dt$$

for all $f \in \mathscr{D}_1(\tau, p, I)$.

Proof. If $L_p(I) \oplus L_p(I)$ is normed by $\|(x, y)\| = \|x\|_p + \|y\|_p$ then $\mathscr{D}_1(\tau, p, I)$ is in isometric correspondence with the graph G of $T_1(\tau, p, I)$, and F may be considered as a continuous linear functional on G. We may extend this linear functional to all of $L_p(I) \oplus L_p(I)$ by the Hahn-Banach Theorem. Restricting this extension to the first (second) component we obtain a continuous linear functional on $L_p(I)$ which may then be represented by an element $g(h)$ of $L_q(I)$. Thus $F(f) = F(\tau f, f) = F(\tau f, 0) + F(0, f) = \int_I[\tau f\bar{g} + f\bar{h}]dt$.

Note that the representation obtained in this lemma is not unique, for if $k \in \mathscr{D}_0(\tau^*, q, I)$, the domain of the minimal operator associated with the adjoint differential expression $\tau^*y = \sum_{j=0}^n (-1)^{n-j}(\bar{p}_jy)^{(n-j)}$, then the pair (g, h) may be replaced by $(g + k, h - \tau^*k)$ without changing F (see Kemp [**1**]).

This representation is still too general to handle so we add a basic assumption:

(A) The interval I is the union of subintervals I_k ($I_j \cap I_k$ consists of at most one point for $j \neq k$) such that for each F_j the restriction of the corresponding g_j to I_k belongs to $\mathscr{D}_1(\tau^*, q, I_k)$.

Under assumption (A) we thus have

$$F_j(f) = \sum_k \int_{I_k} [(\tau f)\bar{g}_j - f\overline{\tau^* g_j} + f\overline{(h_j - \tau^* g_j)}]dt$$
$$= \sum_k \langle f|g_j\rangle_{\tau, I_k} + \int_I f\overline{(h_j - \tau^* g_j)}dt,$$

where

$$(1.2) \qquad \langle y|z\rangle_{\tau, I_k} = \int_{I_k} [(T_1(\tau, p, I_k)y)\bar{z} - y\overline{T_1(\tau^*, q, I_k)z}]dt$$

is the boundary form formula for τ on the interval I_k. Now in case τ has singularities in the interior of I, $\langle y|z\rangle_{\tau, I}$ depends on values of y and z in an arbitrary neighbourhood of an intrinsic boundary $\mathscr{B}(\tau, p, I)$ (see Kemp [**1**]) which may be somewhat complicated. In order to avoid complications which are not essential to the current problem we shall add a further assumption:

(B) The interval I may be subdivided into a finite number l of subintervals I_k such that $\mathscr{B}(\tau, p, I)$ is included in the set of end-points of the I_k's, and the restriction of g_j to I_k belongs to $\mathscr{D}_1(\tau^*, q, I_k)$.

Thus we shall consider operators on $L_p(I)$ $(1 \leqq p \leqq \infty)$ generated by the expression

$$(1.3)\quad \mathscr{L}y = \tau y + \sum_{k=1}^{m} \chi_k[(y|\psi_k) + F_k(y)],$$

where $\psi_k \in L_q(I)$ $(1/p + 1/q = 1)$, $(\phi|\psi_k) = \int_I \phi\bar{\psi}_k dt$, and F_k is a continuous linear functional on

$$(1.4)\quad \mathscr{D}_1(\mathscr{L}, p, I) = \{\phi \in L_p(I) | \phi|_{I_k} \in \mathscr{D}_1(\tau, p, I_k), 1 \leqq k \leqq l\}$$

which annihilates

$$(1.5)\quad \tilde{\mathscr{D}}_0(\mathscr{L}, p, I) = \{\phi \in \mathscr{D}_1(\mathscr{L}, p, I) | \phi|_{I_k} \in \mathscr{D}_0(\tau, p, I_k), 1 \leqq k \leqq l\}.$$

The operator on $L_p(I)$ with domain $\mathscr{D}_1(\mathscr{L}, p, I)$ defined by $\mathscr{L}$ will be referred to as the maximal operator associated with $\mathscr{L}$, and denoted by $T_1(\mathscr{L}, p, I)$.

Since each $\mathscr{D}_1(\tau, p, I_k)$ is dense in $L_p(I_k)$ it is clear that $\mathscr{D}_1(\mathscr{L}, p, I)$ is dense in $L_p(I)$. It is also clear that $T_1(\mathscr{L}, p, I)$ is a finite dimensional perturbation of the direct sum of the operators $T_1(\tau, p, I_k)$, $1 \leqq k \leqq l$. It is not clear that $T_1(\mathscr{L}, p, I)$ is closed but this will follow from our discussion of adjoints.

2. Denseness of certain submanifolds of $\mathscr{D}_1(\mathscr{L}, p, I)$. In this section we prove that the type of submanifold of $\mathscr{D}_1(\mathscr{L}, p, I)$, which arises naturally in the process of taking adjoints, is dense in $L_p(I)$.

LEMMA 2.1. *Let $\mathscr{D}$ ($\mathscr{D}^*$) be a dense linear manifold in (the dual $\mathscr{X}^*$ of) a Banach space $\mathscr{X}$, and $\mathscr{Q}^*$ ($\mathscr{Q}$) a finite dimensional subspace of $\mathscr{X}^*$ ($\mathscr{X}$). Then $\mathscr{D} \cap {}^\perp\mathscr{Q}^*$ ($\mathscr{D}^* \cap \mathscr{Q}^\perp$) is dense in ${}^\perp\mathscr{Q}^*$ ($\mathscr{Q}^\perp$).*

Proof. Since the technique of proof is identical in the two cases, we give only one.

If $\mathscr{Q}^* = \{0\}$ the result is trivial. Suppose first that dim $\mathscr{Q}^* = 1$ and $\mathscr{Q}^*$ is spanned by x_0^* $(\neq 0)$. Let $x_0 (\neq 0) \in {}^\perp\mathscr{Q}^*$ (i.e. $x_0^*(x_0) = 0$). Since $\mathscr{D}$ is dense in $\mathscr{X}$ there exists a sequence $\{d_n\}_1^\infty \subset \mathscr{D}$ which converges to x_0. If $x_0^*(d_n) = 0$ for all n then $\{d_n\}_1^\infty \subset \mathscr{D} \cap {}^\perp\mathscr{Q}^*$. Otherwise let n_0 be the smallest integer such that $x_0^*(d_{n_0}) \neq 0$ and set

$$\tilde{d}_n = d_n - x_0^*(d_n)d_{n_0}/x_0^*(d_{n_0}).$$

Then clearly $\tilde{d}_n \in \mathscr{D} \cap {}^\perp\mathscr{Q}^*$ and since $d_n \to x_0$, $x_0^*(d_n) \to x_0^*(x_0) = 0$ so $\tilde{d}_n \to x_0$. Thus in either case we have, for arbitrary $x_0 \in {}^\perp\mathscr{Q}^*$, a sequence in $\mathscr{D} \cap {}^\perp\mathscr{Q}^*$ which converges to x_0.

Now suppose the result is true provided dim $\mathscr{Q}^* \leqq n - 1$ and consider a case with dim $\mathscr{Q}^* = n$ and $\mathscr{Q}^*$ spanned by $x_1^*, x_2^*, \ldots, x_n^*$. Then

$$\mathscr{D} \cap {}^\perp[\bigvee(x_1^*, \ldots, x_n^*)] = \{\mathscr{D} \cap {}^\perp[\bigvee(x_1^*, \ldots, x_{n-1}^*)]\} \cap {}^\perp\{\bigvee(x_n^*)\}$$

where $\bigvee\{\text{- -}\}$ denotes the span of the indicated vectors. As $\mathscr{D} \cap {}^\perp[\bigvee(x_1^*, \ldots, x_{n-1}^*)]$ is dense by the induction hypothesis, the complete result follows at once from the proof for $n = 1$.

LEMMA 2.2. *Let* $\mathscr{D}_0 \subset \mathscr{D}$ ($\mathscr{D}_0^* \subset \mathscr{D}^*$) *be dense linear manifolds in* (*the dual* $\mathscr{X}^*$ *of*) *a Banach space* $\mathscr{X}$, *which are complete in* $\| \ \|_{\mathscr{D}}$ ($\| \ \|_{\mathscr{D}^*}$). *Let* $F_1, \ldots, F_m$ *be linearly independent continuous linear functionals on* $\mathscr{D}$ ($\mathscr{D}^*$) *which annihilate* $\mathscr{D}_0$ ($\mathscr{D}_0^*$), *and* $x_1^*, \ldots, x_m^*$ ($x_1, \ldots, x_m$) *arbitrary elements of* $\mathscr{X}^*$ ($\mathscr{X}$). *Then*

$$\mathscr{A} = \{x \in \mathscr{D} | F_j(x) + x_j^*(x) = 0, 1 \leqq j \leqq m\}$$
$$(\mathscr{A}^* = \{x^* \in \mathscr{D}^* | F_j(x^*) + x^*(x_j) = 0, 1 \leqq j \leqq m\})$$

is dense in $\mathscr{X}$ ($\mathscr{X}^*$).

Proof. As in the previous lemma the technique of proof for the two cases is identical so we only present one.

We first re-arrange the conditions defining $\mathscr{A}$. Let j_1 be the smallest integer such that $x_{j_1}^* \neq 0$, and if $j_1, \ldots, j_{r-1}$ are known let j_r be the smallest integer $> j_{r-1}$ such that $x_{j_1}^*, \ldots, x_{j_r}^*$ are linearly independent. This process will terminate and yield a set $\{j_1, \ldots, j_k\}$ such that $x_{j_1}^*, \ldots, x_{j_k}^*$ are linearly independent, and if $i_1 < i_2 < \ldots < i_{m-k}$ are such that $\{j_1, \ldots, j_k\} \cup \{i_1, \ldots, i_{m-k}\} = \{1, \ldots, m\}$, then

$$x_{i_k}^* = \sum_{j_s < i_r} a_{rs} x_{j_s}^*$$

for some numbers a_{rs}. Now let

$$y_r^* = x_{j_r}^*, \quad 1 \leqq r \leqq k;$$
$$G_r = F_{j_r}, \quad 1 \leqq r \leqq k;$$
$$G_{k+r} = F_{i_r} - \sum_{j_s < i_r} a_{rs} F_{j_s}, \quad 1 \leqq r \leqq m - k.$$

Then $G_1, \ldots, G_m$ are still linearly independent continuous linear functionals on $\mathscr{D}$ which annihilate $\mathscr{D}_0$; $y_1^*, \ldots, y_k^*$ are linearly independent elements of $\mathscr{X}^*$; and

$$\mathscr{A} = \{x \in \mathscr{D} | G_j(x) + y_j^*(x) = 0, \\ 1 \leqq j \leqq k; G_j(x) = 0, k + 1 \leqq j \leqq m\}.$$

Let $\mathscr{Q}^* = \bigvee\{y_1^*, \ldots, y_k^*\}$ and note that $\mathscr{A} \supset \mathscr{D}_0 \cap {}^{\perp}\mathscr{Q}^*$. Since $\mathscr{D}_0 \cap {}^{\perp}\mathscr{Q}^*$ is dense in ${}^{\perp}\mathscr{Q}^*$ by Lemma 2.1, we have $\overline{\mathscr{A}} \supset {}^{\perp}\mathscr{Q}^*$ so $\mathscr{A}^{\perp} = \overline{\mathscr{A}}^{\perp} \subset ({}^{\perp}\mathscr{Q}^*)^{\perp} = \mathscr{Q}^*$. Thus if $\mathscr{A}$ is not dense there must exist a non-zero $y^* \in \mathscr{Q}^*$ such that $y^*(x) = 0$ for all $x \in \mathscr{A}$. Thus $y^* = \sum_1^k a_j y_j^*$ with not all a_j's vanishing. In particular, for all $x \in \mathscr{A}$, $0 = y^*(x) = \sum_1^k a_j y_j^*(x) = -\sum_1^k a_j G_j(x)$.

Now let $\tilde{G} = -\sum_1^k a_j G_j$ and note that $\{\tilde{G}, G_{k+1}, \ldots, G_m\}$ are linearly independent on $\mathscr{D}$ so the mapping $\pi : \mathscr{D} \to \mathbf{C}^{m-k+1}$ defined by $\pi(f) = (\tilde{G}(f), G_{k+1}(f), \ldots, G_m(f))$ is onto. Thus there exists $f_1 \in \mathscr{D}$ such that $\tilde{G}(f_1) = 1$ and $G_j(f_1) = 0$, $k + 1 \leqq j \leqq m$. Similarly the map $\pi_0 : \mathscr{D}_0 \to \mathbf{C}^k$ defined by $\pi_0(f) = (y_1^*(f), \ldots, y_k^*(f))$ is onto as $y_1^*, \ldots, y_k^*$ are linearly independent and $\mathscr{D}_0$ is dense. Thus there exists $f_0 \in \mathscr{D}_0$ such that $y_j^*(f_0) = -y_j^*(f) - G_j(f)$, $1 \leqq j \leqq k$. Consider now $f = f_0 + f_1$. For all j, $G_j(f) =$

$G_j(f_1)$ and thus for $1 \leqq j \leqq k$, $G_j(f) + y_j^*(f) = G_j(f_1) + y_j^*(f_0) + y_j^*(f_1) = 0$, and for $k + 1 \leqq j \leqq m$, $G_j(f) = 0$. Thus $f \in \mathscr{A}$ and $\tilde{G}(f) = 1$, whereas by construction $\tilde{G}$ annihilates all elements of $\mathscr{A}$. This contradiction implies that there is no such non-zero y^*, and thus that $\mathscr{A}$ is dense.

3. Adjoints. For convenience in the L_2 case we shall define the *adjoint* of an operator T in $L_p(I)$ to be the maximal operator S in $L_q(I)$ $(1/p + 1/q = 1)$ such that

(3.1) $$0 = \int_I ((Tf)\bar{g} - f\overline{Sg})dt$$

for all $f \in \mathscr{D}(T)$ and all $g \in \mathscr{D}(S)$.

We shall now calculate the adjoint of an operator generated by $\mathscr{L}$, which is obtained by restricting $T_1(\mathscr{L}, p, I)$ by certain generalized boundary conditions. This will lead to a definition for adjoint expressions to $\mathscr{L}$, and to minimal operators associated with $\mathscr{L}$.

Note first that the boundary conditions involved in the expression (1.3) for $\mathscr{L}$ are all those for τ on each subinterval I_k. Now for a linear manifold in $\mathscr{D}_1(\mathscr{L}, p, I)$ determined by a linearly independent set of generalized boundary conditions of the form $V_j(y) + (y|\phi_j) = 0$ $1 \leqq j \leqq M$, to be dense, we must insist that any relation of the form $\sum_1^M a_j V_j = 0$, must imply $\sum_1^M \bar{a}_j \phi_j = 0$. Thus we must assume that $V_1, \ldots, V_M$ are linearly independent. Then imitating the argument in the proof of Lemma 2.2 we may replace these conditions by an equivalent set in which $\phi_1, \ldots, \phi_k$ are linearly independent and $\phi_j = 0$, $k + 1 \leqq j \leqq M$. Thus let

(3.2) $$\mathscr{D}(T) = \{y \in \mathscr{D}_1(\mathscr{L}, p, I) | V_j(y) + (y|\phi_j) = 0, \\ 1 \leqq j \leqq k; V_j(y) = 0, k + 1 \leqq j \leqq M\},$$

where $V_1, \ldots, V_M$ are linearly independent continuous linear functionals on $\mathscr{D}_1(\mathscr{L}, p, I)$ which annihilate $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$, and $\phi_1, \ldots, \phi_k$ are linearly independent elements of $L_q(I)$. Let the operator T on $L_p(I)$ with domain $\mathscr{D}(T)$ be defined by $Ty = \mathscr{L}y$.

Let the linearly independent set $V_1, \ldots, V_M$ be extended to a basis $V_1, \ldots, V_N$ for all continuous linear functionals on $\mathscr{D}_1(\mathscr{L}, p, I)$ which annihilate $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$†. Note that $N \leqq 2ln$ where l is the number of subintervals and n is the order of τ. There is then a dual basis $\hat{V}_1, \ldots, \hat{V}_N$ for the continuous linear functionals on $\oplus_1^l \mathscr{D}_1(\tau^*, q, I_k)$ which annihilate $\oplus_1^l \mathscr{D}_0(\tau^*, q, I_k)$ such that

(3.3) $$\langle y|z\rangle_{\tau,p,I} = \sum_1^l \langle y|z\rangle_{\tau,p,I_k} = i \sum_1^N \overline{\hat{V}_j(z)} V_j(y).$$

Now since $V_1, \ldots, V_N$ form a basis each F_j in (1.3) can be expressed in terms

†The existence of such bases is clear if τ is regular on I, and is proved for the singular case (provided the essential resolvent set is non-empty) in Rota [**9**].

of this basis:

$$(3.4)\qquad F_j(y) = \sum_{s=1}^{N} c_{js}V_s(y)$$

for all $y \in \mathscr{D}_1(\mathscr{L}, p, I)$.

THEOREM 3.1. *The adjoint of T is the operator T^* on $L_q(I)$ with domain:*

$$(3.5)\qquad \mathscr{D}(T^*) = \left\{ z \in \oplus_1^l \mathscr{D}_1(\tau^*, q, I) \,|\, \hat{V}_j(z) + i \sum_{s=1}^{m} \bar{c}_{sj}(z|\chi_s) = 0,\right.$$
$$\left. M + 1 \leqq j \leqq N \right\},$$

defined by:

$$(3.6)\qquad T^*z = \tau^*z + \sum_{j=1}^{m} \psi_j(z|\chi_j) + \sum_{j=1}^{k} \phi_j[i\hat{V}_j(z) - \sum_{s=1}^{m} \bar{c}_{sj}(z|\chi_s)],$$

for $z \in \mathscr{D}(T^*)$ where τ^* is the adjoint to τ.

Proof. Note that $\mathscr{D}(T)$ is dense by Lemma 2.2 so the adjoint exists.

We shall construct a basis of $\mathscr{D}_1(\mathscr{L}, p, I)$ modulo $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$ which contains a basis of $\mathscr{D}(T)$. Since $\phi_1, \ldots, \phi_k$ are linearly independent there exist $\Sigma_i \in \tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$ such that $(\Sigma_i|\phi_j) = \delta_{ij}$. Since $V_1, \ldots, V_N$ form a basis there exists a basis $\{v_1, \ldots, v_N\}$ of $\mathscr{D}_1(\mathscr{L}, p, I)$ modulo $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$ such that $V_i(v_j) = \delta_{ij}$. Now let

$$\omega_j = \begin{cases} v_j - \Sigma_j - \sum_{r=1}^{k} (v_j|\phi_r)\Sigma_r, & 1 \leqq j \leqq k, \\ v_j - \sum_{r=1}^{k} (v_j|\phi_r)\Sigma_r, & k + 1 \leqq j \leqq N. \end{cases}$$

Since $\omega_j = v_j$ modulo $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$, $\omega_1, \ldots, \omega_N$ form a basis of $\mathscr{D}_1(\mathscr{L}, p, I)$ modulo $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$ and furthermore:

$$V_s(\omega_j) = V_s(v_j) = \delta_{sj}, \quad 1 \leqq j, s \leqq N,$$
$$(\omega_j|\phi s) = \begin{cases} -\delta_{sj}, & 1 \leqq j \leqq k, \\ 0, & k + 1 \leqq j \leqq N. \end{cases}$$

Thus $V_s(\omega_j) + (\omega_j|\phi_s) = 0, 1 \leqq j \leqq N, 1 \leqq s \leqq k$ and $V_s(\omega_j) = 0, k + 1 \leqq s \leqq M$ for all $j \notin [k + 1, M]$. Thus $\omega_j \in \mathscr{D}(T)$ for all $j \notin [k + 1, M]$. On the other hand if $y \in \mathscr{D}(T)$ then $y \in \mathscr{D}_1(\mathscr{L}, p, I)$ so

$$y = y_0 + \sum_{1}^{N} a_j\omega_j$$

where $y_0 \in \tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$ and $a_j \in \mathbf{C}$.

Now since $y \in \mathscr{D}(T)$, $V_s(y) = \sum_1^N a_jV_s(\omega_j) = a_s$ for all s, and this must vanish for $s \in [k+1, M]$. Thus if $y \in \mathscr{D}(T)$

$$y = y_0 + \sum_{j \notin [k+1, M]} a_j \omega_j,$$

and since y and the sum both belong to $\mathscr{D}(T)$ it follows that $y_0 \in \mathscr{D}(T)$ and further, since $y_0 \in \tilde{\mathscr{D}}_0(\mathscr{L}, p, I)$, that $y_0 \in \tilde{\mathscr{D}}_0(\mathscr{L}, p, I) \cap {}^{\perp}[\bigvee\{\phi_1, \ldots, \phi_k\}]$.

Now in order that $z \in \mathscr{D}(T^*)$ and $T^*z = w$ we must have, by (3.1):

$$\text{(3.7)} \qquad \begin{aligned} 0 &= \int_I [\bar{z}\mathscr{L}y - \bar{w}y]dt \\ &= \int_I [\bar{z}\mathscr{L}y_0 - \bar{w}y_0]dt + \sum_{j \notin [k+1, M]} a_j \int_I [\bar{z}\mathscr{L}\omega_j - \bar{w}\omega_j]dt \end{aligned}$$

for all $y_0 \in \tilde{\mathscr{D}}_0(\mathscr{L}, p, I) \cap {}^{\perp}[\bigvee\{\phi_1, \ldots, \phi_k\}]$ and all $a_j \in \mathbf{C}$.

In particular let us consider the case where all a_j's are 0 and y_0 vanishes outside a compact subinterval $[c, d]$ of the interior of I_r. Let $y_0^{(n)} = h$ so that $y_0^{(l)}(t) = \int_c^t (t-s)^{n-l-1}h(s)ds/(n-l-1)!$ on $[c, d]$ and vanishes outside $[c, d]$. Since $y_0^{(l)}(c) = y_0^{(l)}(d) = 0$, $0 \leqq l \leqq n-1$ we have

$$\text{(3.8)} \qquad \int_c^d s^l h(s)ds = 0, \quad 0 \leqq l \leqq n-1.$$

Furthermore, since y_0 is orthogonal to all ϕ_j, we have

$$\text{(3.9)} \qquad \int_c^d h(t) \int_t^d \frac{(s-t)^{n-1}}{(n-1)!} \overline{\phi_j(s)}\, dsdt = 0, \quad 1 \leqq j \leqq k.$$

Conversely, if h is any locally integrable function vanishing outside $[c, d]$ which satisfies (3.8) and (3.9), then $y_0(t) = \int_c^t(t-s)^{n-1}h(s)ds/(n-1)!$ inside $[c, d]$ and vanishing outside $[c, d]$ is an element of $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I) \cap {}^{\perp}[\bigvee\{\phi_1, \ldots, \phi_1\}]$.

Thus in terms of h, with all a_j's zero, (3.7) reduces to:

$$\begin{aligned} 0 &= \int_c^d \left[\overline{z(t)}\left\{\tau y_0(t) + \sum_{j=1}^m \chi_j(t)(y_0|\psi_j)\right\} - \overline{w(t)}\, y_0(t)\right]dt \\ &= \int_c^d h(t)\left\{p_0(t)\,\overline{z(t)} + \sum_{l=1}^n \int_t^d \frac{(s-t)^{l-1}}{(l-1)!} p_l(s)\,\overline{z(s)}\, ds \right. \\ &\qquad \left. + \sum_{j=1}^m \overline{(z|\chi_j)} \int_t^d \frac{(s-t)^{n-1}}{(n-1)!} \overline{\psi_j(s)}ds - \int_t^d \frac{(s-t)^{n-1}}{(n-1)!} \overline{\omega(s)}ds\right\}dt \end{aligned}$$

for all admissible h. We conclude immediately that

$$\text{(3.10)} \qquad \begin{aligned} &\overline{p_0(t)}\, z(t) + \sum_{l=1}^M \int_t^d \frac{(s-t)^{l-1}}{(l-1)!} \overline{p_l(s)}\, z(s)ds \\ &\quad + \sum_{j=1}^m (z|\chi_j) \int_t^d \frac{(s-t)^{n-1}}{(n-1)!} \psi_j(s)ds - \int_t^d \frac{(s-t)^{n-1}}{(n-1)!} \omega(s)ds \\ &\qquad = \sum_{j=1}^k \hat{c}_j \int_t^d \frac{(s-t)^{n-1}}{(n-1)!} \phi_j(s)ds + \sum_{j=0}^{n-1} c_j t^j \end{aligned}$$

a.e. on $[c, d]$ for some constants $\hat{c}_1, \ldots, \hat{c}_k, c_0, \ldots, c_{n-1}$. It follows that we may alter z on a null set so that (3.10) holds everywhere on $[c, d]$, and thus that $\bar{p}_0 z$ is absolutely continuous there. Differentiating, we obtain the derivative of an absolutely continuous function equal a.e. to an absolutely continuous function. Thus equality holds everywhere and we may repeat the process. After $n-1$ repetitions we obtain the fact that $z \in C^{n-1}([c, d])$, $z^{(n-1)}$ is absolutely continuous there, and

$$(3.11) \quad T^*z = w = \tau^*z + \sum_{j=1}^{m} \psi_j(z|\chi_j) - \sum_{j=1}^{k} \hat{c}_j \phi_j$$

a.e. on $[c, d]$. Since $[c, d]$ is an arbitrary compact subinterval of the interior of I_r, we have the restriction of z to I_r belonging to $\mathscr{D}_1(\tau^*, q, I_r)$ and (3.11) holds on I_r. Furthermore, since r is arbitrary $z \in \oplus_1^l \mathscr{D}_1(\tau^*, q, I_r)$ and T^*z is given by (3.11) throughout I except that the constants $\hat{c}_j$ seemingly may change with r.

However let us, now that we know $z \in \oplus_1^l \mathscr{D}_1(\tau^*, q, I_r)$, consider an arbitrary $y_0 \in \tilde{\mathscr{D}}_0(\mathscr{L}, p, I) \cap \mathscr{D}(T)$. Then (3.7) becomes:

$$\begin{aligned} 0 &= \sum_1^l \int_{I_r} \left[\bar{z}\left\{\tau y_0 + \sum_1^m \chi_j(y_0|\psi_j)\right\} - \bar{w}y_0\right]dt \\ &= \sum_1^l \int_{I_r} \left[\bar{z}\tau y_0 - y_0\overline{\tau^*z} + y_0\overline{\tau^*z} + \bar{z}\sum_1^m \chi_j(y_0|\psi_j) - y_0\bar{w}\right]dt \\ &= \sum_1^l \langle y_0|z\rangle_{\tau,p,I_r} + \int_I y_0\left[\overline{\tau^*z + \sum_1^m \psi_j(z|\chi_j) - w}\right]dt. \end{aligned}$$

Since the first term vanishes, and $\tilde{\mathscr{D}}_0(\mathscr{L}, p, I) \cap \mathscr{D}(T)$ is dense in ${}^{\perp}[\bigvee\{\phi_1, \ldots, \phi_k\}]$ it follows that there exist constants $\hat{c}_1, \ldots, \hat{c}_k$ such that (3.11) holds (a.e.) throughout I.

Now due to the arbitrary nature of the a_j's we must have, from (3.7), that for $j \notin [k+1, M]$

$$\begin{aligned} 0 &= \int_I [\bar{z}T\omega_j - \bar{w}\omega_j]dt \\ &= \int_I \left[\bar{z}\left\{\tau\omega_j + \sum_{s=1}^m \chi_s\left((\omega_j|\psi_s) + \sum_{p=1}^N c_{sp}V_p(\omega_j)\right)\right\}\right. \\ &\qquad\qquad \left. - \omega_j\overline{\left\{\tau^*z + \sum_{s=1}^m \psi_s(z|\chi_s) - \sum_{p=1}^k \hat{c}_p\phi_p\right\}}\right]dt \\ &= i\sum_{s=1}^N V_s(\omega_j)\overline{\hat{V}_s(z)} + \sum_{s=1}^m \overline{(z|\chi_s)}[(\omega_j|\psi_s) + c_{sj}] \\ &\qquad\qquad - \sum_{s=1}^m \overline{(z|\chi_s)}(\omega_j|\psi_s) + \sum_{p=1}^k \hat{c}_p(\omega_j|\phi_p) \\ &= i\,\overline{\hat{V}_j(z)} + \sum_{s=1}^m c_{sj}\overline{(z|\chi_s)} - \begin{cases} \overline{\hat{c}_j} & 1 \leqq j \leqq k, \\ 0 & j \geqq k+1. \end{cases} \end{aligned}$$

Since this applies for $j \notin [k+1, M]$ we have:

$$\hat{c}_j = -i\,\hat{V}_j(z) + \sum_{s=1}^{m} \bar{c}_{sj}(z|\chi_s)$$

for $1 \leqq j \leqq k$, which proves (3.6), and

$$\hat{V}_j(z) + i \sum_{s=1}^{m} \bar{c}_{sj}(z|\chi_s) = 0, \quad M+1 \leqq j \leqq N,$$

which proves (3.5).

COROLLARY 3.2. $T = T^{**}$, *and thus* T *is closed.*

Proof. T^* is clearly closed, and it is densely defined by Lemma 2.2. Thus T^{**} exists, and by Theorem 3.1 it is an operator with domain contained in $\oplus_1^l \mathscr{D}_1(\tau, p, I_r) = \mathscr{D}_1(\mathscr{L}, p, I)$, defined by M linearly independent boundary conditions of the form:

$$\sum_{s=1}^{N} a_{js} V_s(y) + \sum_{s=1}^{m} b_{js}(y|\psi_s) + \sum_{s=1}^{k} d_{js}(y|\phi_s) = 0, \quad 1 \leqq j \leqq M.$$

Since $T \subset T^{**}$ and $\mathscr{D}(T)$ is defined by M linearly independent boundary conditions of this form, it follows that $T = T^{**}$.

Since the application of Theorem 3.1 involves a rearrangement of the boundary conditions defining $\mathscr{D}(T)$ it is convenient to do some reformulation. We note that $T_1(\mathscr{L}, p, I)$ is the maximal operator on $L_p(I)$ associated with the expression $\mathscr{L}$. Thus its adjoint should be the minimal operator on $L_q(I)$ associated with an expression "adjoint" to $\mathscr{L}$. Unfortunately there is no unique adjoint expression.

We first note that the adjoint of $T_1(\mathscr{L}, p, I)$ is the operator S with domain

$$\mathscr{D}(S) = \left\{ z \in \oplus_1^l \mathscr{D}_1(\tau^*, q, I) | V_j(z) + i \sum_{s=1}^{m} \bar{c}_{sj}(z|\chi_s) = 0, \quad 1 \leqq j \leqq N \right\},$$

defined by

$$Sz = \tau^* z + \sum_{j=1}^{m} \psi_j(z|\chi_j).$$

Restricting $T_1(\mathscr{L}, p, I)$ by the imposition of generalized boundary conditions results, as we have seen in Theorem 3.1, in extending S to a larger domain where some of the conditions defining $\mathscr{D}(S)$ are no longer satisfied. Thus choosing an expression $\tilde{\mathscr{L}}$ "adjoint" to $\mathscr{L}$ such that $T_1(\tilde{\mathscr{L}}, q, I)$ is an extension of S, is equivalent to choosing a minimal restriction of $T_1(\mathscr{L}, p, I)$, since the two operators will be adjoint to each other.

Suppose first that we wish a maximal extension of S. It is clear that this must involve terms that are linear in z, and yet vanish when $z \in \mathscr{D}(S)$. Thus the only possibility is to have for our expression adjoint to $\mathscr{L}$:

$$(3.12)\quad \tilde{\mathscr{L}}z = (\tau^*z + \sum_{j=1}^{m} \psi_j(z|\chi_j) + \sum_{j=1}^{N} \phi_j\left[\tilde{V}_j(z) + i\sum_{s=1}^{m} \bar{c}_{sj}(z|\chi_s)\right],$$

for some functions $\phi_1, \ldots, \phi_N \in L_q(I)$. On the other hand, if we choose a minimal restriction T_0 of $T_1(\mathscr{L}, p, I)$ by choosing N linearly independent boundary conditions to define the domain $\mathscr{D}(T_0)$ of T_0, then in Theorem 3.1 we have $M = N$ and the adjoint of T_0 is defined by an expression such as (3.12).

To obtain a more symmetrical appearance to our formulas we shall change notation. Since the ψ_j's and ϕ_j's enter (3.12) in a similar manner we shall write both of them as linear combinations of a single set of functions $\tilde{\chi}_j$, and may assume $\tilde{\chi}_1, \ldots, \tilde{\chi}_{\tilde{m}}$ and $\chi_1, \ldots, \chi_m$ are each linearly independent sets. $\boldsymbol{\chi}$, $\tilde{\boldsymbol{\chi}}$, $(y|\boldsymbol{\chi})$, etc., will denote the obvious column vectors and a superscript "T" or "*" will indicate transpose or conjugate transpose respectively. If $V_1, \ldots, V_N$ and $\tilde{V}_1, \ldots, \tilde{V}_N$ represent "naturally arising" bases for the boundary functionals for τ on $L_p(I)$ and for τ^* on $L_q(I)$ respectively, we denote by $\mathbf{V}(y)$ and $\tilde{\mathbf{V}}(z)$ the obvious column vectors. There is then a non-singular $N \times N$ matrix $\mathbf{C}(\tau)$ such that

$$(3.13)\quad \langle y|z\rangle_{\tau,p,I} = i\,\tilde{\mathbf{V}}(z)^*\mathbf{C}(\tau)\mathbf{V}(y).$$

When τ is clearly understood we shall denote $\mathbf{C}(\tau)$ by just $\mathbf{C}$. There is a $m \times \tilde{m}$ matrix $\mathbf{A}$, a $\tilde{m} \times N$ matrix $\mathbf{D}$, a $\tilde{m} \times m$ matrix $\tilde{\mathbf{A}}$, and a $\tilde{m} \times N$ matrix $\tilde{\mathbf{D}}$, such that

$$(3.14)\quad \mathscr{L}y = \tau y + \boldsymbol{\chi}^T[\mathbf{A}(y|\tilde{\boldsymbol{\chi}}) + i\mathbf{D}\mathbf{V}(y)],$$

$$(3.15)\quad \tilde{\mathscr{L}}z = \tau^*z + \tilde{\boldsymbol{\chi}}^T[\tilde{\mathbf{A}}(z|\boldsymbol{\chi}) + i\tilde{\mathbf{D}}\tilde{\mathbf{V}}(z)].$$

Definition 3.3. *The expression* $\tilde{\mathscr{L}}$ *is adjoint to the expression* $\mathscr{L}$ *if and only if*

$$(3.16)\quad \mathbf{A} - \tilde{\mathbf{A}}^* = i\mathbf{D}\mathbf{C}(\tau)^{-1}\tilde{\mathbf{D}}^*.$$

Note that $\langle z|y\rangle_{\tau^*,q,I} = -\overline{\langle y|z\rangle_{\tau,p,I}} = i\,\overline{\tilde{\mathbf{V}}(z)^*\mathbf{C}(\tau)\mathbf{V}(y)} = i\,\overline{\mathbf{V}(y)^*\mathbf{C}(\tau^*)\tilde{\mathbf{V}}(z)}$ so that $\mathbf{C}(\tau^*) = \mathbf{C}(\tau)^*$ and thus from (3.16), $\tilde{\mathbf{A}} - \mathbf{A}^* = i\,\tilde{\mathbf{D}}\mathbf{C}(\tau^*)^{-1}\mathbf{D}^*$, so $\mathscr{L}$ is adjoint to $\tilde{\mathscr{L}}$.

LEMMA 3.4. *If* $\mathscr{L}$ *and* $\tilde{\mathscr{L}}$ *are adjoint expressions given by* (3.14) *and* (3.15), *then for all* $y \in \mathscr{D}_1(\mathscr{L}, p, I)$ *and all* $z \in \mathscr{D}_1(\tilde{\mathscr{L}}, q, I)$, *the boundary form*

$$\langle y|z\rangle_{\mathscr{L},\tilde{\mathscr{L}}} = (\mathscr{L}y|z) - (y|\tilde{\mathscr{L}}z)$$

is given by

$$(3.17)\quad \langle y|z\rangle_{\mathscr{L},\tilde{\mathscr{L}}} = i[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\boldsymbol{\chi})]^*\mathbf{C}[\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\boldsymbol{\chi}})].$$

Proof. Clearly

$$\begin{aligned}\langle y|z\rangle_{\mathscr{L},\tilde{\mathscr{L}}} &= \langle y|z\rangle_{\tau,p,I} + (z|\boldsymbol{\chi})^*[\mathbf{A}(y|\tilde{\boldsymbol{\chi}}) + i\mathbf{D}\mathbf{V}(y)] \\ &\qquad - \overline{(y|\tilde{\boldsymbol{\chi}})}^{\,T}\,[\overline{\tilde{\mathbf{A}}}\,\overline{(z|\boldsymbol{\chi})} - i\overline{\tilde{\mathbf{D}}}\,\overline{\tilde{\mathbf{V}}(z)}] \\ &= i[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\boldsymbol{\chi})]^*\mathbf{C}[\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\boldsymbol{\chi}})] \\ &\qquad + (z|\boldsymbol{\chi})^*[\mathbf{A} - \tilde{\mathbf{A}}^* - i\mathbf{D}\mathbf{C}^{-1}\tilde{\mathbf{D}}^*](y|\tilde{\boldsymbol{\chi}}),\end{aligned}$$

which completes the proof.

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THEOREM 3.5. *Let* $\mathscr{L}$ *be given by* (3.14) *and* T *a restriction of* $T_1(\mathscr{L}, p, I)$ *with*

$$\mathscr{D}(T) = \{y \in \mathscr{D}_1(\mathscr{L}, p, I) | \mathbf{PV}(y) + \mathbf{Q}(y|\tilde{\chi}) = \mathbf{0}\},$$

where $\mathbf{P}$ *is an* $M \times N$ *matrix of rank* M *and* $\mathbf{Q}$ *is an* $M \times \tilde{m}$ *matrix. Then there exists an expression* $\tilde{\mathscr{L}}$ (*as in* (3.15)) *adjoint to* $\mathscr{L}$ *such that*

$$\mathbf{PC}^{-1}\tilde{\mathbf{D}}^* = \mathbf{Q}. \tag{3.18}$$

Furthermore, for any $\tilde{\mathscr{L}}$ *adjoint to* $\mathscr{L}$ *which satisfies* (3.18), *and any* $(N - M) \times N$ *matrix* $\tilde{\mathbf{P}}$ *of rank* $N - M$ *such that*

$$\mathbf{PC}^{-1}\tilde{\mathbf{P}}^* = \mathbf{0}, \tag{3.19}$$

we have

$$\mathscr{D}(T^*) = \{z \in \mathscr{D}_1(\tilde{\mathscr{L}}, q, I) | \tilde{\mathbf{P}}[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi)] = 0\}$$

and $T^*z = \tilde{\mathscr{L}}z$ *for all* $z \in \mathscr{D}(T^*)$.

Proof. Since $\mathbf{P}$ is of rank M there exists an $N \times M$ matrix $\hat{\mathbf{P}}$ such that $\mathbf{P}\hat{\mathbf{P}} = \mathbf{I}_M$. Thus $\mathbf{Q} = \mathbf{P}\hat{\mathbf{P}}\mathbf{Q}$ and (3.18) will be satisfied if $\mathbf{C}^{-1}\tilde{\mathbf{D}}^* = \hat{\mathbf{P}}\mathbf{Q}$. Now we choose $\tilde{\mathbf{D}} = \mathbf{Q}^*\hat{\mathbf{P}}^*\mathbf{C}^*$ and $\tilde{\mathbf{A}} = \mathbf{A}^* + i\tilde{\mathbf{D}}\mathbf{C}^{*-1}\mathbf{D}^* = \mathbf{A}^* + i\mathbf{Q}^*\hat{\mathbf{P}}^*\mathbf{D}^*$ to obtain an expression $\tilde{\mathscr{L}}$ adjoint to $\mathscr{L}$ which satisfies (3.18).

Now let $\tilde{\mathscr{L}}$ be any such expression and note that

$$\mathscr{D}(T) = \{y \in \mathscr{D}_1(\mathscr{L}, p, I) | \mathbf{P}[\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\chi})] = \mathbf{0}\}.$$

Now there is a non-singular $M \times M$ matrix Λ such that the rows of $\Lambda\mathbf{P}$ are orthonormal row vectors and an $(N - M) \times M$ matrix $\mathbf{R}$ such that the matrix

$$\begin{bmatrix} \Lambda\mathbf{P} \\ \mathbf{R} \end{bmatrix}$$

is unitary. Then $y \in \mathscr{D}(T)$ if and only if

$$\begin{bmatrix} \Lambda\mathbf{P} \\ \mathbf{R} \end{bmatrix} [\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\chi})] = \begin{bmatrix} \mathbf{0} \\ \mathbf{n} \end{bmatrix}$$

where $\mathbf{n}$ is an arbitrary $(N - M) \times 1$ column vector. Thus $y \in \mathscr{D}(T)$ if and only if there exists $\mathbf{n}$ such that

$$\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\chi}) = \mathbf{R}^*\mathbf{n}.$$

From (3.17) it now follows that $\langle y|z\rangle_{\mathscr{L},\tilde{\mathscr{L}}} = 0$ for all $y \in \mathscr{D}(T)$ if and only if

$$\mathbf{0} = i[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi)]^*\mathbf{CR}^*\mathbf{n}$$

for arbitrary $\mathbf{n}$, or equivalently if and only if

$$\mathbf{RC}^*[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi)] = \mathbf{0}.$$

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Thus the operator S defined by $Sz = \tilde{\mathscr{L}}z$ with domain

$$\mathscr{D}(S) = \{z \in \mathscr{D}_1(\tilde{\mathscr{L}}, q, I) | \mathbf{RC}^*[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi)] = \mathbf{0}\}$$

is clearly a restriction of T^*. However, by Theorem 3.1 the domain of T^* is defined by imposing $N - M$ linearly independent generalized boundary conditions on $\mathscr{D}_1(\tilde{\mathscr{L}}, q, I)$, and since $\mathscr{D}(S)$ is obtained in this way and $S \subset T^*$, the two must coincide.

Finally, if $\tilde{\mathbf{P}}$ is an $(N - M) \times N$ matrix of rank $N - M$ such that $\mathbf{PC}^{-1}\tilde{\mathbf{P}}^* = \mathbf{0}$ it follows that the rows of $\tilde{\mathbf{P}}\mathbf{C}^{*-1}$ are orthogonal to the rows of $\mathbf{P}$, and thus spanned by the rows of $\mathbf{R}$. Thus there is a nonsingular matrix $\tilde{\Lambda}$ such that $\tilde{\mathbf{P}}\mathbf{C}^{*-1} = \tilde{\Lambda}\mathbf{R}$ or $\tilde{\mathbf{P}} = \tilde{\Lambda}\mathbf{RC}^*$ and it follows that

$$\mathscr{D}(T^*) = \mathscr{D}(S) = \{z \in \mathscr{D}_1(\tilde{\mathscr{L}}, q, I) | \tilde{\mathbf{P}}[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi)] = \mathbf{0}\},$$

which completes the proof.

Definition 3.6. Associated with the adjoint pair $\mathscr{L}, \tilde{\mathscr{L}}$ given by (3.14) and (3.15) we have the minimal comains

$$\mathscr{D}_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) = \{y \in \mathscr{D}_1(\mathscr{L}, p, I) | \mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\chi}) = \mathbf{0}\},$$
$$D_0(\tilde{\mathscr{L}}, \mathscr{L}, q, I) = \{z \in \mathscr{D}_1(\tilde{\mathscr{L}}, q, I) | \tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi) = \mathbf{0}\}$$

and the operators $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ and $T_0(\tilde{\mathscr{L}}, \mathscr{L}, q, I)$ which are the restrictions of $T_1(\mathscr{L}, p, I)$ to $\mathscr{D}_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$ and $T_1(\tilde{\mathscr{L}}, q, I)$ to $\mathscr{D}_0(\tilde{\mathscr{L}}, \mathscr{L}, q, I)$ respectively.

COROLLARY 3.7. $T_1(\mathscr{L}, p, I)^* = T_0(\tilde{\mathscr{L}}, \mathscr{L}, q, I)$, $T_0(\tilde{\mathscr{L}}, \mathscr{L}, p, I)^* = T_1(\tilde{\mathscr{L}}, q, I)$, *and similarly with the adjoints on the other side.*

Note that for $z \in \mathscr{D}_0(\tilde{\mathscr{L}}, \mathscr{L}, q, I)$,

$$T_0(\tilde{\mathscr{L}}, \mathscr{L}, q, I)z = \tau^*z + \tilde{\chi}^T(\tilde{\mathbf{A}} - i\tilde{\mathbf{D}}\mathbf{C}^{*-1}\mathbf{D}^*)(z|\chi) = \tau^*z + \tilde{\chi}^T\mathbf{A}^*(z|\chi),$$

and for $y \in \mathscr{D}_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)$

$$T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I)y = \tau y + \chi^T(\mathbf{A} - i\mathbf{D}\mathbf{C}^{-1}\tilde{\mathbf{D}}^*)(y|\tilde{\chi}) = \tau y + \chi^T\tilde{\mathbf{A}}^*(y|\tilde{\chi}).$$

Since for any T such that $T_0(\mathscr{L}, \tilde{\mathscr{L}}, p, I) \subset T \subset T_1(\mathscr{L}, p, I)$ we may modify the form of $\mathscr{L}$ by using the conditions defining $\mathscr{D}(T)$, it is possible to achieve a further simplification relative to T.

THEOREM 3.8. *Let T be the restriction of $T_1(\mathscr{L}, p, I)$ to*

$$\mathscr{D}(T) = \{y \in \mathscr{D}_1(\mathscr{L}, p, I) | \mathbf{P}[\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\chi})] = \mathbf{0}\}$$

where $\mathbf{P}$ is $M \times N$ or rank M, and

$$\mathscr{D}(T^*) = \{z \in \mathscr{D}_1(\tilde{\mathscr{L}}, q, I) | \tilde{\mathbf{P}}[\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi)] = \mathbf{0}\}$$

where $\mathbf{PC}^{-1}\tilde{\mathbf{P}}^* = \mathbf{0}$, and $\mathscr{L}$ and $\tilde{\mathscr{L}}$ are adjoint expressions as usual.

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Then there exists a pair of adjoint expressions $\mathscr{L}_0$ *and* $\tilde{\mathscr{L}}_0$, *defined by* (3.14) *and* (3.15) *using matrices* $\mathbf{A}_0$, $\tilde{\mathbf{A}}_0$, $\mathbf{D}_0$, *and* $\tilde{\mathbf{D}}_0$, *such that*

$$\begin{aligned} &\mathscr{L}_0 y = \mathscr{L} y \quad \text{for all } y \in \mathscr{D}(T),\\ &\tilde{\mathscr{L}}_0 z = \tilde{\mathscr{L}} z \quad \text{for all } z \in \mathscr{D}(T^*),\\ &\tilde{\mathbf{A}}_0 = \mathbf{A}_0^*,\\ &\mathbf{D}_0\mathbf{C}^{-1}\tilde{\mathbf{D}}_0^* = \mathbf{0}, \end{aligned}$$

and $\mathbf{D}$ *and* $\tilde{\mathbf{D}}$ *may be replaced by* $\mathbf{D}_0$ *and* $\tilde{\mathbf{D}}_0$ *respectively in the definitions of* $\mathscr{D}(T)$ *and* $\mathscr{D}(T^*)$.

Furthermore, $\mathbf{A}_0$ *has no non-zero entries off the diagonal, and that all non-zero entries are unity.*

Proof. It is clear that for any $m \times M$ matrix Λ and $m \times (N - M)$ matrix $\tilde{\Lambda}$ we have

$$\mathscr{L}_0 y = \tau y + \chi^T[\mathbf{A}(y|\tilde{\chi}) + i\mathbf{D}\mathbf{V}(y) + i\Lambda\mathbf{P}\{\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}^*(y|\tilde{\chi})\}] = \mathscr{L} y$$

for all $y \in \mathscr{D}(T)$, and

$$\tilde{\mathscr{L}}_0 z = \tau^* z + \tilde{\chi}^T[\tilde{\mathbf{A}}(z|\chi) + i\tilde{\mathbf{D}}\tilde{\mathbf{V}}(z) + i\tilde{\Lambda}\tilde{\mathbf{P}}\{\tilde{\mathbf{V}}(z) + \mathbf{C}^{*-1}\mathbf{D}^*(z|\chi)\}] = \tilde{\mathscr{L}} z$$

for all $z \in \mathscr{D}(T^*)$. Thus

$$\begin{aligned} \mathbf{A}_0 &= \mathbf{A} + i\Lambda\mathbf{P}\mathbf{C}^{-1}\tilde{\mathbf{D}}^*\\ \mathbf{D}_0 &= \mathbf{D} + \Lambda\mathbf{P}\\ \tilde{\mathbf{A}}_0 &= \tilde{\mathbf{A}} + i\tilde{\Lambda}\tilde{\mathbf{P}}\mathbf{C}^{*-1}\mathbf{D}^*\\ \tilde{\mathbf{D}}_0 &= \tilde{\mathbf{D}} + \tilde{\Lambda}\tilde{\mathbf{P}}. \end{aligned}$$

Since

$$\begin{aligned} \mathbf{A}_0 - \tilde{\mathbf{A}}_0^* - i\mathbf{D}_0\mathbf{C}^{-1}\tilde{\mathbf{D}}_0^* &= \mathbf{A} + i\Lambda\mathbf{P}\mathbf{C}^{-1}\tilde{\mathbf{D}}^* - \tilde{\mathbf{A}}^* + i\mathbf{D}\mathbf{C}^{-1}\tilde{\mathbf{P}}^*\tilde{\Lambda}^*\\ &\qquad - i[\mathbf{D} + \Lambda\mathbf{P}]\mathbf{C}^{-1}[\tilde{\mathbf{D}}^* + \tilde{\mathbf{P}}^*\tilde{\Lambda}^*]\\ &= \mathbf{A} - \tilde{\mathbf{A}}^* - i\mathbf{D}\mathbf{C}^{-1}\tilde{\mathbf{D}}^* - i\Lambda[\mathbf{P}\mathbf{C}^{-1}\tilde{\mathbf{P}}^*]\tilde{\Lambda}^* = \mathbf{0}, \end{aligned}$$

it follows that $\mathscr{L}_0$ and $\tilde{\mathscr{L}}_0$ are an adjoint pair. Thus in order to have $\tilde{\mathbf{A}}_0 = \mathbf{A}_0^*$ we need only show that Λ and $\tilde{\Lambda}$ can be chosen so that $\mathbf{D}_0\mathbf{C}^{-1}\tilde{\mathbf{D}}_0^* = 0$.

Recall from the proof of Theorem 3.5 that there are non-singular matrices $\mathbf{R}$ and $\mathbf{S}$ such that the matrix

$$\begin{bmatrix} \mathbf{R}\mathbf{P} \\ \mathbf{S}\tilde{\mathbf{P}}\mathbf{C}^{*-1} \end{bmatrix}$$

is unitary, and thus that

$$\mathbf{I} = \mathbf{P}^*\mathbf{R}^*\mathbf{R}\mathbf{P} + \mathbf{C}^{-1}\tilde{\mathbf{P}}^*\mathbf{S}^*\mathbf{S}\tilde{\mathbf{P}}\mathbf{C}^{*-1}.$$

Thus if we set

$$\Lambda = -\mathbf{D}\mathbf{P}^*\mathbf{R}^*\mathbf{R} \quad \text{and} \quad \tilde{\Lambda} = -\tilde{\mathbf{D}}\mathbf{C}^{*-1}\mathbf{C}^{-1}\tilde{\mathbf{P}}^*\mathbf{S}^*\mathbf{S}$$

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we have

$$\begin{aligned}\mathbf{D}_0\mathbf{C}^{-1}\tilde{\mathbf{D}}_0{}^* &= [\mathbf{D} - \mathbf{D}\mathbf{P}^*\mathbf{R}^*\mathbf{R}\mathbf{P}]\mathbf{C}^{-1}[\tilde{\mathbf{D}} - \tilde{\mathbf{D}}\mathbf{C}^{*-1}\mathbf{C}^{-1}\tilde{\mathbf{P}}^*\mathbf{S}^*\mathbf{S}\tilde{\mathbf{P}}]^* \\ &= \mathbf{D}[\mathbf{I} - \mathbf{P}^*\mathbf{R}^*\mathbf{R}\mathbf{P}]\mathbf{C}^{-1}[\mathbf{I} - \tilde{\mathbf{P}}^*\mathbf{S}^*\mathbf{S}\tilde{\mathbf{P}}\mathbf{C}^{*-1}\mathbf{C}^{-1}]\tilde{\mathbf{D}}^* \\ &= \mathbf{D}[\mathbf{I} - \mathbf{P}^*\mathbf{R}^*\mathbf{R}\mathbf{P} - \mathbf{C}^{-1}\tilde{\mathbf{P}}^*\mathbf{S}^*\mathbf{S}\tilde{\mathbf{P}}\mathbf{C}^{*-1} \\ &\qquad + \mathbf{P}^*\mathbf{R}^*\mathbf{R}[\mathbf{P}\mathbf{C}^{-1}\tilde{\mathbf{P}}^*]\mathbf{S}^*\mathbf{S}\tilde{\mathbf{P}}\mathbf{C}^{*-1}]\mathbf{C}^{-1}\tilde{\mathbf{D}}^* = \mathbf{0}.\end{aligned}$$

Finally, if we set $\chi = \mathbf{G}\chi_0$ and $\tilde{\chi} = \mathbf{H}\tilde{\chi}_0$ where $\mathbf{G}$ and $\mathbf{H}$ are non-singular, we replace $\mathbf{A}_0$ by $\mathbf{G}^T\mathbf{A}_0\overline{\mathbf{H}}$, $\mathbf{D}_0$ by $\mathbf{G}^T\mathbf{D}_0$, $\tilde{\mathbf{A}}_0$ by $\mathbf{H}^T\tilde{\mathbf{A}}_0\overline{\mathbf{G}}$, and $\tilde{\mathbf{D}}_0$ by $\mathbf{H}^T\tilde{\mathbf{D}}_0$. This does not destroy the required properties for any $\mathbf{G}$ and $\mathbf{H}$ so they may be chosen to reduce $\mathbf{A}_0$ to a matrix with zeros off the diagonal and entries on the diagonal either unity or zero.

4. Self-adjointness. If we consider $p = q = 2$ then an expression $\mathscr{L}$ may give rise to self-adjoint operators provided that $T_1(\mathscr{L}, 2, I)^* \subset T_1(\mathscr{L}, 2, I)$. This means that for all y such that $\tilde{\mathbf{V}}(y) + \mathbf{C}^{*-1}\mathbf{D}^*(y|\chi) = \mathbf{0}$ we must have

$$\tau y + \chi^T[\mathbf{A}(y|\tilde{\chi}) + i\mathbf{D}\mathbf{V}(y)] = \tau^* y + \tilde{\chi}^T\mathbf{A}^*(y|\chi). \tag{4.1}$$

Since this domain is dense, and the differential operator $\tau - \tau^*$ cannot have a finite dimensional range on a dense domain unless it vanishes, we have $\tau^* = \tau$. Thus we also have, from (4.1) that

$$\chi^T[\mathbf{A}(y|\tilde{\chi}) + i\mathbf{D}\mathbf{V}(y)] = \tilde{\chi}^T\mathbf{A}^*(y|\chi), \tag{4.2}$$

for all y such that $\tilde{\mathbf{V}}(y) + \mathbf{C}^{*-1}\mathbf{D}^*(y|\tilde{\chi}) = \mathbf{0}$.

Now since $\tau^* = \tau$ it is natural to use $\tilde{\mathbf{V}} = \mathbf{V}$. Thus since $\langle y|z\rangle_{\tau,2,I} = -\overline{\langle z|y\rangle}_{\tau,2,I}$ we have $\mathbf{C}^* = \mathbf{C}$ and (4.2) becomes

$$\chi^T[\mathbf{A} - i\mathbf{D}\mathbf{C}^{-1}\mathbf{D}^*](y|\tilde{\chi}) = \tilde{\chi}^T\mathbf{A}^*(y|\chi). \tag{4.3}$$

It follows that the entries in χ and $\tilde{\chi}$ together are linearly dependent. Thus we may as well write them both as linear combinations of a single linearly independent set χ_0. We shall assume that this has been done, so that $\chi = \tilde{\chi}$ in the first place. Then the linear independence of the entries in χ and the density of the y's implies from (4.3) that

$$\mathbf{A} - \mathbf{A}^* = i\mathbf{D}\mathbf{C}^{-1}\mathbf{D}^*. \tag{4.4}$$

This discussion proves:

THEOREM 4.1. *If $T_1(\mathscr{L}, 2, I)^* \subset T_1(\mathscr{L}, 2, I)$ then $\tau = \tau^*$. If in addition we use $\tilde{\mathbf{V}} = \mathbf{V}$ and write $\tilde{\chi}$ and χ as linear combinations of a single linearly independent set which we denote again by χ, then $\mathscr{L}$ is self-adjoint in the sense that $\mathscr{L}, \mathscr{L}$ is an adjoint pair. Furthermore the restriction*

$$T_0(\mathscr{L}, 2, I) = T_1(\mathscr{L}, 2, I)^*$$

of $T_1(\mathscr{L}, 2, I)$ to

$$\mathscr{D}_0(\mathscr{L}, 2, I) = \{y \in \mathscr{D}_1(\mathscr{L}, 2, I) | \mathbf{V}(y) + \mathbf{C}^{-1}\mathbf{D}^*(y|\chi) = \mathbf{0}\}$$

is a symmetric operator.

Throughout the remainder of this section we shall assume that $\mathscr{L}$ is of the form described in Theorem 4.1, i.e. $p = q = 2$, $\tau = \tau^*$, $\tilde{\mathbf{V}} = \mathbf{V}$, and $\mathscr{L}y = \tau y + \chi^T[\mathbf{A}(y|\chi) + i\mathbf{D}\mathbf{V}(y)]$ where $\mathbf{A}$ and $\mathbf{D}$ satisfy (4.4), and χ is a $m \times 1$ column vector with linearly independent entries.

COROLLARY 4.2. *T is a self-adjoint restriction of $T_1(\mathscr{L}, 2, I)$ if and only if N is even and there is a $(N/2) \times N$ matrix $\mathbf{P}$ of rank $N/2$ such that*

$$\mathbf{P}\mathbf{C}^{-1}\mathbf{P}^* = \mathbf{0} \tag{4.5}$$

and $\mathbf{D}(T) = \{y \in \mathscr{D}_1(\mathscr{L}, 2, I) | \mathbf{P}(\mathbf{V}(y) + \mathbf{C}^{-1}\mathbf{D}^*(y|\chi)) = \mathbf{0}\}$.

Proof. This follows immediately from Theorems 3.5 and 4.1.

We may also, with reference to a self-adjoint T, prove an analogue of Theorem 3.8 and replace $\mathscr{L}$ by a simpler expression $\mathscr{L}_0$ which is equivalent to $\mathscr{L}$ in $\mathscr{D}(T)$.

THEOREM 4.3. *If T is a self-adjoint restriction of $T_1(\mathscr{L}, 2, I)$ as described in Corollary 4.2, then there is a self-adjoint expression $\mathscr{L}_0$:*

$$\mathscr{L}_0 y = \tau y + \chi^T[\mathbf{A}_0(y|\chi) + i\mathbf{D}_0\mathbf{V}(y)] \tag{4.6}$$

such that

$$\begin{aligned} &\mathscr{L}_0 y = \mathscr{L} y \quad \text{for all } y \in \mathscr{D}(T), \\ &\mathbf{A}_0 = \mathbf{A}_0^*, \\ &\mathbf{D}_0\mathbf{C}^{-1}\mathbf{D}_0^* = \mathbf{0}, \end{aligned} \tag{4.7}$$

and $\mathscr{D}(T) = \{y \in \mathscr{D}_1(\mathscr{L}_0, 2, I) | \mathbf{P}(\mathbf{V}(y) + \mathbf{C}^{-1}\mathbf{D}_0^*(y|\chi)) = \mathbf{0}\}$, *where* $\mathbf{P}\mathbf{C}^{-1}\mathbf{P}^* = \mathbf{0}$. *Furthermore,* $\mathbf{A}_0$ *is a diagonal matrix with diagonal entries* ± 1 *or* 0.

Proof. Clearly

$$\mathscr{L}_0 y = \tau y + \chi^T[\mathbf{A}(y|\chi) + i\mathbf{D}\mathbf{V}(y) + i\Lambda\mathbf{P}(\mathbf{V}(y) + \mathbf{C}^{-1}\mathbf{D}^*(y|\chi))] = \mathscr{L} y$$

for all $y \in \mathbf{D}(T)$. Thus $\mathbf{A}_0 = \mathbf{A} + i\Lambda\mathbf{P}\mathbf{C}^{-1}\mathbf{D}^*$ and $\mathbf{D}_0 = \mathbf{D} + \Lambda\mathbf{P}$. It is easy to verify that $\mathscr{L}_0$ is self-adjoint, and that the expression for $\mathbf{D}(T)$ in (4.7) follows from (4.5). Thus we must verify that it is possible to choose Λ so that

$$\begin{aligned} 0 = \mathbf{A}_0 - \mathbf{A}_0^* &= i\mathbf{D}_0\mathbf{C}^{-1}\mathbf{D}_0^* \\ &= i[\mathbf{D}\mathbf{C}^{-1}\mathbf{D}^* + \mathbf{D}\mathbf{C}^{-1}\mathbf{P}^*\Lambda^* + \Lambda\mathbf{P}\mathbf{C}^{-1}\mathbf{D}^*] \end{aligned} \tag{4.8}$$

using (4.5).

Now replacing $\mathbf{V}(y)$ by a new basis for the boundary functionals for τ on $L_2(I)$, say $\mathbf{V}(y) = \mathbf{W}\mathbf{V}_0(y)$ where $\mathbf{W}$ is non-singular, results in replacing $\mathbf{P}$ by $\mathbf{P}_1 = \mathbf{P}\mathbf{W}$, $\mathbf{D}$ by $\mathbf{D}_1 = \mathbf{D}\mathbf{W}$, and $\mathbf{C}$ by $\mathbf{C}_1 = \mathbf{W}^*\mathbf{C}\mathbf{W}$, without changing $\mathbf{A}$ or Λ. We choose $\mathbf{W}$ so that the Hermitian symmetric matrix $\mathbf{C}_1$ is diagonal, with ± 1's on the diagonal. Thus $\mathbf{C}_1$ is also unitary.

On substitution (4.8) becomes

$$(4.9)\qquad \mathbf{0} = \mathbf{D}_1\mathbf{C}_1^{-1}\mathbf{D}_1^* + \mathbf{D}_1\mathbf{C}_1^{-1}\mathbf{P}_1^*\Lambda^* + \Lambda\mathbf{P}_1\mathbf{C}_1^{-1}\mathbf{D}_1^*$$

Since (4.5) implies $\mathbf{P}_1\mathbf{C}_1^{-1}\mathbf{P}_1^* = \mathbf{0}$, if $\mathbf{R}$ is a non-singular matrix such that the rows of $\mathbf{RP}_1$ are orthonormal (i.e. $\mathbf{RP}_1\mathbf{P}_1^*\mathbf{R}^* = \mathbf{I}$) then

$$\mathbf{U} = \begin{bmatrix} \mathbf{RP}_1 \\ \mathbf{RP}_1\mathbf{C}_1^{-1} \end{bmatrix}$$

satisfies $\mathbf{UU}^* = \mathbf{I}$ ($\mathbf{C}_1 = \mathbf{C}_1^{-1}$ so $\mathbf{C}_1^2 = \mathbf{I}$).

Thus $\mathbf{U}$ is unitary and

$$\mathbf{I} = \mathbf{U}^*\mathbf{U} = \mathbf{P}_1^*\mathbf{R}^*\mathbf{RP}_1 + \mathbf{C}_1^{-1}\mathbf{P}_1^*\mathbf{R}^*\mathbf{RP}_1\mathbf{C}_1^{-1}.$$

Thus if we set

$$\Lambda = -\mathbf{D}_1\mathbf{P}_1^*\mathbf{R}^*\mathbf{R}$$

the right side of (4.9) is

$$\begin{aligned}&\mathbf{D}_1\mathbf{C}_1^{-1}\mathbf{D}_1^* - \mathbf{D}_1\mathbf{C}_1^{-1}\mathbf{P}_1^*\mathbf{R}^*\mathbf{RP}_1\mathbf{D}_1^* - \mathbf{D}_1\mathbf{P}_1^*\mathbf{R}^*\mathbf{RP}_1\mathbf{C}_1^{-1}\mathbf{D}_1^* \\ &\quad = \mathbf{D}_1\mathbf{C}_1^{-1}[\mathbf{I} - \mathbf{P}_1^*\mathbf{R}^*\mathbf{RP}_1 - \mathbf{C}_1^{-1}\mathbf{P}_1^*\mathbf{R}^*\mathbf{RP}_1\mathbf{C}_1^{-1}]\mathbf{D}_1^* = \mathbf{0}.\end{aligned}$$

Thus (4.9) and (4.8) are satisfied.

Finally, if we replace χ by $\mathbf{H}\chi_0$ where $\mathbf{H}$ is non-singular we replace $\mathbf{A}_0$ by $\mathbf{H}^T\mathbf{A}_0\overline{\mathbf{H}}$ and $\mathbf{D}_0$ by $\mathbf{H}^T\mathbf{D}_0$. This does not affect the Hermitian symmetry of $\mathbf{H}_0$ nor equations (4.7). Clearly $\mathbf{H}$ may be chosen so that $\mathbf{H}^T\mathbf{A}_0\overline{\mathbf{H}}$ is diagonal with diagonal entries ± 1 and 0.

5. Regular problems. We have now set up a consistent system of operators generated by finite dimensional perturbations of differential expressions. Let us now examine the spectral theory of some of these operators.

The case to be considered is the simplest one, where the differential operator τ is regular throughout $I = [a, b]$, and the subdivision of the interval is necessitated by the desire to impose boundary conditions at intermediate points. Thus let $a = a_0 < a_1 < \ldots < a_l = b$ be this partition into subintervals $I_j = [a_{j-1}, a_j]$ $j = 1, \ldots, l$. If τ is of order n a basic set of boundary conditions will be:

$$(5.1)\qquad \begin{aligned} V_{(i-1)n+j}(y) &= y^{(j-1)}(a_{i-1}+), \quad 1 \leqq i \leqq l, 1 \leqq j \leqq n; \\ V_{ln+(i-1)n+j}(y) &= y^{(j-1)}(a_i-), \quad 1 \leqq i \leqq l, 1 \leqq j \leqq n. \end{aligned}$$

Let u_j, $1 \leqq j \leqq n$, be solutions of $\tau y - \lambda y = 0$ on I such that $u_j^{(k-1)}(a_0, \lambda) = \delta_j^k$, $1 \leqq j, k \leqq n$.

Let $W(t, \lambda)$ be the Wronskian of $u_1(t, \lambda), \ldots, u(t, \lambda)$ and $p_0(t)$ be the leading coefficient of τ. Defining

$$(5.2)\qquad K(t, s, \lambda) = \frac{\begin{vmatrix} u_1(s,\lambda) & \dots u_n(s,\lambda) \\ u_1'(s,\lambda) & \dots u_n'(s,\lambda) \\ u_1^{(n-2)}(s,\lambda) & \dots u_n^{(n-2)}(s,\lambda) \\ u_1(t,\lambda) & \dots u_n(t,\lambda) \end{vmatrix}}{p_0(s) W(s,\lambda)}$$

we find that a solution of $\mathscr{L}_0(y) - \lambda y = f$ (where $\mathscr{L}_0$ is as in Theorem 3.8) must be given on I_i by

$$(5.3)\qquad \begin{aligned} y(t,\lambda) = \sum_{j=1}^{n} c_{(i-1)n+j} u_j(t,\lambda) & - \int_{a_{i-1}}^{t} K(t,s,\lambda)\chi^T(s)ds[\mathbf{A}_0(y|\tilde{\chi}) + i\mathbf{D}_0\mathbf{V}(y)] \\ & + \int_{a_{i-1}}^{t} K(t,s,\lambda) f(s) ds, \end{aligned}$$

for some constants $c_1, \dots, c_{ln}$.

Now if $y(t, \lambda)$ is to lie in the domain $\mathscr{D}(T)$, where T is the operator of Theorem 3.8 with P being $ln \times 2ln$, we must be able to determine $\mathbf{V}(y)$, $(y|\tilde{\chi})$, and the constants c_j, such that

$$(5.4)\qquad \mathbf{P}[\mathbf{V}(y) + \mathbf{C}^{-1}\tilde{\mathbf{D}}_0{}^*(y|\tilde{\chi})] = \mathbf{0}.$$

From (5.3) we obtain:

$$(5.5)\qquad V_{(i-1)n+j}(y) = \sum_{\sigma=1}^{n} c_{(i-1)n+\sigma} u_\sigma{}^{(j-1)}(a_{i-1}+, \lambda)$$

for $1 \leqq j \leqq n$ and $1 \leqq i \leqq l$ (nl equations);

$$(5.6)\qquad \begin{aligned} V_{ln+(i-1)n+j}(y) = \sum_{\sigma=1}^{n} c_{(i-1)n+\sigma} u_\sigma{}^{(j-1)}(a_i, \lambda) & \\ - \int_{a_{i-1}}^{a_i} \frac{\partial^{j-1}K}{\partial t^{j-1}}(a_i, s, \lambda)\chi^T(s)ds \cdot [\mathbf{A}_0(y|\tilde{\chi}) + i\mathbf{D}_0\mathbf{V}(y)] & \\ + \int_{a_{i-1}}^{a_i} \frac{\partial^{j-1}K}{\partial t^{j-1}}(a_i, s, \lambda) f(s) ds & \end{aligned}$$

for $1 \leqq j \leqq n$ and $1 \leqq i \leqq l$ (nl equations);

$$(5.7)\qquad \begin{aligned} (y|\tilde{\chi}_\sigma) = \sum_{i=1}^{l}\sum_{j=1}^{n} c_{(i-1)n+j} \int_{a_{i-1}}^{a_i} u_j(t,\lambda)\overline{\tilde{\chi}_\sigma(t)}dt & \\ - \sum_{i=1}^{l} \int_{a_{i-1}}^{a_i} \overline{\tilde{\chi}_\sigma(t)} \int_{a_{i-1}}^{t} K(t,s,\lambda)\chi^T(s)ds \cdot dt[\mathbf{A}_0(y|\tilde{\chi}) + i\mathbf{D}_0\mathbf{V}(y)] & \\ + \sum_{i=1}^{l} \int_{a_{i-1}}^{a_i} \overline{\tilde{\chi}_\sigma(t)} \int_{a_{i-1}}^{t} K(t,s,\lambda) f(s) ds dt & \end{aligned}$$

for $1 \leqq \sigma \leqq \tilde{m}$ ($\tilde{m}$ equations). Since (5.4) constitutes nl equations we have $3nl + \tilde{m}$ linear equations in the $3nl + \tilde{m}$ unknowns $\mathbf{V}(y)$, c_j, and $(y|\tilde{\chi})$.

Provided the determinant of coefficients, $\Delta(\lambda)$, is non-zero these equations can be solved in terms of the non-homogeneous terms:

$$\int_{a_{i-1}}^{a_i} \frac{\partial^{j-1}K}{\partial t^{j-1}}(a_i, s, \lambda)f(s)ds, \quad 1 \leqq i \leqq l, 1 \leqq j \leqq n,$$

and

$$\sum_{i=1}^{r} \int_{a_{i-1}}^{a_i} \tilde{\chi}_\sigma(t) \int_{a_{i-1}}^{t} K(t, s, \lambda)f(s)dsdt, \quad 1 \leqq \sigma \leqq \tilde{m}.$$

Note that the coefficients entering into $\Delta(\lambda)$ are all entire functions of λ, so $\Delta(\lambda)$ will be entire. The zeros of $\Delta(\lambda)$ are precisely the eigenvalues of T (since for those values of λ there is a nontrivial solution of (5.4), (5.5), (5.6), (5.7) for the case $f = 0$, and thus an element of $D(T)$ satisfying $\mathcal{L}_0 y - \lambda y = 0$). Thus T can have an at most countable number of eigenvalues, which can only accumulate at infinity.

If λ is not an eigenvalue, and we solve the above equations for our unknowns and substitute in (5.3) we must arrive at an expression of the form:

$$(5.8) \qquad y(t, \lambda) = \sum_{j=1}^{n} \frac{u_j(t, \lambda)}{\Delta(\lambda)} M_{ij}(f) - \int_{a_{i-1}}^{t} \frac{K(t, s, \lambda)\chi^T(s)ds}{\Delta(\lambda)} \mathbf{N}_i(f) + \int_{a_{i-1}}^{t} K(t, s, \lambda)f(s)ds$$

on I_i, where $M_{ij}(f)$ is a linear combination of the nonhomogeneous terms with coefficients which are entire functions of λ, and $\mathbf{N}_i(f)$ is a m dimensional column vector with entries which are linear combinations of the nonhomogeneous terms with entire functions of λ as coefficients. Leaving out the coefficients, and the $\Delta(\lambda)$ in the denominator, the first two terms in (5.8) involve terms of the following types:

$$(5.9.I) \qquad \int_{a_{\rho-1}}^{a_\rho} u_j(t, \lambda) \frac{\partial^{\sigma-1}K}{\partial t^{\sigma-1}}(a_\rho, s, \lambda)f(s)ds, \quad 1 \leqq \rho \leqq l, 1 \leqq j, \sigma \leqq n;$$

$$(5.9.II) \qquad \int_{a_{\rho-1}}^{a_\rho} u_j(t, \lambda)\left[\int_s^{a_\rho} K(x, s, \lambda)\overline{\tilde{\chi}_\sigma(x)}dx\right]f(s)ds,$$
$$1 \leqq \rho \leqq l, 1 \leqq j \leqq n, 1 \leqq \rho \leqq \tilde{m};$$

$$(5.9.III) \qquad \int_{a_{\rho-1}}^{a_\rho}\left[\int_{a_{i-1}}^{t} K(t, x, \lambda)\chi_\nu(x)dx\right]\frac{\partial^{\sigma-1}K}{\partial t^{\sigma-1}}(a_\rho, s, \lambda)f(s)ds,$$
$$1 \leqq \rho \leqq l, 1 \leqq \nu \leqq n, 1 \leqq \sigma \leqq \tilde{m};$$

$$(5.9.IV) \qquad \int_{a_{\rho-1}}^{a_\rho}\left[\int_{a_{i-1}}^{t} K(t, x, \lambda)\chi_\nu(x)dx\right]\left[\int_s^{a_\rho} K(z, s, \lambda)\overline{\tilde{\chi}_\sigma}(z)dz\right]f(s)ds,$$
$$1 \leqq \rho \leqq l, 1 \leqq \nu \leqq m, 1 \leqq \sigma \leqq \tilde{m}.$$

Thus if we include all terms we have

$$(5.10)\quad y(t, \lambda) = \int_I \mathscr{G}(t, s, \lambda) f(s) ds.$$

For $t \in I_i$, $\mathscr{G}(t, s, \lambda)$ is a combination of the kernels involved in (5.9) with an additional term which is $K(t, s, \lambda)$ for $a_{i-1} \leqq s \leqq t$ and 0 otherwise. It is clear from known properties of K that there is a constant A_i such that

$$\max_{t \in I_i} |y(t, \lambda)| \leqq A_i \|f\|_p^p$$

and for any $\epsilon > 0$, there is $a_i(\epsilon)$ such that if t_1 and t_2 are in I_i and $|t_1 - t_2| < a_i(\epsilon)$, then $|y(t_1, \lambda) - y(t_2, \lambda)| < \epsilon$. Thus the operator $\mathscr{G}(\epsilon)$ defined by the kernel $\mathscr{G}(t, s, \lambda)$ is completely continuous.

Thus in the case where T is self-adjoint, so $\Delta(\lambda)$ cannot be identically zero, the theory of completely continuous operators can be used to show that T has an infinite set of eigenvalues, and that the eigenfunctions form a basis for $L_2(I)$.

The above discussion is summarized in the following theorem.

THEOREM 5.1. *If τ is regular throughout I and $\mathbf{V}(y)$ has $2N$ entries, then any operator T defined by $\mathscr{L}$ and N boundary conditions has an at most countable number of eigenvalues, which can only accumulate at infinity. If λ is not an eigenvalue of T then $(T - \lambda I)^{-1}$ is completely continuous and is an integral operator. If in addition T is self-adjoint, then it has a countable number of eigenvalues and the corresponding eigenfunctions form a basis of $L_2(I)$.*

6. Examples. (i) The simplest example is one with a first order differential operator on a finite interval which is not split into subintervals by singularities or boundary conditions. On $[0, 1]$ let $\tau y = -iy'$, $V_1(y) = y(0)$, $V_2(y) = y(1)$, $\mathbf{C} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. Take $\mathscr{L}y = -iy' + \chi^T[\mathbf{A}(y|\chi) + i\mathbf{d}_1 y(0) + i\mathbf{d}_2 y(1)]$. If $\mathscr{L}$ is self-adjoint and T is a self-adjoint operator obtained by imposing a boundary condition on $\mathscr{L}$, then $\mathscr{L}$ can be replaced by $\mathscr{L}_0$ where

$$\mathscr{L}_0 y = -iy' + \chi_0^T[\mathbf{A}_0(y|\chi_0) + i\mathbf{d}(y(0) + \phi y(1))]$$

where $\mathbf{A}_0$ is diagonal and the only non-zero entries are $+1$ or -1, $\mathbf{d}$ is an $m \times 1$ column vector, and $|\phi| = 1$. The boundary condition must be of the form

$$y(0) + \theta y(1) + (1 - \theta\bar{\phi})\mathbf{d}^*(y|\chi_0) = 0.$$

Note that if $\mathbf{d} = \mathbf{0}$, the self-adjoint operator T defined by $\mathscr{L}_0$ and this boundary condition is a perturbation of the self-adjoint operator S generated by τ and the same boundary condition. If $\mathbf{d} \neq \mathbf{0}$, T is not a perturbation, in the usual sense, of any operator arising from τ.

If $\mathbf{A}_0 = \mathbf{0}$ then $\mathscr{L}_0 y = -iy' + i(y(0) + \phi y(1))\chi_1$ and the boundary condi-

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tion is $y(0) + \theta y(1) + (1 - \theta\bar{\phi})(y|\chi_1) = 0$ where $\chi_1 = \chi_0{}^T\mathbf{d}$ is a single function. In this case, if $\theta = 1, \phi = -1, \chi_1 = i/\sqrt{2}$, eigenvalues are determined by $\tan \lambda + 2\lambda/(\lambda^2 - 1) = 0$.

(ii) We illustrate the effect of splitting the interval by using $\tau y = -iy'$ on $[-1, 1]$ with a split at 0. Here $V_1(y) = y(-1)$, $V_2(y) = y(0-)$, $V_3(y) = y(0+)$, $V_4(y) = y(1)$, and

$$\mathbf{C} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}.$$

Then $\mathscr{L}y = -iy' + \chi^T[\mathbf{A}(y|\chi) + i\mathbf{D}\mathbf{V}(y)]$ requires two boundary conditions to give an operator with empty essential spectrum. It may be possible, of course, to choose these conditions so that one applies only to $[-1, 0]$ and the other to $[0, 1]$ so that the resulting operator is the direct sum of two operators of the type considered in (i). However since in general the boundary conditions involve $(y|\chi) = \int_{-1}^{1} y\, \bar{\chi} dt$, such a situation will be unusual. Also, the imposition of the condition $y(0+) = y(0-)$, which may be possible, will not result in the same condition on the adjoint.

Suppose in particular that we wish $\mathscr{L}$ to be self-adjoint and choose a self-adjoint restriction T. Then the corresponding $\mathscr{L}_0$ will be:

$$\mathscr{L}_0 y = -iy' + \chi_0{}^T[\mathbf{A}_0(y|\chi_0) + i\mathbf{D}_0\mathbf{V}(y)]$$

where $\mathbf{A}_0$ is diagonal with entries ± 1 or 0 along the diagonal and $\mathbf{D}_0 = [d_{ij}]$ $1 \leqq i \leqq m$, $1 \leqq j \leqq 4$ must satisfy $d_{i1}\bar{d}_{j1} + d_{i3}\bar{d}_{j3} = d_{i2}\bar{d}_{j2} + d_{i4}\bar{d}_{j4}$ for all i and j and the boundary conditions must be of the form:

$$\mathbf{P}[\mathbf{V}(y) + \mathbf{C}^{-1}\mathbf{D}_0{}^*(y|\chi)] = \mathbf{0}.$$

where

$$\mathbf{P} = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \end{bmatrix}$$

satisfies $p_{i1}\bar{p}_{j1} + p_{i3}\bar{p}_{j3} = p_{i2}\bar{p}_{j2} + p_{i4}\bar{p}_{j4}$ for $i = 1, 2$ and $j = 1, 2$. Now in order to have the condition $y(0-) - y(0+) = 0$ or $V_2(y) - V_3(y) = 0$ we must have $p_{11} = 0$, $p_{12} = 1$, $p_{13} = -1$, $p_{14} = 0$ and $d_{i3} = -d_{i2}$ for all i. Thus $d_{i1}\bar{d}_{j1} + (-d_{i2})(-\bar{d}_{j2}) = d_{i2}\bar{d}_{j2} + d_{i4}\bar{d}_{j4}$ for all i and j implies that $d_{i1}\bar{d}_{j1} = d_{i4}\bar{d}_{j4}$ for all i and j. Furthermore $p_{23} = -p_{22}$ so that a row operation on P will make $p_{23} = p_{22} = 0$. Then $|p_{24}| = |p_{21}|$, so we may take $p_{21} = 1$ and $p_{24} = \theta$ with $|\theta| = 1$.

This has the effect, when the boundary conditions are used, of eliminating any appearance of $y(0+)$ and $y(0-)$ in $\mathscr{L}_0$, and thus the split of the interval is purely artificial.

(iii) As another example we consider $\tau y = -y''$ on the interval $[0, \infty)$. Here $V_1(y) = y(0)$, $V_2(y) = y'(0)$ and $\mathbf{C} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$.

With $\mathscr{L}y = -y'' + \chi^T[\mathbf{A}(y|\chi) + i\mathbf{d}_1 y(0) + i\mathbf{d}_2 y'(0)]$, if we wish T to be a self-adjoint operator obtained from $\mathscr{L}$ by the imposition of a boundary condition we have

$$\mathscr{L}_0 y = -y'' + \chi_0{}^T[\mathbf{A}_0(y|\chi_0) + i\mathbf{d}_{01}y(0) + i\mathbf{d}_{02}y'(0)]$$

where $\mathbf{A}_0$ is diagonal with entries ± 1 or 0 on the diagonal and $\mathbf{d}_{01} = [d_{j1}]$, $\mathbf{d}_{02} = [d_{j2}]$ with $d_{j1}\bar{d}_{k2} - d_{j2}\bar{d}_{k1} = 0$. If there is any index j such that $d_{j1} \neq 0$ then $d_{j2} = \alpha d_{j1}$ and for $k = j$ we obtain $(2i \text{ Im } \alpha)|d_{j1}|^2 = 0$, so α is real and $d_{k2} = \alpha d_{k1}$ for all k. If there is any index j such that $d_{j2} \neq 0$ then $d_{j1} = \beta d_{j2}$ with β real and $d_{k1} = \beta d_{k2}$ for all k. Thus to include both cases we write $\mathbf{D} = \mathbf{d}[\cos\theta \sin\theta]$ where $\mathbf{d}$ is a $m \times 1$ column vector.

Then the boundary condition is of the form [from similar analysis):

$$\cos\phi\, y(0) + \sin\phi\, y'(0) + i \sin(\phi - \theta)(y|\mathbf{d}^T\chi_0) = 0.$$

Here the i may be absorbed into the vector $\mathbf{d}$ to have:

$$\mathscr{L}_0 y = -y'' + \chi_0{}^T[\mathbf{A}_0(y|\chi_0) + \mathbf{d}\ (\cos\theta\, y(0) + \sin\theta\, y'(0))],$$
$$\cos\phi\, y(0) + \sin\phi\, y'(0) + \sin(\theta - \phi)(y|\mathbf{d}^T\chi_0) = 0.$$

The result is similar if $\tau y = -y''$ is replaced by $\tau y = -y'' + q(t)y$ where $q(t)$ is such that τ is limit point at ∞.

For the simple case $\tau y = -y''$ it is easy to see that the "perturbation" may introduce a finite number of eigenvalues, but the continuous spectrum on $[0, \infty)$ is preserved.

(iv) If $\mathbf{D}_0 = 0$ (as in Theorem 3.8) the operators obtained are, in fact, perturbations of operators associated with τ. For example let S be an operator with pure point spectrum $\{\lambda_j\}_1^\infty$ associated with τ, and assume the eigenfunctions ψ_j of S span $L_2(I)$. We consider the operator T with domain $\mathscr{D}(T) = \mathscr{D}(S)$ given by

$$Ty = \tau y + \chi^T\mathbf{A}(y|\tilde{\chi})$$

we may expand in terms of the eigenfunctions ψ_i of S. Suppose $\chi = \sum \mathbf{x}_j\psi_j$, $\tilde{\chi} = \sum \tilde{\mathbf{x}}_j\psi_j{}^*$ where $\psi_j{}^*$ are the eigenfunctions associated with S^* and form a biorthogonal set with the ψ_j. Suppose $y = \sum y_j\psi_j$ is an eigenfunction of T associated with the eigenvalue λ. Then

$$0 = \sum_j(\lambda_j - \lambda)y_j\psi_j + \sum_j \psi_j\mathbf{x}_j{}^T\mathbf{A} \sum_k y_k\bar{\tilde{\mathbf{x}}}_k = 0.$$

So for each j,

$$(\lambda_j - \lambda)y_j + \sum_k (\tilde{\mathbf{x}}_k{}^*\mathbf{A}^*\mathbf{x}_j)y_k = 0,$$

or equivalently $(\lambda_j - \lambda)y_j + \mathbf{x}_j{}^T\mathbf{A}(y|\tilde{\chi}) = 0$ so that if $\lambda \neq \lambda_j$ for any j,

$$y_j = \frac{\mathbf{x}_j{}^T\mathbf{A}(y|\tilde{\chi})}{\lambda - \lambda_j}.$$

Now

$$(y|\tilde{\chi}) = \sum_j y_j \bar{\bar{\mathbf{x}}}_j = \sum_j \frac{\mathbf{x}_j{}^T\mathbf{A}(y|\tilde{\chi})\bar{\bar{\mathbf{x}}}_j}{\lambda - \lambda_j}$$

constitutes a set of $\tilde{m}$ homogeneous linear equations for the components of $(y|\tilde{\chi})$.

If we are to have an eigenvalue $\lambda = \lambda_N$ then we have for $j \neq N$

$$y_j = \frac{\mathbf{x}_j{}^T\mathbf{A}(y|\tilde{\chi})}{\lambda_N - \lambda_j},$$

and from $j = N$,

$$\mathbf{x}_N{}^T\mathbf{A}(y|\tilde{\chi}) = 0.$$

Now

$$(y|\tilde{\chi}) = y_N\bar{\bar{\mathbf{x}}}_N + \sum_{j \neq N} \frac{\bar{\bar{\mathbf{x}}}_j{}^T\mathbf{A}(y|\tilde{\chi})\bar{\bar{\mathbf{x}}}_j}{\lambda_N - \lambda_j}$$

constitute $\tilde{m}$ homogeneous linear equations for the $\tilde{m}$ components of $(y|\tilde{\chi})$ and for y_N, and $\mathbf{x}_N{}^T\mathbf{A}(y|\tilde{\chi}) = 0$ completes the set.

(v) If, on $L_2(-\infty, \infty)$ we seek to perturb $\tau y = -iy'$ then $\mathscr{D}_1(\tau, 2, I) = \mathscr{D}_0(\tau, 2, I)$. Thus the matrices $\mathbf{D}$ and $\tilde{\mathbf{D}}$ are irrelevant, and we have self-adjoint operators with domain $\mathscr{D}_1(\tau, 2, I)$ of the form

$$\mathscr{L}y = \tau y + \chi^T\mathbf{A}(y|\chi)$$

for any Hermitian symmetric matrix $\mathbf{A}$. Replacing χ by $\chi_0 = \mathbf{B}\chi$ we may assume $\mathbf{A}$ is diagonal with entries ± 1 or 0, and drop the entries in χ_0 corresponding to the 0 entries in $\mathbf{A}$. Let ϵ_j denote the diagonal entries of $\mathbf{A}$.

If we seek to solve $\mathscr{L}y - \lambda y = f$ by using the Fourier transform:

$$\mathscr{F}(h) = \hat{h}(t) - \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-ist}h(s)ds$$

we obtain

$$t\hat{y}(t) + \sum_{j=1}^{m} \hat{\chi}_j(t)\epsilon_j(y|\chi_j) - \lambda\hat{y}(t) = \hat{f}(t).$$

Since the Fourier transform is unitary, $(y|\chi_j) = (\hat{y}|\hat{\chi}_j)$ and

$$(t - \lambda)\hat{y}(t) = \hat{f}(t) - \sum_{j=1}^{m} \hat{\chi}_j(t)\epsilon_j(\hat{y}|\hat{\chi}_j).$$

If λ is not real

$$\hat{y}(t) = \frac{\hat{f}(t)}{t - \lambda} - \sum_{j=1}^{m} \frac{\hat{\chi}_j(t)\epsilon_j(\hat{y}|\hat{\chi}_j)}{t - \lambda}$$

will clearly be in L_2, and

$$(\hat{y}|\hat{\chi}_k) = \left(\frac{\hat{f}(t)}{t-\lambda}\,\middle|\,\hat{\chi}_k\right) - \sum_{j=1}^{m}\left(\frac{\hat{\chi}_j(t)}{t-\lambda}\,\middle|\,\hat{\chi}_k\right)\epsilon_j(\hat{y}|\hat{\chi}_j)$$

$(1 \leqq k \leqq m)$ constitute m equations for the m unknowns $(y|\chi_k) = (\hat{y}|\hat{\chi}_k)$.
Since $\mathscr{L}$ is self-adjoint, its eigenvalues are real and must be such that

$$\hat{y}(t) = -\sum_{j=1}^{m}\frac{\hat{\chi}_j(t)\epsilon_j(\hat{y}|\hat{\chi}_j)}{t-\lambda}$$

belongs to $L_2(-\infty, \infty)$. Thus the only possibilities are those λ for which there exist numbers c_j such that

$$f(t) = -\sum_{j=1}^{m}\frac{\epsilon_j c_j\hat{\chi}_j(t)}{t-\lambda} \in L_2(-\infty,\infty),$$

and this λ can only be an eigenvalue if in addition $c_j = (f|\hat{\chi}_j)$. In particular if $m = 1$, in order that λ be an eigenvalue we must have $(\chi = \chi_1)$, $\hat{\chi}(t)/(t-\lambda) \in L_2(-\infty,\infty)$ and

$$\int_{-\infty}^{\infty}\frac{|\hat{\chi}(t)|^2 dt}{t-\lambda} = -1.$$

Then the corresponding eigenfunction has Fourier transform $-c\epsilon\hat{\chi}(t)/(t-\lambda)$ for c chosen to normalize the eigenfunction.

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