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RESEARCH ARTICLE

Some properties of convex and increasing convex orders under Archimedean copula

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Abstract
In this paper, the ordering properties of convex and increasing convex orders of the dependent random variables are studied. Some closure properties of the convex and increasing convex orders under independent random variables are extended to the dependent random variables under the Archimedean copula. Two applications are provided to illustrate our results.

1. Introduction

Various concepts of stochastic ordering provide a variety of useful comparisons in insurance actuarial science [3, 8–10, 28], risk management [5, 12], and reliability [14, 20, 21, 24], etc. Many researchers have studied convex order and its related orders. These researches mainly focus on stochastic comparisons of independent random variables. For example, Alain and Liu [1] studied the relationships of their random partial sums among the stochastic ordering of independent random variables. Pellerey [25] considered the invariance under convolutions and mixing of the increasing convex order based on independent random variables. Denuit *et al.* [13] obtained the comparison properties of the *m*-convex order for compound sums based on nonnegative independent and identically distributed random variables.

In the real world, interdependent random variables are often encountered. For this reason, it is important to investigate stochastic comparisons of dependent random variables. Copula is widely applied to describe the dependence relationship among random variables because it provides the dependence structure of random variables. Archimedean copulas are a very important class of copulas, which are widely used in many fields of society. For a comprehensive review of copulas and their applications, see [6, 7, 11]. Under the Archimedean copula, Zuo *et al.* [29] established a bivariate joint model on the basis of the marginal distributions. Ji and Liu [18] made use of the three-dimensional Archimedean copula method to assess the $PM_{2.5}$ pollution risk. Fang and Huang [16] used the Archimedean copula for two random vectors to study two inequalities based on majorization. For series-parallel and parallel-series systems, Fang *et al.* [15] investigated various stochastic comparisons based on the Archimedean copula, and Barmalzan *et al.* [4] investigated the hazard rate order and reversed hazard rate order when these components conforming to the general scale model have the dependence under the Archimedean copula. Fang *et al.* [17] discussed stochastic comparisons of the largest order statistics arising from two sets of distribution-free random variables under the Archimedean copula. Ariyafar *et al.* [2] compared the aggregation and minimum of these portfolios with respect to the Laplace transform order using

Archimedean copulas. We find that for dependent random variables, fewer researches considered the properties of convex order and increasing convex order. We consider these issues in this paper.

The rest of this article is organized as follows. Section 2 introduces some related definitions. The closure properties and characterizations of the convex order and increasing convex order under the Archimedean copula dependence, which are the generalizations of some theorems in Shaked and Shanthikumar [26] to dependent case, are developed in Sections 3 and 4. In Section 5, two applications are given. In Section 6, some conclusions are provided.

2. Preliminaries

In this section, we recall the concepts of copula, convex order, increasing convex order. Let $\mathbb{R} = (-\infty, \infty)$ and $\mathbb{R}^+ = [0, \infty)$, $\mathbb{R}^n = \{(z_1, \ldots, z_n) : z_i \in (-\infty, \infty), \text{ for all } i\}$. In this paper, we assume that all expectations exist wherever they are given.

2.1. Copula

Definition 2.1. ([27]): *If for all $t \in (a, b)$, all of derivatives of a function h(t) exist and satisfy $(-1)^k h^{(k)}(t) \geq 0, k \in \{0, 1, \ldots\}$, where $h^{(k)}(\cdot)$ denotes the kth derivative of $h(\cdot)$. Then $h(\cdot)$ is said to be completely monotone on an interval (a, b), $a, b \in \mathbb{R}$.*

Definition 2.2. *For a completely monotone function $\phi : (0, 1] \to [0, \infty)$ with $\phi(1) = 0$ and $\phi(0) = \infty$, then $C_\phi : [0, 1]^n \to [0, 1]$, $C_\phi(\boldsymbol{u}) = \phi^{-1}(\phi(u_1) + \cdots + \phi(u_n))$ is called an Archimedean copula with strict generator ϕ, where $\phi^{-1}(u)$ is the pseudo-inverse of ϕ.*

Let F be an n-dimensional distribution function with marginal distributions $F_1, \ldots, F_n$. A copula associated with F is a distribution function $C : [0, 1]^n \to [0, 1]$ satisfying

$$F(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)).$$

Similarly, a survival copula associated with $\bar{F}$ is a survival function $C : [0, 1]^n \to [0, 1]$ satisfying: $\bar{F}(x_1, \ldots, x_n) = C(\bar{F}_1(x_1), \ldots, \bar{F}_n(x_n))$. For comprehensive discussions on "copula," one may refer to Widder [19, 23, 27].

According to the Bernstein's Theorem 12*a* in Widder [27], we know that, if ϕ^{-1} is a completely monotone function, then there must be a distribution function $L_{\phi^{-1}}$ such that:

$$\phi^{-1}(x) = \int_0^\infty e^{-\alpha x} dL_{\phi^{-1}}(\alpha).$$

Therefore, we have

$$C_\phi(\boldsymbol{u}) = \int_0^\infty \prod_{i=1}^n e^{-\alpha \phi(u_i)} dL_{\phi^{-1}}(\alpha). \tag{1}$$

2.2. Stochastic orders

In this subsection, we give the definitions of the convex order and the increasing convex order.

Definition 2.3. *The random variable X is said to be smaller than the random variable Y in the convex order, written $X \leq_{cx} Y$, if $E[\psi(X)] \leq E[\psi(Y)]$ for all convex functions $\psi : \mathbb{R} \to \mathbb{R}$.*

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Definition 2.4. *The random variable X is said to be smaller than the random variable Y in the increasing convex order, written* $X \leq_{icx} Y$*, if* $E[\psi(X)] \leq E[\psi(Y)]$ *for all increasing convex functions* $\psi : \mathbb{R} \rightarrow \mathbb{R}$*.*

Comprehensive discussion on various stochastic orderings and relations between them may refer to [22, 26].

3. The convex order

In this section, some results of convex order under independent random variables are extended to the case of dependent random variables with the Archimedean copula. In order to extend those results, we need the following lemmas.

Lemma 3.1. *Let* $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ *be the joint survival function of* $(U_1, \ldots, U_n)$*, where C is the Archimedean copula with generator* ϕ*. For* $i = 1, \ldots, n$*, let* $U_i(\alpha)$*'s be independent random variables with survival functions* $\exp\{-\alpha\phi(\bar{F}_{U_i})\}$*. Then, for all* $\alpha > 0$ *and all continuous functions* $g : \mathbb{R}^n \rightarrow \mathbb{R}$*, we have*

$$E\left[g\left(U_1, \ldots, U_n\right)\right] = \int_0^\infty E\left[g\left(U_1(\alpha), \ldots, U_n(\alpha)\right)\right] dL_{\phi^{-1}}(\alpha).$$

Proof. Using Eq. (1), for all $\alpha > 0$, the joint survival function of $(U_1, \ldots, U_n)$ can be written as:

$$\begin{aligned}\bar{F}_{U_1,\ldots,U_n}(u_1, \ldots, u_n) &= \phi^{-1}(\phi(\bar{F}_{U_1}(u_1)) + \cdots + \phi(\bar{F}_{U_n}(u_n))) \\ &= \int_0^\infty e^{-\alpha\phi(\bar{F}_{U_1}(u_1))} \times \cdots \times e^{-\alpha\phi(\bar{F}_{U_n}(u_n))} dL_{\phi^{-1}}(\alpha). \end{aligned} \tag{2}$$

Considering that the derivative of the survival function is a negative probability density function, therefore the probability density function of $(U_1, \ldots, U_n)$ is given by:

$$\begin{aligned} f_{U_1,\ldots,U_n}(u_1, \ldots, u_n) &= (-1)^n \frac{\partial^n}{\partial u_1 \ldots \partial u_n} \bar{F}_{U_1,\ldots,U_n}(u_1, \ldots, u_n) \\ &= \int_0^\infty \prod_{k=1}^n h_{U_k}(u_k; \alpha) dL_{\phi^{-1}}(\alpha), \end{aligned}$$

where

$$h_{U_k}(u_k; \alpha) = -\alpha f_{U_k}(u_k) \exp\left(-\alpha\phi(\bar{F}_{U_k}(u_k))\right)\phi'(\bar{F}_{U_k}(u_k)), 1 \leq k \leq n.$$

Clearly, for all $\alpha > 0$, $h_{U_i}(u_i; \alpha)$ is the probability density function of $U_i(\alpha)$. Hence for any continuous function $g : \mathbb{R}^n \rightarrow \mathbb{R}$, we have

$$\begin{aligned} E\left[g\left(U_1, \ldots, U_n\right)\right] &= \int_{u_1} \ldots \int_{u_n} g\left(u_1, \ldots, u_n\right) \int_0^\infty \prod_{i=1}^n h_{U_i}\left(u_i; \alpha\right) dL_{\phi^{-1}}(\alpha) du_1 \ldots du_n \\ &= \int_0^\infty \int_{u_1} \cdots \int_{u_n} g\left(u_1, \ldots, u_n\right) \prod_{i=1}^n h_{U_i}\left(u_i; \alpha\right) du_1 \ldots du_n dL_{\phi^{-1}}(\alpha) \\ &= \int_0^\infty E\left[g\left(U_1(\alpha), \ldots, U_n(\alpha)\right)\right] dL_{\phi^{-1}}(\alpha), \end{aligned} \tag{3}$$

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where we use the Fubini theorem in the second equation and the definition of expectation in the third equation.

Theorem 4.A.8 in Shaked and Shanthikumar [26] provided the following result. □

Lemma 3.2. *Let X, Y, and Θ be random variables such that, for all θ in the support of Θ, we have that $[X \mid \Theta = \theta] \leq_{icx} [Y \mid \Theta = \theta]$. Then $X \leq_{icx} Y$. That is, the increasing convex order is closed under mixtures.*

The following theorem shows that the convex order is still closed to convolution operation under the Archimedean copula, and it is an extension of Theorem 3.A.12(d) in Shaked and Shanthikumar [26].

Theorem 3.3. *Let $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ and $C(\bar{\boldsymbol{F}}_{\boldsymbol{V}_n}(\boldsymbol{v}_n))$ be the joint survival functions of $(U_1, \ldots, U_n)$ and $(V_1, \ldots, V_n)$ respectively, where C is the Archimedean copula with generator ϕ. For all $\alpha > 0$, let the random variables $U_1(\alpha), \ldots, U_n(\alpha)$ $[V_1(\alpha), \ldots, V_n(\alpha)]$ be independent, and the survival function of $U_i(\alpha)$ $[V_i(\alpha)]$ be $\exp\{-\alpha\phi(\bar{F}_{U_i})\}[\exp\{-\alpha\phi(\bar{F}_{V_i})\}]$, respectively. If $U_i(\alpha) \leq_{cx} V_i(\alpha)$ for $i = 1, 2, \ldots, n$, then*

$$\sum_{i=1}^{n} U_i \leq_{cx} \sum_{i=1}^{n} V_i.$$

Proof. Since $U_i(\alpha) \leq_{cx} V_i(\alpha)$ for $i = 1, 2, \ldots, n$, it can be obtained from Theorem 3.A.12(d) in Shaked and Shanthikumar [26] that:

$$\sum_{i=1}^{n} U_i(\alpha) \leq_{cx} \sum_{i=1}^{n} V_i(\alpha).$$

Therefore for all convex functions φ, we have:

$$E\left[\varphi\left(\sum_{i=1}^{n} U_i(\alpha)\right)\right] \leq E\left[\varphi\left(\sum_{i=1}^{n} V_i(\alpha)\right)\right]. \tag{4}$$

Using Lemma 3.1 and Eq. (4), we have:

$$\begin{aligned} E\left[\varphi\left(\sum_{i=1}^{n} U_i\right)\right] &= \int_0^\infty E\left[\varphi\left(\sum_{i=1}^{n} U_i(\alpha)\right) dL_{\phi^{-1}}(\alpha)\right. \\ &\leq \int_0^\infty E\left[\varphi\left(\sum_{i=1}^{n} V_i(\alpha)\right) dL_{\phi^{-1}}(\alpha)\right. \\ &= E\left[\varphi\left(\sum_{i=1}^{n} V_i\right)\right], \text{ for all convex functions } \varphi. \end{aligned}$$

The proof is completed.

Now we extend Theorem 3.3 to random sums case. □

Theorem 3.4. *Suppose that $\{U_i, i \geq 1\}$ and $\{V_i, i \geq 1\}$ are two sequences of random variables, $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ and $C(\bar{\boldsymbol{F}}_{\boldsymbol{V}_n}(\boldsymbol{v}_n))$ are the joint survival functions of $(U_1, \ldots, U_n)$ and $(V_1, \ldots, V_n)$ respectively, where C is the Archimedean copula with generator ϕ. For all $\alpha > 0$, let the random variables*

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$\{U_i(\alpha), i \geq 1\}$ $[\{V_i(\alpha), i \geq 1\}]$ be independent, and the survival function of $U_i(\alpha)$ $[V_i(\alpha)]$ be $\exp\{-\alpha\phi(\bar{F}_{U_i})\}[\exp\{-\alpha\phi(\bar{F}_{V_i})\}]$, respectively. Assume that the nonnegative integer-valued random variable N is independent of U_i's, V_i's, $U_i(\alpha)$'s and $V_i(\alpha)$'s, respectively. Then

$$U_i(\alpha) \leq_{cx} V_i(\alpha), \quad i = 1, 2, \ldots \Rightarrow \sum_{i=1}^{N} U_i \leq_{cx} \sum_{i=1}^{N} V_i.$$

Proof. Since $U_i(\alpha) \leq_{cx} V_i(\alpha), i = 1, 2, \ldots$, we have from Theorem 3.3 that for all convex functions φ,

$$E\left[\varphi\left(\sum_{i=1}^{n} U_i\right)\right] \leq E\left[\varphi\left(\sum_{i=1}^{n} V_i\right)\right], \quad n = 1, 2, \ldots.$$

Therefore, we have that for all convex functions φ,

$$\begin{aligned} E\left[\varphi\left(\sum_{i=1}^{N} U_i\right)\right] &= \sum_{n=1}^{\infty} P(N=n) E\left[\varphi\left(\sum_{i=1}^{n} U_i\right)\right] \\ &\leq \sum_{n=1}^{\infty} P(N=n) E\left[\varphi\left(\sum_{i=1}^{n} V_i\right)\right] \\ &= E\left[\varphi\left(\sum_{i=1}^{N} V_i\right)\right]. \end{aligned}$$

This shows that $\sum_{i=1}^{N} U_i \leq_{cx} \sum_{i=1}^{N} V_i$.

The following theorem shows that Theorem 3.A.13 in Shaked and Shanthikumar [26] can be extended to the Archimedean copula case. □

Theorem 3.5. *Suppose that $\{U_i, i \geq 1\}$ and $\{V_i, i \geq 1\}$ are two sequences of nonnegative identically distributed random variables, $C(\bar{\boldsymbol{F}}_{U_n}(\boldsymbol{u}_n))$ and $C(\bar{\boldsymbol{F}}_{V_n}(v_n))$ are the joint survival functions of $(U_1, \ldots, U_n)$ and $(V_1, \ldots, V_n)$ respectively, where C is the Archimedean copula with generator ϕ. For all $\alpha > 0$, let the random variables $\{U_i(\alpha), i \geq 1\}$ $[\{V_i(\alpha), i \geq 1\}]$ be independent, and the survival function of $U_i(\alpha)$ $[V_i(\alpha)]$ be $\exp\{-\alpha\phi(\bar{F}_{U_i})\}[\exp\{-\alpha\phi(\bar{F}_{V_i})\}]$, respectively. Let N_1 and N_2 be two positive integer-valued random variables that are independent of U_i's, V_i's, $U_i(\alpha)$'s, and $V_i(\alpha)$'s, respectively. If $U_i(\alpha) \leq_{cx} V_i(\alpha)$ for $i = 1, 2, \ldots$, and $N_1 \leq_{cx} N_2$, then*

$$\sum_{i=1}^{N_1} U_i \leq_{cx} \sum_{i=1}^{N_2} V_i.$$

Proof. To prove this theorem, we know from Theorem 3.4 that we only need to show:

$$\sum_{i=1}^{N_1} U_i \leq_{cx} \sum_{i=1}^{N_2} U_i.$$

Let h be a convex function, $T_n = \sum_i^n U_i$ and define $g(n) = E[h(T_n)]$. Then

$$g(n+1) - g(n) = E[h(T_n + U_{n+1}) - h(T_n)].$$

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Note that h is a convex function, and

$$E\left[h\left(T_n+U_{n+1}\right)-h\left(T_n\right)\middle| T_n=z\right]=E\left[h\left(z+U_{n+1}\right)-h\left(z\right)\right]$$
$$=f(z)(say),$$

then $f(z)$ is increasing in z. Since T_n increases in n in the usual stochastic order, it can be obtained that $g(n+1)-g(n)=E\left[f\left(T_n\right)\right]$ increases in n. Hence, $g(n)$ is a convex function of n. Therefore, for $N_1 \leq_{cx} N_2$, we have:

$$E\left[h\left(\sum_{i=1}^{N_1} U_i\right)\right] \leq E\left[h\left(\sum_{i=1}^{N_2} U_i\right)\right].$$

That is,

$$\sum_{i=1}^{N_1} U_i \leq_{cx} \sum_{i=1}^{N_2} U_i. \tag{5}$$

The proof is completed. In particular, taking $V_i = U_i$ in Theorem 3.5, we obtain the following corollary. □

Corollary 3.6. *Suppose that* $\{U_i, i \geq 1\}$ *is a sequence of nonnegative identically distributed random variables,* $C(\bar{\boldsymbol{F}}_{U_n}(\boldsymbol{u}_n))$ *is the joint survival function of* $(U_1,\ldots,U_n)$*, where* C *is the Archimedean copula with generator* ϕ*. Let* N_1 *and* N_2 *be two positive integer-valued random variables, and be independent of* U_i*'s. Then*

$$N_1 \leq_{cx} N_2 \Longrightarrow \sum_{i=1}^{N_1} U_i \leq_{cx} \sum_{i=1}^{N_2} U_i.$$

Remark 3.7. Corollary 3.6 is the generalization of Theorem 8.A.13(b) in Shaked and Shanthikumar [26].

Corollary 3.8. *Under the setup of Theorem 3.5, if there is some positive integer* k *such that:*

$$\sum_{i=1}^{k} U_i(\alpha) \leq_{cx} V_1(\alpha) \text{ and } N_1 \leq_{cx} kN_2,$$

then

$$\sum_{j=1}^{N_1} U_j \leq_{cx} \sum_{j=1}^{N_2} V_j.$$

Proof. Note that $N_1 \leq_{cx} kN_2$, by Corollary 3.6, we have $\sum_{j=1}^{N_1} U_j \leq_{cx} \sum_{j=1}^{kN_2} U_j$. It is observed that $\sum_{j=1}^{kN_2} U_j = \sum_{i=1}^{N_2} \sum_{j=k(i-1)+1}^{ki} U_j$. Under the condition that $\sum_{i=1}^{k} U_i(\alpha) \leq_{cx} V_1(\alpha)$, using Theorem 3.4, we obtain $\sum_{i=1}^{N_2} \sum_{j=k(i-1)+1}^{ki} U_j \leq_{cx} \sum_{i=1}^{N_2} V_i$. Therefore,

$$\sum_{j=1}^{N_1} U_j \leq_{cx} \sum_{j=1}^{N_2} V_j,$$

which is the desired result.

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The above corollary is an extension of Theorem 3.A.14 in Shaked and Shanthikumar [26]. Similarly, Theorem 3.A.15 and 3.A.16 of Shaked and Shanthikumar [26] can be extended to the Archimedean copula case. These results are not stated. □

4. The increasing convex order

In this section, some increasing convex order results under independent random variables are extended to the case of dependent random variables with the Archimedean copula.

Theorem 4.1. *Let* $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ *and* $C(\bar{\boldsymbol{F}}_{\boldsymbol{V}_n}(\boldsymbol{v}_n))$ *be the joint survival functions of* $(U_1, \ldots, U_n)$ *and* $(V_1, \ldots, V_n)$ *respectively, where* C *is the Archimedean copula with generator* ϕ. *For all* $\alpha > 0$, *let the random variables* $U_1(\alpha), \ldots, U_n(\alpha)$ $[V_1(\alpha), \ldots, V_n(\alpha)]$ *be independent, and the survival function of* $U_i(\alpha)$ $[V_i(\alpha)]$ *be* $\exp\{-\alpha\phi(\bar{F}_{U_i})\}[\exp\{-\alpha\phi(\bar{F}_{V_i})\}]$, *respectively. If* $U_i(\alpha) \leq_{icx} V_i(\alpha)$ *for* $i = 1, 2, \ldots, n$, *then*

$$\sum_{i=1}^{n} U_i \leq_{icx} \sum_{i=1}^{n} V_i.$$

The above theorem is the generalization of Theorem 4.A.8(d) in Shaked and Shanthikumar [26]. The proof of this theorem is similar to the proof of the Theorem 3.3, so it is omitted. Furthermore, we have the following result.

Theorem 4.2. *Let* $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ *and* $C(\bar{\boldsymbol{F}}_{\boldsymbol{V}_n}(\boldsymbol{v}_n))$ *be the joint survival functions of* $(U_1, \ldots, U_n)$ *and* $(V_1, \ldots, V_n)$ *respectively, where* C *is the Archimedean copula with generator* ϕ. *For all* $\alpha > 0$, *let the random variables* $U_1(\alpha), \ldots, U_n(\alpha)$ $[V_1(\alpha), \ldots, V_n(\alpha)]$ *be independent, and the survival function of* $U_i(\alpha)$ $[V_i(\alpha)]$ *be* $\exp\{-\alpha\phi(\bar{F}_{U_i})\}[\exp\{-\alpha\phi(\bar{F}_{V_i})\}]$, *respectively. Assume that the nonnegative integer-valued random variable* N *is independent of* U_i*'s,* V_i*'s,* $U_i(\alpha)$*'s and* $V_i(\alpha)$*'s, respectively. Then*

$$U_i(\alpha) \leq_{icx} V_i(\alpha), \quad i = 1, 2, \ldots \Rightarrow \sum_{i=1}^{N} U_i \leq_{icx} \sum_{i=1}^{N} V_i.$$

Proof. Note that the condition that:

$$U_i(\alpha) \leq_{icx} V_i(\alpha), i = 1, 2, \ldots, n,$$

by Theorem 4.1, we have:

$$\sum_{i=1}^{n} U_i \leq_{icx} \sum_{i=1}^{n} V_i,$$

for each $n \in \{1, 2, \ldots\}$. This means,

$$\left(\sum_{i=1}^{N} U_i | N = n\right) \leq_{icx} \left(\sum_{i=1}^{N} V_i | N = n\right),$$

for each $n \in \{1, 2, \ldots\}$. Using Lemma 3.2, the desired result is immediately obtained.

In addition, we can extend Theorem 4.A.9 in Shaked and Shanthikumar [26]. The proof is similar to the proof of Theorem 3.5, so it is omitted. □

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Theorem 4.3. *Suppose that* $\{U_i, i \geq 1\}$ *and* $\{V_i, i \geq 1\}$ *are two sequences of nonnegative identically distributed random variables,* $C(\bar{\boldsymbol{F}}_{U_n}(u_n))$ *and* $C(\bar{\boldsymbol{F}}_{V_n}(v_n))$ *are the joint survival functions of* $(U_1, \ldots, U_n)$ *and* $(V_1, \ldots, V_n)$ *respectively, where* C *is the Archimedean copula with generator* ϕ. *For all* $\alpha > 0$, *let the random variables* $U_1(\alpha), U_2(\alpha), \ldots$ $[V_1(\alpha), V_2(\alpha), \ldots]$ *be independent, and the survival function of* $U_i(\alpha)$ $[V_i(\alpha)]$ *be given by* $\exp\{-\alpha\phi(\bar{F}_{U_i})\}[\exp\{-\alpha\phi(\bar{F}_{V_i})\}]$, *respectively. Let* N_1 *and* N_2 *be two positive integer-valued random variables, and be independent of* U_i*'s,* V_i*'s,* $U_i(\alpha)$*'s and* $V_i(\alpha)$*'s, respectively. If* $U_i(\alpha) \leq_{icx} V_i(\alpha)$ *for* $i = 1, 2, \ldots$, *and* $N_1 \leq_{icx} N_2$, *then*

$$\sum_{i=1}^{N_1} U_i \leq_{icx} \sum_{i=1}^{N_2} V_i.$$

Particularly, taking $V_k = U_k$ in Theorem 4.3, we obtain the following result, which is the extension of Theorem 8.A.13(a) in Shaked and Shanthikumar [26].

Corollary 4.4. *Suppose that* $\{U_i, i \geq 1\}$ *is a sequence of nonnegative identically distributed random variables,* $C(\bar{\boldsymbol{F}}_{U_n}(\boldsymbol{u}_n))$ *is the joint survival function of* $(U_1, \ldots, U_n)$, *where* C *is the Archimedean copula with generator* ϕ. *Let* N_1 *and* N_2 *be two positive integer-valued random variables that are independent of* U_i*'s. Then*

$$N_1 \leq_{icx} N_2 \Longrightarrow \sum_{i=1}^{N_1} U_i \leq_{icx} \sum_{i=1}^{N_2} U_i.$$

Corollary 4.5. *Under the setup of Theorem 4.3, if there is some positive integer* k *such that:*

$$\sum_{i=1}^{k} U_i(\alpha) \leq_{icx} V_1(\alpha) \quad \text{and} \quad N_1 \leq_{icx} kN_2,$$

then

$$\sum_{j=1}^{N_1} U_j \leq_{icx} \sum_{j=1}^{N_2} V_j.$$

The above corollary is an extension of Theorem 4.A.12 in Shaked and Shanthikumar [26]. Similarly, Theorem 4.A.13 and 4.A.14 of Shaked and Shanthikumar [26] can be extended to the Archimedean copula case. These results are not stated.

The following theorem shows that Theorem 4.A.15 in Shaked and Shanthikumar [26] can be extended to the Archimedean copula dependent case.

Theorem 4.6. *Let* $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ *and* $C(\bar{\boldsymbol{F}}_{\boldsymbol{V}_n}(\boldsymbol{v}_n))$ *be the joint survival functions of* $(U_1, \ldots, U_n)$ *and* $(V_1, \ldots, V_n)$ *respectively, where* C *is the Archimedean copula with generator* ϕ. *For all* $\alpha > 0$, *let the random variables* $U_1(\alpha), \ldots, U_n(\alpha)$ $[V_1(\alpha), \ldots, V_n(\alpha)]$ *be independent, and the survival function of* $U_i(\alpha)$ $[V_i(\alpha)]$ *be* $\exp\{-\alpha\phi(\bar{F}_{U_i})\}[\exp\{-\alpha\phi(\bar{F}_{V_i})\}]$, *respectively. If* $U_i(\alpha) \leq_{icx} V_i(\alpha)$ *for* $i = 1, 2, \ldots, n$, *then for every increasing and componentwise convex function* g, *we have:*

$$g(U_1, U_2, \ldots, U_n) \leq_{icx} g(V_1, V_2, \ldots, V_n).$$

Proof. For any increasing convex function φ, according to Lemma 3.1, we have:

$$E\{\varphi[g(U_1, \ldots, U_n)]\} = \int_0^{\infty} E\{\varphi[g(U_1(\alpha), \ldots, U_n(\alpha))]\} \, dL_{\phi^{-1}}(\alpha), \tag{6}$$

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and

$$E\{\varphi[g(V_1,\dots,V_n)]\} = \int_0^\infty E\{\varphi[g(V_1(\alpha),\dots,V_n(\alpha))]\}\,dL_{\phi^{-1}}(\alpha). \tag{7}$$

Because $U_k(\alpha) \le_{icx} V_k(\alpha), 1 \le k \le n$ for all $\alpha > 0$, we get from Theorem 4.A.15 in Shaked and Shanthikumar [26] that

$$g(U_1(\alpha),\dots,U_n(\alpha)) \le_{icx} g(V_1(\alpha),\dots,V_n(\alpha)).$$

This means that for any increasing convex function φ, we have:

$$E[\varphi(g(U_1(\alpha),\dots,U_n(\alpha)))] \le E[\varphi(g(V_1(\alpha),\dots,V_n(\alpha)))]. \tag{8}$$

Using (6), (7) and (8), we obtain:

$$E[\varphi(g(U_1,\dots,U_n))] \le E[\varphi(g(V_1,\dots,V_n))].$$

Hence we have $g(U_1,\dots,U_n) \le_{icx} g(V_1,\dots,V_n)$.

In particular, Theorem 4.1 can be obtained by taking $g(U_1,\dots,U_n) = \sum_{i=1}^n U_i$. Similarly, we can extend Theorem 4.A.16 in Shaked and Shanthikumar [26]. The proof is omitted. □

Corollary 4.7. *Under the setup of Theorem 4.6. Then*

$$U_i(\alpha) \le_{icx} V_i(\alpha), \quad i = 1,2,\dots,n \Rightarrow max\{U_1,U_2,\cdots,U_n\} \le_{icx} max\{V_1,V_2,\cdots,V_n\}.$$

The following corollary shows the comparison results of the sums and products of the dependent random variables, which have very important applications in the fields of finance and insurance.

Corollary 4.8. *Under the setup of Theorem 4.6, assume that $a_1,\dots,a_n$ are nonnegative real numbers, and $U_i(\alpha) \le_{icx} V_i(\alpha), i = 1,2,\dots,n$. Then*

(i) $\sum_{k=1}^n a_k U_k \le_{icx} \sum_{k=1}^n a_k V_k$;
(ii) if $(U_1,\dots,U_n)$ and $(V_1,\dots,V_n)$ are nonnegative, we have: $\prod_{k=1}^n a_k U_k \le_{icx} \prod_{k=1}^n a_k V_k$.

Remark 4.9. It should be emphasized that all the above theorems and corollaries still hold if $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ and $C(\bar{\boldsymbol{F}}_{\boldsymbol{V}_n}(\boldsymbol{v}_n))$ are replaced by the joint distribution functions $C\left(\boldsymbol{F}_{\boldsymbol{U}_n}(\boldsymbol{u}_n)\right)$ and

$C\left(\boldsymbol{F}_{\boldsymbol{V}_n}\left(\boldsymbol{v}_n\right)\right)$, and the survival functions of $U_i(\alpha)$ and $V_i(\alpha)$ are replaced by the distribution functions $\exp\left\{-\alpha\phi\left(F_{U_i}\right)\right\}$ and $\exp\left\{-\alpha\phi\left(F_{V_i}\right)\right\}$.

5. Two applications

In this section, two potential applications of our results are presented. One application is to compare the lifetimes of two systems subjected to shocks. Compound Poisson processes are usually used to describe the wear accumulated by systems during time. Suppose that the system is subjected to shocks arriving according to the Poisson process $\{N(t), t \geq 0\}$, and that the ith shock causes a nonnegative damage U_i $[V_i]$. Then $\sum_{i=1}^{N(t)} U_i$ and $\sum_{i=1}^{N(t)} V_i$ denote the total wear accumulated up to time t by the two systems, respectively. It is usually assumed that $\{U_1, U_2, \dots\}$ $[\{V_1, V_2, \dots\}]$ is the sequence of independent random variables. In practice, however, this assumption of independence often fails. The following theorem, which is obtained by using Theorem 3.4, can be used to compare these two totals.

Theorem 5.1. *Suppose that $\{U_i,\ i \geq 1\}$ and $\{V_i,\ i \geq 1\}$ are two nonnegative dependent sequences of random variables, $C(\bar{\boldsymbol{F}}_{\boldsymbol{U}_n}(\boldsymbol{u}_n))$ and $C(\bar{\boldsymbol{F}}_{\boldsymbol{V}_n}(\boldsymbol{v}_n))$ are the joint survival functions of $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ respectively, where C is the Archimedean copula with generator ϕ. For all $\alpha > 0$, let the random variables $\{U_i(\alpha),\ i \geq 1\}$ $[\{V_i(\alpha),\ i \geq 1\}]$ be independent, and the survival function of $U_i(\alpha)$ $[V_i(\alpha)]$ be $\exp\{-\alpha\phi(\bar{F}_{U_i}(u_i))\}$ $[\exp\{-\alpha\phi(\bar{F}_{V_i}(v_i))\}]$, respectively. Assume that the Poisson process N(t) is independent of U_i's, V_i's, $U_i(\alpha)$'s and $V_i(\alpha)$'s, respectively. Then,*

$$U_i(\alpha) \leq_{cx} V_i(\alpha), \quad i = 1, 2, \dots \Rightarrow \sum_{i=1}^{N(t)} U_i \leq_{cx} \sum_{i=1}^{N(t)} V_i.$$

Another application is to compare the average life span of two populations. Assume that each population has n individuals. U_i $[V_i]$ denotes the life span of the ith individual. Then $\sum_{i=1}^{n} U_i/n$ and $\sum_{i=1}^{n} V_i/n$ denote the average life span of the two populations. Similar to the first application, we assume that $U_1, \dots, U_n$ $[V_1, \dots, V_n]$ are dependent. We apply Theorem 4.1 to obtain the following result for the comparison of two populations in the sense of the increasing convex order.

Theorem 5.2. *Suppose that $\{U_i, n \geq i \geq 1\}$ and $\{V_i, n \geq i \geq 1\}$ are two sequences of nonnegative random variables, $C(\bar{\boldsymbol{F}}_{U_n}(u_n))$ and $C(\bar{\boldsymbol{F}}_{V_n}(v_n))$ are the joint survival functions of $(U_1, \dots, U_n)$ and $(V_1, \dots, V_n)$ respectively, where C is the Archimedean copula with generator ϕ. Let the random variables $U_1(\alpha), U_2(\alpha), \dots$ $[V_1(\alpha), V_2(\alpha), \dots]$ be independent, and the survival function of $U_i(\alpha)$ $[V_i(\alpha)]$ be given by $\exp\{-\alpha\phi(\bar{F}_{U_i}(u_i))\}$ $[\exp\{-\alpha\phi(\bar{F}_{V_i}(v_i))\}]$, respectively, for all $\alpha > 0$. If $U_i(\alpha) \leq_{icx} V_i(\alpha)$ for $i = 1, 2, \dots, n$, then:*

$$\sum_{i=1}^{n} U_i/n \leq_{icx} \sum_{i=1}^{n} V_i/n.$$

6. Conclusions

It is an important issue to compare the stochastic ordering for the functions of random variables especially in the insurance and economic context for the close relationship between stochastic orders and risk measurement. Among various stochastic orders, convex order and increasing convex order have many applications in the fields of finance and insurance. Given that Archimedean copulas are often used to describe the dependence of the risk factors and claim amounts, it is meaningful to study the properties of convex order and increasing convex order under the Archimedean copula dependence.

Under the Archimedean copula, the results of some well-known independent random variables on convex order and increasing convex order are generalized to the dependent case, which include the functions and random sums of dependent random variables. Two applications in reliability are also provided to illustrate the main results.

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