PIECEWISE ANALYTIC SOLUTIONS OF MIXED BOUNDARY VALUE PROBLEMS

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1. Introduction. In a mixed problem one is required to find a solution of a system of partial differential equations when the values of certain combinations of the derivatives are given on two or more distinct intersecting surfaces. If the differential equations arise from some physical process, the correct boundary conditions are usually apparent. Many particular problems of this type have been solved by special methods such as separation of variables and the method of images. However, no general criterion has been given for what constitutes a correctly set mixed problem. In fact such problems have usually been formulated in connection with hyperbolic differential equations with data prescribed on two surfaces (called the initial and boundary surfaces).

Mixed problems for normal hyperbolic linear equations of the second order have been treated by Kryzyanski and Schauder **(7)**, by Ladyzhenskaya **(8)**, and in **(3**; **9**; **10**; **11)**. As for higher-order equations, little work has been done. Campbell and Robinson **(2)** have solved a mixed problem for a hyperbolic equation of general order in two independent variables by an extension of Picard's iteration method. Duff **(4)** treats mixed problems for weakly hyperbolic analytic systems in N independent variables. Later in **(5)** he solves a mixed problem for a single weakly hyperbolic equation of degree R in N independent variables by replacing this equation by an equivalent system of first-order equations.

Herein we study mixed problems for weakly hyperbolic systems of general order in N independent variables. In the first part, we treat linear analytic systems of the first order in which we have a characteristic boundary surface. By modifying the procedure given in **(4)**, we construct a piecewise analytic solution which takes the desired values. Next, we reduce a high-order linear system to a first-order system and apply the previous results.

In the second part (beginning with §4) we consider mixed problems for non-linear analytic systems. The method of attack is to derive theorems for quasi-linear systems which we apply later. In order to apply these theorems, we use the fact that a first-order non-linear system can be reduced to a quasi-linear system. Finally, we deduce several theorems for higher-order non-linear

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systems of equations, replacing them by systems of non-linear first-order equations.

2.1. A mixed boundary value problem. Consider the system of R linear partial differential equations of the first order

$$(2.1.1)\qquad a_{rs}{}^{i}\frac{\partial u_s}{\partial x^i} + b_{rs}u_s = f_r \qquad (r, s = 1, \ldots, R; i = 1, \ldots, N)$$

in R dependent variables u_r and N independent variables x^i. Here summation over the repeated indices s and i is understood. All coefficients are real analytic functions of the variables x^i.

A characteristic surface **(12**, p. 26**)**, $\phi(x^i) = 0$, of the equations (2.1.1) is a solution of

$$(2.1.2)\qquad |A| = |(a_{rs}{}^{i}\,\partial\phi/\partial x^i)| = 0.$$

If A has rank $R - \mu$ at the points of $\phi = 0$, then μ will be called the *multiplicity* of the characteristic surface $\phi = 0$.

Since the characteristic equation (2.1.2) is of degree R, there will be a number of characteristic surfaces, G, of (2.1.1) which pass through the edge $C = S \cap T$ into the domain defined by the initial surface S: $t = 0$ and the boundary surface T: $x = 0$. We denote the regions between the characteristic surfaces G_i and G_{i+1} and bounded by the edge C by D_i.

We seek a solution of the differential equations which is defined in D and has the following properties:

(1) The solution functions u_r are analytic in the closure of each sector domain D_i.

(2) The u_r are continuous across each G_i $(i = 1, \ldots, k)$.

(3) The u_r take given analytic values on S.

(4) The u_r satisfy k_0 independent linear boundary conditions on T.

We emphasize that not all of the characteristic surfaces lying in D need be selected in this way. Hereafter we shall have no need to mention any but the k_0 selected characteristic surfaces.

The above problem has been studied by Duff **(4)**. In his paper the boundary surface T is not characteristic. Here we allow T to be characteristic and of multiplicity L.

The system (2.1.1) can be reduced to the following form **(4)**:

$$(2.1.3)\qquad \frac{\partial w_r}{\partial t} = \lambda_r\frac{\partial w_r}{\partial x} + b_{rs}{}^{\rho}\frac{\partial w_s}{\partial x^\rho} + c_{rq}{}^{\rho}\frac{\partial u_q}{\partial x^\rho} + f_{rq}u_q + e_{rs}w_s$$

$$(\rho = 3, \ldots, N; r, s = 1, \ldots, R; q = 1, \ldots, L),$$

$$(2.1.4)\qquad \frac{\partial u_q}{\partial t} = \sum_{i=1}^{q-1}\left(g_{qi}\frac{\partial u_i}{\partial x} + h_{qi}\frac{\partial u_i}{\partial x^\rho}\right) + k_{qr}w_r.$$

In (2.1.3) there is no summation over the repeated index r, while in (2.1.4) when $q = 1$, the expression in parentheses is not present. Also b_{rs}, c_{rq}, e_{rs}, f_{rs}, g_{qi}, h_{qi}, and k_{qr} are analytic functions of the variables s, t, and x.

From (2.1.3) and (2.1.4) we see that a characteristic surface $\phi(t, x, x^\rho)$ satisfies

$$(\partial\phi/\partial t)^L \det(\delta_{rs} - \lambda_r \phi_x \delta_{rs} - b_{rs}{}^\rho \phi_{x^\rho}) = 0.$$

The K_0 boundary conditions are

$$w_r = \textstyle\sum_\sigma c_{r\sigma} w_\sigma + g_r \qquad (r = 1, \ldots, K_0; \sigma = K_0 + 1, \ldots, R). \tag{2.1.5}$$

The datum functions $g_r(t, x^\rho)$ are real and analytic on T and since our solution is to be continuous, we postulate that $g_r(0, x^\rho) = 0$. The initial conditions are given by

$$u_s = 0, \quad w_r = 0 \qquad (s = 1, \ldots, L; r = 1, \ldots, R). \tag{2.1.6}$$

2.2. The discontinuity expansion. In order to expand the functions about the selected characteristic surfaces it is convenient to make the following change of variables:

$$T_i = t + \phi^i(x, x^\rho), \quad S_i = x, \quad x_i{}^\rho = x^\rho \qquad (i = 1, 2, \ldots, K_0),$$

where $t + \phi^i = 0$ is the ith characteristic surface through the edge C. Then equation (2.1.3) becomes

$$a_{rs}{}^i \frac{\partial w_s}{\partial T_i} = \lambda_r \frac{\partial w_r}{\partial S_i} + b_{rs}{}^\rho \frac{\partial w_s}{\partial x_i{}^\rho} + C_{rq}{}^\rho \frac{\partial u_q}{\partial x_i{}^\rho} + e_{rs} w_s + f_{rq} u_q + c_{rq}{}^\rho \frac{\partial u_q}{\partial T_i} \frac{\partial \phi^i}{\partial x_i{}^\rho}, \tag{2.2.1}$$

where

$$a_{rs}{}^i = (\delta_{rs} - \phi_x{}^i \lambda_r \delta_{rs} - b_{rs}{}^\rho \phi_{x^\rho}{}^i). \tag{2.2.2}$$

On the other hand, equation (2.1.4) can be put in the form

$$\frac{\partial u_q}{\partial t} = \sum_{j=1}^{q-1} \left(\bar{g}_{qj} \frac{\partial u_j}{\partial S_i} + \bar{h}_{qj}{}^\rho \frac{\partial u_j}{\partial x_i{}^\rho} \right) + \bar{K}_{rq} w_r. \tag{2.2.3}$$

This follows because of the special form of the equations (2.1.5) and the fact that $\partial v/\partial T_i = \partial v/\partial t$. Using (2.2.3) we can rewrite (2.2.1) as follows:

$$a_{rs}{}^i \frac{\partial w_s}{\partial T_i} = \lambda_r \frac{\partial w_s}{\partial S_i} + b_{rs}{}^\rho \frac{\partial w_s}{\partial x_i{}^\rho} + e_{rs} w_s + f_{rq} u_q + C_{rq}{}^\rho \bar{F}_q \phi_{x^\rho}{}^i. \tag{2.2.4}$$

Since $t + \phi^i = 0$ is a characteristic surface of multiplicity one, $(a_{rs}{}^i)$ is of rank $R - 1$. Let $n_r{}^i$ be the null vector of $(a_{rs}{}^i)$; thus

$$\sum_{r=1}^{R} n_r{}^i a_{rs}{}^i = 0.$$

Multiplying (2.2.4) by n_r^i and making the replacements indicated below, we obtain

$$(2.2.5)\qquad \frac{\partial \bar{w}_i^{\,i}}{\partial S_i} = \bar{b}_{is}\frac{\partial w_s}{\partial x_i^{\,\rho}} + \bar{c}_{iq}^{\,\rho}\frac{\partial u_q}{\partial x_i^{\,\rho}} + \bar{e}_{is}w_s + \bar{f}_{iq}u_q + \bar{c}_{iq}^{\,\rho}F_q\,\phi_{x^\rho}^{\,i}$$

where F_q is linear in $\partial u_j/\partial S_i$, $\partial u_j/\partial x_i^\rho$, and w_r and we have set

$$(2.2.6)\qquad \bar{w}_i^{\,i} = \frac{1}{\lambda_i}\sum_{s=1}^{R}\lambda_s n_s^{\,i} w_s \qquad (i = 1, 2, \ldots, K_0).$$

There is no summation over the repeated index i.

Let us also set

$$(2.2.7)\qquad \bar{w}_r^{\,i} = \frac{1}{a_{rr}^{\,i}}\sum_{s=1}^{R} a_{rs}w_s \qquad (r \neq i;\ r = 1, 2, \ldots, R).$$

For a fixed i, the equations (2.2.6) and (2.2.7) define a transformation from the variables w_s to the variables $\bar{w}_s^{\,i}$. We must show that this transformation is non-singular.

We note that

$$\left(\frac{\partial \phi^i}{\partial x}\right)_{x=0} = \left(\frac{1}{\lambda_i}\right)_{x=0}, \qquad \left(\frac{\partial \phi^i}{\partial x^\rho}\right)_{x=0} = 0$$

since the characteristic surfaces pass through the edge C. Thus $(a_{rs}^{\,i})_{x=0}$ is a diagonal matrix whose diagonal elements $(a_{rr}^{\,i})_{x=0}$ are $1 - \lambda_r/\lambda_i$ $(r \neq i)$ and whose ith diagonal element is zero. Thus we can choose $(n_r^{\,i})_{x=0} = \delta_{ir}$ and since the null vectors are ratios of analytic functions and are finite on the edge C, they are analytic functions in a neighbourhood of the edge C. If we can show that the determinant of the transformation (2.2.6)–(2.2.7) is not zero, then the transformation is non-singular. But the above facts show that the value of the determinant is unity. This determinant is an analytic function and is non-zero in the neighbourhood of the edge C. Thus there exists an analytic inverse transformation for each i. Let it be given by

$$(2.2.8)\qquad w_r = \sum_{s=1}^{R}\beta_{rs}^{\,i}\,\bar{w}_s^{\,i}.$$

By using equations (2.2.6)–(2.2.8), we can reduce equations (2.2.4) and (2.2.5) to the following:

$$(2.2.9)\qquad \frac{\partial \bar{w}_r^{\,i}}{\partial T_i} = \sum_{s=1}^{R}\lambda_r\beta_{rs}^{\,i}\frac{\partial \bar{w}_s^{\,i}}{\partial S_i} + L_r^{\,i}(\bar{w}_r^{\,i}, u_q^{\,i}) \qquad (r = 1, \ldots, R;\ r \neq i),$$

$$(2.2.10)\qquad \frac{\partial \bar{w}_i^{\,i}}{\partial S_i} = L_i^{\,i}(\bar{w}_r^{\,i}, u_q^{\,i}),$$

where the $L_s^{\,i}$ are linear differential operators in $\partial/\partial x^\rho$, $\partial/\partial S_i$; however only the variables $u_q^{\,i}$ are differentiated with respect to S_i.

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Let a discontinuity (**4**, pp. 148–156) across G_i of the nth time derivative of a function U be denoted by

$$[D^n u]_i. \tag{2.2.11}$$

This is really the coefficient of T_i^n in the power series for u. Since all discontinuities to be considered are finite and analytic along the G_i, the total discontinuity $[u]_0$ taken across C of a function defined on T is the sum of the limits of the jumps across the G_i:

$$[u]_0 = \sum_{i=1}^{K_0} [u]_{i|S_i=0}. \tag{2.2.12}$$

Let us split (2.2.9) into two groups corresponding to the selected and non-selected characteristic surfaces as follows:

$$\frac{\partial \bar{w}_r{}^i}{\partial T_i} = \sum_{s=1}^{R} \lambda_r \, \beta_{rs}{}^i \frac{\partial w_s{}^i}{\partial S_i} + L_r{}^i(\bar{w}_r{}^i, u_q{}^i) \qquad (r = 1, \ldots, K_0;\ r \neq 1), \tag{2.2.13}$$

$$\frac{\partial \bar{w}_\sigma{}^i}{\partial T_i} = \sum_{s=1}^{R} \lambda_\sigma \, \beta_{\sigma s}{}^i \frac{\partial w_s{}^i}{\partial S_i} + L_\sigma{}^i(\bar{w}_r{}^i, u_q{}^i) \qquad (\sigma = K_0 + 1, \ldots, R). \tag{2.2.14}$$

To these equations we adjoin (2.2.3), which now has the following form (by the use of (2.2.8):

$$\frac{\partial u_q{}^i}{\partial t} = \sum_{j=1}^{q-1} \left[\bar{g}_{qj} \frac{\partial u_j{}^i}{\partial S_i} + \bar{h}_{qj} \frac{\partial u_j{}^i}{\partial x_i^\rho} \right] + \sum_{s=1}^{K} K_{qs} \, {}^i w_s{}^i \, . \tag{2.2.15}$$

The coefficients of g_r, when g_r is expanded in a series of powers of t, will be denoted by g_{rn}; see (2.15).

Zero-order jumps are zero since the solutions are continuous across the chosen characteristic surfaces.

We calculate first-order jumps as follows. From (2.2.14), the discontinuity across G_i is

$$[D\bar{w}_r{}^i]_i = 0 \tag{2.2.16}$$

since all terms on the right are continuous by hypothesis. Similarly, from (2.2.15),

$$[D\bar{w}_\sigma{}^i]_i = 0, \tag{2.2.17}$$

$$[Du_q{}^i]_i = 0. \tag{2.2.18}$$

Thus all first-order jumps vanish except possibly $[D\bar{w}_i{}^i]_i$. To find this quantity, we differentiate (2.2.10) with respect to T_i and take the jump across G_i, obtaining

$$\frac{\partial}{\partial S_i} [D\bar{w}_i{}^i]_i = \frac{\partial}{\partial t} L_i{}^i = a_{ii}{}^\rho \frac{\partial}{\partial x_i{}^\rho} [D\bar{w}_i{}^i]_i + b_{ii}[D\bar{w}_i{}^i]_i. \tag{2.2.19}$$

Here the appropriate coefficients in $L_i{}^i$ have been exhibited. All other terms, being continuous, drop out when the jump operator is applied. This equation has the Cauchy–Kowalewsky normal form (**12**, p. 14) with respect to the edge C in the variables S_i and x_i, since C has the equation $S_i = 0$ on the surface G_i: $T_i = 0$. An initial condition on C for (2.2.19) is now to be found. If we differentiate (2.2.14) with respect to t, take jumps, and then use (2.2.12), we have

$$\text{(2.2.20)}\qquad \begin{aligned}[D\bar{w}_i{}^i]_{i|S_i=0} &= [D\bar{w}_i{}^i]_0 - \sum_{j\neq i} [D\bar{w}_i{}^{(i)}]_{j|S_j=0} \\ &= [D\bar{w}_i{}^i]_0 = \sum_{\sigma} c_{i\sigma}[Dw_\sigma]_0 + g_{i1} = g_{i1}.\end{aligned}$$

Here we have used the fact that

$$(\bar{w}_r{}^i)_{x=s_i=0} = w_r, \qquad (D\bar{w}_r{}^i)_{x=s_i=0} = (Dw_r)_{x=s_i=0}$$

for all i. This follows from the definition (2.2.7) and (2.2.8) of the $\bar{w}_r{}^i$. With this initial condition the single partial differential equation (2.2.19) has a unique solution on G_i. This completes the calculation of the first-order jumps, and it may be noted that the non-homogeneous term g_{i1} induces a first-order contribution only from the corresponding proper variable w_i over the corresponding surface G_i.

If the first non-zero term in the series for g_i is of a higher order n, the only non-zero nth order discontinuity is of the kind just mentioned.

The discontinuities of higher order are found in succession by this process. Suppose all jumps of order $n - 1$ or less are known; let us find those of order n. Differentiating (2.2.13), (2.2.14), and (2.2.15) $n - 1$ times with respect to t and taking jumps over G_i, we obtain

$$\text{(2.2.21)}\qquad \begin{cases} [D^n\bar{w}_r{}^i]_i = \sum_{s=1}^{R} \lambda_r \beta_{rs} \dfrac{\partial}{\partial S_i} [D^{n-1}\bar{w}_s{}^i] + D^{n-1}L_r{}^i + \dots, \\ [D^n\bar{w}_\sigma{}^i]_i = \sum_{s=1}^{R} \lambda_r \beta_{rs} \dfrac{\partial}{\partial S_i} [D^{n-1}\bar{w}_s{}^i]_i + D^{n-1}L_\sigma{}^i + \dots, \\ [D^n u_q{}^i]_i = [D^{n-1}\bar{F}_q]_i. \end{cases}$$

Now the right-hand sides are all known in terms of discontinuities of order $\leqslant n - 1$ already calculated and so the left side is determined.

To find the remaining quantity $[D^n\bar{w}_i{}^i]_i$, we differentiate (2.2.10) n times with respect to t and then the discontinuity across G_i. The result is

$$\text{(2.2.22)}\qquad \begin{aligned}\frac{\partial}{\partial S_i} [D^n\bar{w}_i{}^i]_i = [D^nL_i{}^i]_i = \sum_r a_{ir}{}^\rho \frac{\partial}{\partial x_i{}^\rho} [D^n\bar{w}_i{}^i]_i \\ + \sum_r b_{ir}[D^n\bar{w}_r{}^i]_i + K\end{aligned}$$

where K consists of terms of discontinuity of order n other than $[D^n\bar{w}_i{}^i]_i$ and terms of discontinuity of order less than n. However, both of these are known and we obtain a non-homogeneous equation for $D^n\bar{w}_i{}^i$. The initial condition is now found from (2.1.5) by differentiating n times with respect to t and taking jumps. Using the fact that $\bar{w}_r^{\,i}$ and $D^n\bar{w}_r^{\,i}$ are equal to w_r and $D^n w_r$ respectively on the boundary, we have from (2.2.12)

$$[D^n\bar{w}_i^{\,i}]_{i|S_i=0} = [D^n w_i]_0 - \sum_{j\neq i}^{K_0} [D^n w_i]_{j|S_j=0} \tag{2.2.23}$$

and by Leibnitz's formula used in connection with (2.1.5)

$$[D^n w_i] = \sum_{\sigma,m} c_{i\sigma m} \binom{n}{m} [D^m w_\sigma]_0 + g_{in}. \tag{2.2.24}$$

The right-hand sides are known and the initial values determined. Since (2.2.22) is a non-homogeneous version of (2.2.19), the existence of an analytic solution on G_i follows as in the first-order case.

The recursive construction being complete both for the u_q and w_r, we can now define the series of which the solution functions are composed. The series $\bar{w}_r^{\,i}$ is given by

$$\bar{w}_r^{\,i} = \sum_{n=0}^{\infty} [D^n\bar{w}_r^{\,i}]_i \,(t + \phi^i(x, x^\rho))^n. \tag{2.2.25}$$

The $u_q^{\,i}$ have an analogous expression for their expansion about the ith characteristic surface. If we let $v_r^{\,i}$ stand for either of these expansions, then, as indicated previously, the final formal solution is

$$\begin{aligned} v_r &= \sum_{i=1}^{h} v_r^{\,i} \quad \text{in } D_h \qquad (h = 1, 2, \ldots, K_0),\\ v_r &= 0 \qquad\quad\ \text{in } D_0. \end{aligned} \tag{2.2.26}$$

This solution can be expressed in terms of the w_r by the use of equations (2.2.7) and (2.2.8). To complete our existence proof we must show that these series have a common domain of convergence.

2.3. Convergence of the discontinuity expansion. It can be shown that each of the series (2.2.26) is dominated by the solution of a certain problem wherein only two characteristic surfaces G, T appear and one boundary condition is present. Hence the original problem has a solution. We omit the proof. The interested reader can find the details in (**11**, p. 150). To sum up, we have

THEOREM I. *Let a non-characteristic surface S: $t = 0$ and a characteristic surface T: $x = 0$ relative to the analytic system* (2.1.3) *and* (2.1.4) *intersect in an edge C from which issue into a quadrant at least K_0 distinct characteristic surfaces G_i. Then there exists a solution continuous in the quadrant and analytic except across the G_i which takes given Cauchy data on S, and for which the selected variables w_i take values on T determined by the linear boundary conditions* (2.1.5).

2.4. Uniqueness of the series solution. The expansions of Theorem I imply that the solutions are analytic not only in each sector domain D_i, but also in the closure of D. We can therefore only assert, in general, that the series solution found above is unique in the class of vector functions U having this strong piecewise analyticity. That it is unique in this class follows from the well-defined nature of the construction of the coefficients $[D^n \bar{w}_r^{\ i}]_i$ of the power series.

One use in which uniqueness of the solution in a wider class of real vector functions can be shown is the case when all the roots are real and all the negative roots are select. By the use of Holmgren's theorem we can prove uniqueness within the class of once continuously differentiable vector functions. The details are given in (**4**, p. 154). However, a slight modification is necessary, since we allow some of the characteristic roots to be zero.

THEOREM II. *If all the λ_r are real and all negative λ_r are select, the piecewise analytic solution of Theorem I is the only C' solution of the problem.*

An interesting result follows from the above theorem. Suppose that all the u_r are given by two different sets of analytic functions on the hyperplane S in the regions $x > 0$ and $x < 0$. Although the prescribed values differ in the two regions, they are continuous and have continuous first derivatives on the edge C: $x = 0, t = 0$. If all the characteristic surfaces which pass through the edge C lie above the x-coordinate hyperplane (the region $t > 0$), we can construct a piecewise analytic C' solution in the region $t = 0, x > 0$, which is defined in the region $t > 0$ and takes the prescribed values. A second piecewise solution can also be constructed if we proceed in the clockwise sense, that is, we now begin in the region $x < 0, t = 0$. To construct these solutions, we select a hyperplane H which passes through the edge C and lies between the x-coordinate hyperplane and the first characteristic surface. Then by the Cauchy–Kowalewsky theorem, we can construct a unique analytic solution taking the given initial values, that is, the prescribed values in the region $x > 0, t = 0$. In this manner we determine the values of the u_r on the hyperplane H. Next we make a transformation of coordinates. Let the new $x = 0$ hyperplane be H, while the other coordinate hyperplanes are the same. In this new coordinate system, by considering all the characteristic surfaces as the select ones, we can construct the solutions as in §2.2. By Theorem II the two solutions are identical. We summarize these facts in the following theorem.

THEOREM III. *Let there be given a linear analytic system of R differential equations in R dependent variables and N independent variables and Cauchy data on an initial surface S of dimension $N - 1$. The Cauchy data is C' across an edge C of dimension $N - 2$. If the system has R distinct characteristic surfaces, there exists a unique C' solution taking these piecewise analytic data. It is analytic except across the characteristic surfaces radiating from C, where it is C'.*

3.1. The general-order system. By using the Cauchy–Kowalewsky theorem, as in (**12**), a non-homogeneous linear mixed system can be reduced to a homogeneous mixed system with zero initial conditions. The explicit form of this mixed system of R partial differential equations in R dependent variables and N independent variables is

$$\partial^{k_r} u_r / \partial t^{k_r} = L u_s \tag{3.1.1}$$

where L represents the following linear operator:

$$\begin{aligned} L = \sum_{s=1}^{R} (a_{rs}^{i_1, i_2+1} D_{i_1, i_2+1}^{k_s} + a_{rs}^{i_1, i_2, i_3+1} D_{i_1, i_2, i_3+1}^{k_s} \\ + \ldots + a_{rs}^{i_1, i_2, \ldots, i_l+1} D_{i_1, i_2, \ldots, i_l+1}^{k_s} \\ + \sum_{s=1}^{R} a_{rs}^{i_1, i_2, \ldots, i_N} D_{i_1, i_2, \ldots, i_N}^{i_1+i_2+\ldots+i_N} \end{aligned} \tag{3.1.2}$$

and summation over the repeated indices $i_1, \ldots, i_l$ is understood to be over the range $0 \leqslant i_1 + \ldots + i_l \leqslant k_s - 1$. Here, $x^2 = x$, $x^1 = t$, and

$$a_{rs}^{i_1, i_2, \ldots, i_l} = a_{rs}^{i_1, i_2, \ldots, i_l, 0, 0, \ldots, 0}$$

which has zeros everywhere after the lth place. The subscripts on the superscripts indicate the independent variables, that is $D_{i_r}{}^k$ stands for the kth partial derivative of u with respect to x^r. All coefficients are real analytic functions of the variables. The initial conditions on $t = 0$ are

$$\partial^{k_r - s} u_r / \partial t^{k_r - s} = 0 \qquad (s = 1, \ldots, k_r;\ r = 1, \ldots, R). \tag{3.1.3}$$

A characteristic surface, $\phi(t, x, x^\rho) = 0$, of (3.1.1) satisfies

$$\begin{aligned} \det\left(-\left(\frac{\partial \phi}{\partial t}\right)^{k_s} \delta_{rs} + a_{rs}^{i_1, i_2+1} \left(\frac{\partial \phi}{\partial t}\right)^{i_1} \left(\frac{\partial \phi}{\partial x}\right)^{i_2+1} \right. \\ \left. + \ldots + a_{rs}^{i_1, \ldots, i_N+1} \left(\frac{\partial \phi}{\partial t}\right)^{i_1} \ldots \left(\frac{\partial \phi}{\partial x^N}\right)^{i_N} \right) = 0. \end{aligned} \tag{3.1.4}$$

This is a first-order non-linear partial differential equation of degree

$$k_1 + k_2 + \ldots + k_R = k \text{ for } \phi.$$

Let the initial surface $t = 0$ and the boundary surface T: $x = 0$, both non-characteristic, meet in an edge C. We note that by the theory of first-order partial differential equations, the characteristic surfaces through C are composed of the characteristic strips of the characteristic equations (3.1.4). On the initial "curve" C the values for the strip elements are found from (3.1.2) and the condition

$$\frac{\partial \phi}{\partial t} dt + \frac{\partial \phi}{\partial x} dx + \sum_{\rho=3}^{N} \frac{\partial \phi}{\partial x^\rho} dx^\rho = 0. \tag{3.1.5}$$

Since for $\rho = 3, \ldots, N$ the dx^ρ are independent and $dt = dx = 0$ on C, we have $\partial\phi/\partial x^\rho = 0$, and only $\partial\phi/\partial t$ and $\partial\phi/\partial x$ are different from zero. Setting

$$\lambda = \frac{\partial \phi / \partial t}{\partial \phi / \partial x}, \tag{3.1.6}$$

we have by (3.1.5) that det $A(\lambda) = 0$ where

$$(3.1.7) \qquad A(\lambda) = \text{diag}(\lambda^{k_1}, \lambda^{k_2}, \ldots, \lambda^{k_r}) - \left(\sum_{j=0}^{k_s-1} a_{rs}^{j,k_s-j} \lambda^j\right).$$

This determines the initial values of λ on C and in a neighbourhood of C. Thus, in general, there will be many characteristic surfaces containing the edge C. Note that $-1/\lambda$ is the slope of the tangent hyperplane to the characteristic surface at a point on the edge C.

3.2. Reduction to a first-order system. The characteristic surface through C corresponding to 0 divides the first quadrant into distinct regions as in §2.1. In order to study a mixed boundary value problem similar to that in §2.1 we are going to reduce the system (3.1.1) to the form (2.1.3) and (2.1.4). This will enable us to use the results of §2, which in turn will help us to determine the appropriate conditions to impose on the boundary surface S.

The first stage in this reduction process is the replacement of the mixed-order system (3.1.1) by a linear system of first-order equations (**6**, p. 283). In any derivative of u_r appearing in (3.1.1) there is a largest integer l such that u_r is not differentiated with respect to any x^s where $s > l$. If

$$i_1 + i_2 + \ldots + i_l = k_r,$$

we replace $D^{k_r}{}_{i_1,i_2,\ldots,i_l}$ by $\partial u_r{}^{i_1,i_2,\ldots,i_{l-1}}/\partial x^l$ where $D^{k_r-1}_{i_1,i_2,\ldots,i_{l-1}}$ has been replaced by $u_r{}^{i_1,\ldots,i_{l-1}}$. Here $u_r{}^{i_1,\ldots,i_l}$ denotes $u_r{}^{i_1,\ldots,i_l 0,0,\ldots 0}$ where there are zeros in every place in the superscripts after the lth place. On the other hand, if $i_1 + i_2 + \ldots + i_l < k_r$, then we set

$$(3.2.1) \qquad D^{k_r}_{i_1,\ldots,i_l} u_r = u_r^{i_1,\ldots,i_l}.$$

After making the above replacements in (3.1.1), we obtain the following system of equations:

$$(3.2.2) \qquad \frac{\partial u_r^{k_r-1}}{\partial t} = \sum_{s=1}^{R} \left(a_{rs}^{i_1,i_2+1} \frac{\partial u_s^{i_1,i_2}}{\partial x} + \ldots + a_{rs}^{i_1,i_2,\ldots,i_l+1} \frac{\partial u_s^{i_1,i_2,\ldots,i_l}}{\partial x}\right.$$
$$+ \sum_{s=1}^{R} a_{rs}^{i_1,i_2,\ldots,i_N} u_s^{i_1,i_2,\ldots,i_N} \qquad (r = 1, 2, \ldots, R),$$

$$(3.2.3) \qquad \frac{\partial u_r^{j,i_2,\ldots,i_l}}{\partial t} = \frac{\partial u_r^{j+1,i_2,\ldots,i_l-j-1}}{\partial x} \qquad (j = 0, 1, \ldots, i_l - 1),$$

$$(3.2.4) \qquad \frac{\partial u_r{}^{i-1}}{\partial t} = u_r{}^i \qquad (i - 1, \ldots, k_r - 1).$$

Here l is the largest integer such that $i_l > 0$. We also replace the initial conditions (2.1.2) by

$$(3.2.5) \quad \text{(a)} \quad u_r^{i_1,\ldots,i_N} = D^{i_1+i_2+\ldots+i_N}_{i_1,i_2,\ldots,i_N} u_r \quad (i_2 + \ldots + i_N \geqslant 1;\ 1 \leqslant i_1 + i_2 + \ldots + i_N \leqslant k_r - 1),$$
$$\text{(b)} \quad u_r^{k_r-1} = 0 \qquad (i = 1, \ldots, k_r).$$

Thus any solution $u_\tau(t, x, x^\rho)$ of (3.1.1) satisfying the initial conditions (3.1.2) is also a solution of the above system (3.2.2)–(3.2.5) and satisfies the initial conditions (3.2.5) (since the order of differentiation can be reversed).

Conversely any solution of (3.2.2)–(3.2.5) which satisfies (3.2.5) is a solution of (3.1.1) and satisfies (3.1.2), that is, u_τ satisfies (3.1.1). In order to prove this, we merely have to show that (3.2.1) can be deduced from (3.2.3)–(3.2.4) and the initial conditions (3.2.5), for then we can replace $u_\tau^{i_1, i_2, \ldots, i_l}$ in (3.2.2) and obtain (3.1.1). This process is analogous to the proof given in **(6)** for a simpler system.

The fact that the systems (3.2.2)–(3.2.4) can, in general, have solutions which are not solutions of (3.1.1) (for we can remove the restrictions (3.2.5) imposed on the boundary conditions) leads us to conjecture that the system (3.2.3)–(3.2.4) has other characteristic surfaces besides those of (3.2.1). In order to verify this fact, we find the differential equation satisfied by a characteristic surface, $\psi(t, x, x^\rho) = 0$, of (3.2.2)–(3.2.4), and compare this differential equation with (3.1.4), the equation for the characteristic surfaces of (3.2.1). We omit the details of the required calculations and merely state the results.

LEMMA 3.2.2. *If a system of mixed-order linear partial differential equations of the type* (3.1.1) *is reduced to a first-order system* (3.2.2)–(3.2.4) *by introducing new variables, as in* (3.2.1), *then the resulting system has the characteristic surfaces of the original system and also a new characteristic surface of high multiplicity which satisfies the equation* $\partial\phi/\partial t = 0$.

Therefore the only new characteristic surface through the edge C is $x = 0$. Since no new characteristic surfaces are introduced in the "quadrant" between the initial and boundary surfaces, we can apply the results of §1, if we can reduce our new system to the required form.

3.3. Reduction to canonical form. We first rewrite (3.2.2) and (3.2.3) in matrix notation retaining only those terms that involve differentiations with respect to t and x. These equations then have the form

$$\partial U/\partial t = A\ \partial U/\partial x \tag{3.3.2}$$

where the obvious interpretations are to be given to the matrix A and the vector U.

In order to diagonalize A, we find its characteristic roots, $\lambda = \lambda(t, x, x^\rho)$, which satisfy the equation

$$\det(A - \lambda I) = 0. \tag{3.3.3}$$

Simplifying the above equation, we obtain

$$\det(A - \lambda I) = (-)^{k_1+k_2+\ldots+k_R} \det A(\lambda), \tag{3.3.4}$$

where $A(\lambda)$ is defined in (3.1.7).

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If we assume that all characteristic surfaces through the edge C are real and distinct in a neighbourhood of the origin, then $A(\lambda)$ has $k = k_1 + k_2 + \ldots + k_r$ real and distinct characteristic roots, λ_r, and is similar to a diagonal matrix with λ_r on the diagonal. Alternatively, we could assume that the elementary divisors of A are all simple. In either case, the fact that A is similar to the diagonal matrix of its characteristic roots implies that there exist k linearly independent characteristic vectors of the matrix A. Thus, we can diagonalize A if we can find its characteristic vectors.

After some calculation we obtain all the vectors, which we write as

$$N = (\lambda_j^{\alpha(i)} n^j_{\beta(i)}) \qquad (i, j = 1, 2, \ldots, k). \tag{3.3.5}$$

Here (n_i^j) are the right-hand null vectors of $A(\lambda_j)$ while $\alpha(i)$ and $\beta(i)$ are integral-valued functions defined as follows:

$$\alpha(k_1 + k_2 + \ldots + k_r + i) = k_{r+1} - i \tag{3.3.6}$$
$$(k_0 = 0;\, i = 1, 2, \ldots, k_{r+1};\, r = 0, 1, 2, \ldots, R-1),$$

$$\beta(i) = r \quad \text{if } k_1 + k_2 + \ldots + k_{r-1} + 1 \leqslant i \leqslant k_r \tag{3.3.7}$$
$$(r = 1, 2, \ldots, R).$$

If we define the matrix Λ by

$$\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_k), \tag{3.3.8}$$

then $AN = N\Lambda$ or $N^{-1}AN = \Lambda$, that is, A is similar to a diagonal matrix of its characteristic roots.

Now set $U = NW$, where

$$W' = (w_1, w_2, \ldots, w_k) \tag{3.3.9}$$

and substitute in (3.3.2). We see that

$$\frac{\partial W}{\partial t} = N^{-1}AN\frac{\partial W}{\partial x} + N^{-1}\left(A\frac{\partial N}{\partial x} - \frac{\partial N}{\partial t}\right)W \tag{3.3.10}$$
$$= \Lambda\frac{\partial W}{\partial x} + N^{-1}\left(A\frac{\partial N}{\partial x} - \frac{\partial N}{\partial t}\right)W.$$

By applying the above process to equations (3.2.2) and the equations of (3.2.3) in which $\partial/\partial x$ appears, it is evident that they are reduced to the same form as (2.1.3). The remaining part of (3.2.3) and also (3.2.4) are similar to (2.1.4). We now have the entire set of first-order equations to which our system (3.2.2)–(3.2.4), already shown to be equivalent to the higher-order system in §3.2, has been reduced. In order to apply Theorem I, which would show the existence of a piecewise analytic solution, we need to know if it is possible to solve for the select w in terms of the non-select w on T: $x = 0$, as in (2.1.5), if certain linear combinations of the partial derivatives of u_r are given on T.

3.4. Boundary conditions. In order to study the above problem, we write the matrix equation $U = NW$ as follows:

$$(3.4.1)\qquad \frac{\partial^{k_r-1}u_r}{\partial t^{k_r-j}\partial x^{j-1}} = \sum_{\rho=1}^{k} \lambda_\rho^{k_r-j}\, n_r^{\ \rho} w_\rho \qquad (j = 1, \ldots, k_r;\ r = 1, \ldots, R).$$

Now, suppose $w_1, \ldots, w_l$ are select and that the following linear combination of the derivatives of u_r is given on $x = 0$:

$$(3.4.2)\qquad \sum_{j=1}^{q} C_{ij}(t, x^\rho)\, \frac{\partial^{\,k_{r(ij)}-1} u_{r(ij)}}{\partial x^{s_{ij}-1}\partial t^{k_{r(ij)}-s_{ij}}} = f_i(t, x^\rho)$$

$$(i = 1, \ldots, l;\ \rho = 3, \ldots, N;\ 1 \leqslant r(ij) \leqslant R;\ 1 \leqslant s_{ij} \leqslant k_r).$$

The $C_{ij}(t, x^\rho)$, $f_i(t, x^\rho)$ are analytic functions of t, x^ρ and $f_i(t, x^\rho) = 0$ when $t = 0$ in compliance with (2.1.6). We wish to solve for $w_1, \ldots, w_l$ in terms of $w_{l+1}, \ldots, w_k$ as in (2.1.5). By substituting the value of

$$\frac{\partial^{k_{r(ij)}-1} u_{r(ij)}}{\partial x^{s_{ij}-1}\partial t^{k_{r(ij)}-s_{ij}}}$$

obtained from (3.4.1) in (3.4.2), we see that this is possible provided that

$$(3.4.3)\qquad \det\left(\sum_{j=1}^{q} C_{ij}\, \lambda_\rho^{k_{r(ij)}-s_{ij}} n_{r(ij)}\right) \neq 0$$

$$(i = 1, 2, \ldots, l;\ \rho = 1, 2, \ldots, l)$$

in a neighbourhood of the origin. Since all functions involved are continuous, this will be true if (2.4.3) holds for $x = 0$, $t = 0$.

In this case Theorem I shows that the first-order system has a solution which is continuous in the first quadrant, and analytic, except across $G_1, \ldots, G_l$, the characteristic surfaces corresponding to $w_1, \ldots, w_l$, which takes given Cauchy data on S and takes values on T determined by the linear boundary conditions (3.4.1) (when these are transformed by (3.2.1)). However, this shows that u_r, where $u_r \in C^{k_r-1}$, is a piecewise analytic solution (with discontinuities across $G_1, \ldots, G_l$) of the original system (3.1.1) which takes given Cauchy data on S and satisfies linear boundary conditions of the type (3.4.2) on T.

Note that if $q = l$, $r(ij) = r_{\ j}$, and $s_{ij} = s_j$ (that is, the derivatives which appear in the equation (3.4.2) with $i = 1$ also appear in all the other equations, $i = 2, 3, \ldots, l$; only the C_{ij} and the f_i change), then (3.4.3) can be written as

$$\det(C_{ij})\, \det(\lambda_\rho^{k_{r(j)}-s_j} n_{rj}^{\ \rho}) \neq 0,$$

that is,

$$(3.4.4)\qquad \begin{array}{l} \text{(a)}\ \det(C_{ij}) \neq 0 \\ (\partial)\ \det(\lambda_\rho^{k_{r(j)}-s_j} n_{r(j)}) \neq 0 \end{array} \qquad (i, j, \rho = 1, 2, \ldots, l).$$

But if $\det(C_{ij}) \neq 0$, we could have found the value of

$$\partial^{k_{r(j)}-1} u_{r(j)} / \partial t^{k_{r(j)}-s_j} \partial x^{s_j-1}$$

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on $x = 0$ from (3.4.2). Also, by integrating this value $k_{r(j)} - s_j$ times with respect to t, it is known on $x = 0$. Conversely, if

$$\partial^{sj-1} u_{rj} / \partial x^{sj-1}$$

is given on $x = 0$, then

$$\partial^{k_r(j)-1} u_{r(j)} / \partial t^{k_r(j)-s_j} x^{sj-1}$$

is known. Thus if equation (3.4.4) hold, we see that the solution can satisfy boundary conditions of the type

$$\text{(3.4.5)} \qquad \sum_{j=1}^{l} C_{ij} \frac{\partial^{sj-1} u_{r(j)}}{\partial x^{sj-1}} = f_i \; (i = 1, \ldots, l; 1 \leqslant r(j) \leqslant R; 1 \leqslant s_j \leqslant k_{r(j)}).$$

As a special case of the preceding result, suppose any l of $\partial^j u_r / \partial x^j$ $(j = 0, \ldots, k_r - 1)$ are specified on T. We define the order of $\partial^j u_r / \partial x^j$ as follows:

$$\text{(3.4.6)} \qquad O\left(\frac{\partial^j u_r}{\partial x^j}\right) = k_1 + k_2 + \ldots + k_{r-1} + j + 1$$

$$(j = 0, \ldots, k_r - 1; k_0 = 0).$$

Then a piecewise analytic solution can be found taking these values on T, provided that the submatrix of N, whose columns are determined by the select $w_1, \ldots, w_l$ and whose rows are determined by the value of $O\ (\partial^j u_r / \partial x^j)$, is not zero. This is merely a restatement of conditions (3.4.4b).

Let us consider a single equation of order N. In this case $R = 1$, $k_1 = N$, and the 1×1 matrix $A(\lambda)$ is given by

$$\text{(3.4.7)} \qquad A(\lambda) = \lambda^N - \sum_{j=0}^{N-1} a_{11}^{j\,N-j} \lambda^j;$$

see (3.1.7). We see that a null vector n_1 of $A(\lambda)$ satisfies the equation

$$\text{(3.4.8)} \qquad -\lambda n_1 + \sum_{s=1}^{N} a_{11}^{N-s,s} \lambda^{N-s} n_1 = 0.$$

Choosing $n_1 = 1$, we see that (3.3.11) becomes

$$\text{(3.4.9)} \qquad N = (n_{ij})$$

where $n_{ij} = (\lambda_i)^j$. As noted in the preceding paragraph, if any l (where l is the number of selected characteristic surfaces) of $\partial^j u_j / \partial x^j$ $(j = 0, \ldots, N - 1)$, are specified on $x = 0$, there exists a piecewise analytic solution which takes on these values on T provided that

$$\text{(3.4.10)} \qquad \det\left((\lambda_i)^{r_i}\right) \neq 0 \qquad (i = 1, \ldots, l).$$

However, it is well known that such a determinant is not zero.

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Note that if we write down R equations, the rth containing the dependent variable u_r ($r = 1, \ldots, R$) only, of orders k_r, and we select characteristic surfaces, say l_r, from the rth equation and we specify l_r of $\partial^i u_r/\partial x^i$ ($j = 0, \ldots, k_r - 1$), then there will exist a piecewise analytic solution taking these values. For we merely apply the theory developed for a single equation of order N to each of R equations considered separately. Naturally we would expect that the same result holds if the R equations are considered as a system. The next case we consider shows that this conjecture is true.

Suppose that the equation involving $\partial^{k_r}u_r/\partial t^{k_r}$ does not contain

$$\partial^{k_s}u_s/\partial t^{k_s-j}\partial x^j \qquad (s \neq r, j = 1, \ldots, k_s).$$

Then (3.1.7) becomes

$$A(\lambda) = \mathrm{diag}\left(\lambda^{k_1} - \sum_{j=0}^{k_1} a_{11}^{jk_1-j}\lambda^j, \ldots, \lambda^{k_r} - \sum_{j=0}^{k_r} a_{RR}^{k\,k_R-j}\lambda^j\right). \tag{3.4.11}$$

For any root λ_s of

$$\lambda^{k_s} - \sum_{j=1}^{k_s} a_{ss}^{j\,k_s-j}\lambda^j = 0 \qquad (s = 1, 2, \ldots, R),$$

the vector $n_s = 1$ and $n_r = 0$, $r \neq s$, is a null vector of $A(\lambda_s)$. Therefore (3.3.5) becomes

$$\mathrm{diag}(M_1, M_2, \ldots, M_R) \tag{3.4.12}$$

where the matrix M_r can be written

$$M_r = (\mu_i^{\,i-1}) \qquad (i = 1, \ldots, k_r).$$

Here the μ's are the solution of

$$\lambda^{k_r} - \sum_{j=0}^{k_r-1} a_{rr}^{j\,k_r-1} = 0 \qquad (r = 1, 2, \ldots, R).$$

The determinant N is a product of Vandermonde determinants by Lemma 3.2.1.

Suppose that l characteristic surfaces are selected and l of

$$\partial^j u_r/\partial x^j \qquad (j = 0, \ldots, K; r = 1, \ldots, R)$$

are specified on T. Then there exists a piecewise analytic solution taking these values on T if the determinant of the submatrix of N whose rows are determined by the l given $\partial^j u_r/\partial x^i$ and whose columns are determined by the w corresponding to the selected characteristic surfaces is not zero. However, this determinant is a product of determinants of the type (3.4.9) which are not zero.

3.5. Statement of results. We shall say that the system (3.1.1) is hyperbolic in the weak sense with respect to S and T provided that no two of the characteristic surfaces through the edge C touch at any point of the initial edge C. An equivalent statement is that all (characteristic) roots of $A(\lambda)$ are real and distinct in a neighbourhood of the edge C; cf. (3.1.8). For the two-dimensional case, this condition is termed "hyperbolic in the narrow sense by Petrowsky (**12**, p. 60). This is equivalent to Leray's condition of "regularly hyperbolic" only in two dimensions (**9**). In higher-dimensional spaces the possibility of coalescence of roots when T is revolved on S means that Leray's condition will hold only if the system is weakly hyperbolic with respect to S and every surface T meeting S.

THEOREM IV. *Let the system* (3.1.1) *be hyperbolic in the weak sense. Let non-characteristic surfaces S: $t = 0$ and T: $x = 0$ relative to the analytic system* (3.1.1) *intersect in an edge C from which issue into a quadrant*

$$k_0 \leqslant k_1 + k_2 + \ldots + k_R$$

distinct characteristic surfaces G. Suppose that Cauchy data are given on S and in addition the following boundary conditions are assigned on T:

$$\sum_{j=1}^{q} C_{ij}(t, x^\rho)\, \frac{\partial^{k_{r(ij)}-1} u_{r(ij)}}{\partial x^{s_{ij}-1}\, \partial t^{k_{r(ij)}-s(ij)}} = g_i(t, x^\rho)$$

$$(i = 1, 2, \ldots, K_0;\ \rho = 3, \ldots, N;\ 1 \leqslant r(ij) \leqslant R;\ 1 \leqslant s(ij) \leqslant r(ij)).$$

The Cauchy data and the boundary conditions agree on the edge C. Here C_{ij} and g_{ij} are analytic functions of t and x^ρ. Let $n_r^{\ \rho}$ be the right-hand null vectors of $A(\lambda^\rho)$. Suppose that

$$\det\left(\sum_{j=1}^{q} C_{ij}\, \lambda_\rho^{k_{r(ij)}-s(ij)}\, n_{r(ij)}\right) \neq 0 \qquad (i = 1, \ldots, k_0;\ \rho = 1, \ldots, k_0).$$

Then there exists a unique piecewise analytic solution u_r assuming the given initial and boundary values, and analytic except across the G_i, where

$$u_r \in C^{k_r-1} \qquad (r = 1, \ldots, R).$$

COROLLARY IV(a). *Let the system satisfy the hypothesis of the above theorem except that the following boundary conditions are given on T:*

$$\sum_{j=1}^{k_0} C_{ij}\, \frac{\partial^{s_j-1} u_{rj}}{\partial x^{s_j-1}} = g_i \qquad (i = 1, 2, \ldots, K_0;\ 1 \leqslant r_j \leqslant R;\ 1 \leqslant s_j \leqslant r_j).$$

Here C_{ij} and g_i are analytic functions of the t and x and $\det(C_{ij}) \neq 0$. *Suppose that the submatrix of N* (3.3.11) *whose* ith *columns are determined by the λ_i corresponding to the selected w_i and whose rows are determined by the prescribed $\partial^j u_r/\partial x^j$ (the row is given by $O\ (\partial^j u_r/\partial x^j)$, where O is defined in* (3.4.6)) *is not zero. Then there exists a unique piecewise analytic solution u_r assuming the given initial and boundary values and analytic except across the G_i where $u_r \in C^{k_r-1}$.*

COROLLARY IV(b) (see **5**). *Given an analytic linear differential equation of order N in u which is hyperbolic in the weak sense with respect to the initial and* *boundary surfaces S: $t = 0$ and T: $x = 0$. Let K_0 ($K_0 \leqslant N$) characteristic surfaces G_i issuing from $C = S \cap T$ into a quadrant be selected, and any K_0 of the N quantities*

$$u, u_x^{(1)}, u_x^{(2)}, \ldots, u_x^{(N-1)}$$

be assigned on T in addition to the Cauchy data on S. Then there exists a unique piecewise analytic solution u assuming the given initial and boundary values, and analytic except across the G_i, where it is of class C^{N-1}.

COROLLARY IV(c). *Let the system (3.1.1) be hyperbolic in the weak sense. In addition, suppose that*

$$a_r^{k_s-j,j} = 0 \qquad (s \neq r;\, r, s = 1, 2, \ldots, R;\, j = 1, \ldots, k_s).$$

Let non-characteristic surfaces S: $t = 0$ and T: $x = 0$ relative to the analytic system intersect in an edge C from which issue into a quadrant K_0 distinct characteristic surfaces G_i. Let K_0, where $K_0 = \sum k_s$, of the quantities

$$u_r, \partial u_r/\partial x, \ldots, \partial^{k_r-1}u_r/\partial x^{k_r-1} \qquad (r = 1, 2, \ldots, R)$$

be assigned on T in addition to the Cauchy data on S. Then there exists a piecewise analytic solution u_r assuming the given initial and boundary values and analytic except across the G_i where it is of class C^{k_r-1} ($r = 1, 2, \ldots, R$).

4.1. First-order quasi-linear systems. A quasi-linear system of partial differential equations in R dependent variables u_i and N independent variables t, x, and x^ρ ($\rho = 3, \ldots, N$) is of the form

$$\frac{\partial u_r}{\partial t} = a_{rs}(t, x, x^\rho, u_i)\frac{\partial u_s}{\partial x} + a_{rs}^{\rho}(t, x, x^\rho, u_i)\frac{\partial u_s}{\partial x} + b_r(t, x, x^\rho, u_i) \qquad (i, r, s = 1, \ldots, K). \tag{4.1.1}$$

Here a_{rs}, a_{rs}^ρ, and b_r are analytic functions of t, x, x^ρ ,and u_i in some neighbourhood of the origin. The u_r satisfy the initial conditions:

$$(u_r)_{t=0} = 0. \tag{4.1.2}$$

An analytic surface $t + \phi^i(x, x^\rho) = 0$ is a characteristic surface relative to a solution $u_1, \ldots, u_R$ if we cannot solve for the derivatives of all the u_r across the surface. That is, if we make the following change of variables:

$$T_i = t + \phi^i(x, x^\rho), \qquad S_i = x, \qquad Y_i^\rho = x^\rho, \tag{4.1.3}$$

we shall not be able to solve for all the derivatives $\partial u_r/\partial T_i$. The coefficients of $\partial u_r/\partial T_i$ can be found by expressing the various derivatives with respect to t, x, and x^ρ appearing in (4.1.1) in terms of the derivatives with respect to T_i, S_i, and Y_i^ρ. Then the conditions for a characteristic surface become

$$C^i = |\delta_{rs} - a_{rs}\,\phi_x^{\ i} - a_{rs}^{\ \rho}\,\phi_{x^\rho}^{\ i}| = 0. \tag{4.1.4}$$

Note that we cannot determine the characteristic surfaces until we know the solution u_r of (4.1.1).

We are only going to consider the characteristic surfaces through the edge C: $x = 0$, $t = 0$; thus we must have

(4.1.5) $$(\phi_{x^\rho})_{x=0} = 0.$$

Now $C^i = C^i(x, t, x^\rho, u_r, \phi_x{}^i, \phi_{x^\rho}{}^i)$, defined in (4.1.4), is an analytic function. Along the edge C we have

$$C^i(0, 0, x^\rho, 0, (\phi_x{}^i)_0, 0) = 0$$

since $(\phi_{x^\rho}{}^i)_{x=t=0} = 0$ and $(u_r)_{x=t=0} = 0$. The implicit function theorem tells us that there exists a unique solution of equations (4.1.4) with $t = 0$:

(4.1.6) $$\partial\phi^i/\partial x = F^i(x, x^\rho, \phi_{x^\rho}{}^i, u_r)$$

in the neighbourhood of the edge C and such that

$$(\phi_x{}^i)_{x=t=0} = 1/\lambda_i$$

provided that

$$(\partial C^i/\partial\phi_x{}^i)_{x=t=0} \neq 0.$$

If we assume that the function

(4.1.7) $$A(\lambda) = |\lambda\delta_{rs} - a_{rs}(0, x, x^\rho, 0)|$$

has R distinct roots λ_i, none of which is zero, then

(4.1.8) $$A(\lambda) = \prod_{i=1}^{R} (\lambda - \lambda_i).$$

Now (4.1.8) can be rewritten

(4.1.9) $$\left|\delta_{rs} - a_{rs}\frac{1}{\lambda}\right| = \frac{A(\lambda)}{\lambda^R} = \prod_{s=1}^{R} (1 - \lambda_s/\lambda).$$

Thus from (1.1.4), (1.1.7), and (1.1.9), we see that (denoting $\phi_x{}^i$ by v)

$$\frac{\partial C^i}{\partial v} = \prod_{s\neq i} (1 - \lambda_s/\lambda_i),$$

since $(\phi_x{}^i)_{x=0} = 1/\lambda_i$. Therefore an equation of the form (4.1.6) holds.

The tangent planes to the characteristic surfaces along the edge C are determined from the initial data and have the equation

(4.1.10) $$t + \frac{1}{\lambda_i}x = 0.$$

Suppose $L \leqslant R$ of these planes are in the "first quadrant." We select $K_0 \leqslant R$ of the corresponding characteristic surfaces (that is the surfaces to be determined from the corresponding equations (1.1.7)) and prescribe the following boundary conditions on $x = 0$:

(4.1.11) $$g_i(t, x^\rho, u_r) = 0 \qquad (i = 1, \ldots, K_0)$$

where the g_i are analytic functions of the variables t, x^ρ, u_r. For continuity we assume $g_i(0, x^\rho, u_r) = 0$. As in Part one, we shall try to find a piecewise analytic solution of the differential equations which is continuous across the selected characteristic surfaces. However, in the present case the location of the characteristic surfaces is not known. We can overcome this difficulty by adding equation (4.1.6) to the system of partial differential equations and considering the selected characteristic surfaces as being expanded in a power series in x. It will be shown that we can calculate recursively the coefficients in the power series for u_r and the equations of the characteristic surfaces. The details are given below.

Let $(a_{rs})_{t=0} = a_{rs}{}^0(0, x, x^\rho, 0)$. Then we can rewrite (1.1.1) in the form

$$\frac{\partial u_r}{\partial t} = a_{rs}{}^0 \frac{\partial u_s}{\partial x} + a_{rs}{}^\rho \frac{\partial u_s}{\partial x^\rho} + b_r + C_{rs} \frac{\partial u_s}{\partial x} \tag{4.1.12}$$

where $C_{rs} = a_{rs} - a_{rs}{}^0$. We determine the matrix $N = (n_{rs})$ and the matrix $N^{-1} = (n^{rs})$ from the relation

$$N^{-1}A^0N = \operatorname{diag}(\lambda_1, \ldots, \lambda_R) \tag{4.1.13}$$

where $A^0 = (a_{rs}{}^0)$ and $\lambda_i = \lambda_i(0, x, x^\rho, 0)$. We now make the following change of variables:

$$u_r = n_{rs} w_s. \tag{4.1.14}$$

Then (4.1.12) becomes

$$n_{rs} \frac{\partial w_s}{\partial t} = a_{rs}{}^0 n_{rs} \frac{\partial w_s}{\partial x} + a_{rs}{}^0 w_s \frac{\partial n_{rs}}{\partial x} + a_{rs}{}^\rho \left[n_{rs} \frac{\partial w_s}{\partial x^\rho} + \frac{\partial n_{rs}}{\partial x} w_s \right] + C_{rs} \left[n_{rs} \frac{\partial w_s}{\partial x} + w_s \frac{\partial n_{rs}}{\partial x} \right] + b_r.$$

After multiplying by n^{kr} and grouping the terms involving $\partial n_{rs}/\partial x$ with $n^{kr} b_r$, the above equations assume the form

$$\frac{\partial w_r}{\partial t} = \lambda_r \frac{\partial w_r}{\partial x} + \alpha_{rs}{}^\rho \frac{\partial w_s}{\partial x} + \beta_r + \gamma_{rs} \frac{\partial w_s}{\partial x}. \tag{4.1.15}$$

Let w_{ri} be the expansion of w_r about G_i (see 2.1 for the definition of G_i) which is defined in every D_j, $j \geqslant i$, and equal to zero for $j < i$; then, as before, the solution in D_h will be

$$\sum_{i=1}^{h} w_{ri}.$$

This solution satisfies (1.1.16)—that is,

$$\sum_{i=1}^{h} \frac{\partial w_{ri}}{\partial t} = \lambda_r \sum_{i=1}^{h} \frac{\partial w_{ri}}{\partial x} + \alpha_{rs}{}^\rho (w_r{}^h) \sum_{i=1}^{h} \frac{\partial w_{si}}{\partial w^\rho} + \beta_r(w_r{}^h) + \gamma_{rs}(w_r{}^h) \sum_{i=1}^{h} \frac{\partial w_{si}}{\partial x}. \tag{4.1.16}$$

Here $\alpha_{rs}{}^{\rho}(w_{rh}) = \alpha_{rs}{}^{\rho}(t, x, x^{\rho}, w_{r1} + w_{r2} + \ldots + w_{rh})$ and the other coefficients β_r and γ_r are to be interpreted in a similar manner. If in equation (1.1.17) we replace h by $h - 1$ and subtract, we obtain the following equations for the w_{rh}:

$$(4.1.17)\quad \frac{\partial w_{rh}}{\partial t} = \lambda_r \frac{\partial w_{rh}}{\partial x} + [\alpha_{rs}{}^{\rho}(w_r{}^h) - \alpha_{rs}{}^{\rho}(w_r{}^{h-1})]\sum_{i=1}^{h-1} \frac{\partial w_{si}}{\partial x^{\rho}} + \alpha_{rs}{}^{\rho}(w_r{}^h)\frac{\partial w_{sh}}{\partial x^{\rho}} + \beta(w_r{}^h) - \beta_r(w_r{}^{h-1}) + [\gamma_{rs}(w_r{}^h) - \gamma_{rs}(w_r{}^{h-1})]\sum_{i=1}^{h-1}\frac{\partial w_{si}}{\partial x^{\rho}} + \gamma_{rs}(w_r{}^h)\frac{\partial w_{sh}}{\partial x^{\rho}}.$$

If we make the transformation of variables given by (4.1.3) in (4.1.17) and simplify, we obtain

$$(4.1.18)\quad \left[\frac{1}{\lambda_r(0, S_i, Y_i{}^{\rho}, 0)} - \frac{1}{\lambda_h(0, 0, Y_i{}^{\rho}, 0)}\right]\frac{\partial w_{rh}}{\partial t} = \frac{\partial w_{rh}}{\partial S_h} + \bar{\alpha}_{rj}{}^{\rho}\sum_{i=1}^{h-1}\frac{\partial w_{ji}}{\partial Y} + \bar{\beta}_r + \left[\bar{\alpha}_{rj}\sum_{i=1}^{h-1}\frac{\partial w_{ji}}{\partial t}\phi_Y{}^h + \bar{\gamma}_{rj}\sum_{i=1}^{h-1}\left(\frac{\partial w_{ji}}{\partial S_h} + \frac{\partial w_{ji}}{\partial t}\phi_S{}^h\right)\right] + \frac{1}{\lambda_r}\gamma_{rj}\left(\frac{\partial w_{jh}}{\partial S_h} + \frac{\partial w_j}{\partial t}\phi_r{}^h\right) + \frac{\partial w_{rh}}{\partial t}\left(\phi_s{}^h - \frac{1}{\lambda_h}\right) + \frac{\alpha_{rj}{}^{\rho}}{\lambda_r}\left(\frac{\partial w_{jh}}{\partial Y} + \frac{\partial w_{jh}}{\partial t}\phi_Y{}^h\right)$$

where we denote the derivative of functions with respect to $Y_h{}^{\rho}$ and S_h by $\partial/\partial Y$ or ϕ_Y and $\partial/\partial S$ or ϕ_S respectively and where

$$\bar{\alpha}_{rs} = \frac{1}{\lambda_r}[\alpha_{rs}(w_r{}^h) - \alpha_{rs}(w_r{}^{h-1})];$$

$\bar{\beta}_r$ and $\bar{\gamma}_r$ have an analogous interpretation.

4.2. The series expansion. Suppose $w_1, w_2, w_3, \ldots, w_{K_0}$ are the variables corresponding to the selected characteristic surfaces. In order to find the series expansion, we shall have to solve (4.1.11) for $w_1, \ldots, w_{K_0}$ in terms of the remaining variables in the neighbourhood of the edge C. From (4.1.11) and (4.1.14), this will be possible if

$$(4.2.1)\qquad \det[(\partial g_i/\partial u_r)n_{rk}] \neq 0 \qquad (r = 1, \ldots, R; i, k = 1, \ldots, K_0)$$

in the neighbourhood of the edge C. If (4.2.1) holds, then we obtain equations of the following form for the w_r on $x = 0$:

$$(4.2.2)\qquad w_r = f_r(t, x^{\rho}, w_s) \qquad (r = 1, \ldots, R; s = K_0 + 1, \ldots, R).$$

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Let ϕ^i be expanded as

$$\sum_{j=0}^{\infty} b_j{}^i S_i$$

and the w_{rh} as follows:

$$w_{rh} = \sum_{m,n=0}^{\infty} a_{mn}^{rh} S_h^m T_h^n. \tag{4.2.3}$$

Then we can show that the $b_j{}^i$ and a_{mn} can be calculated from the following set of equations and boundary conditions:

$$\phi_{S_i}^i = F_i(S_i, Y_i^\rho, \phi_Y{}^i, w_{r1} + w_{r2} + \ldots + w_{ri}) \quad (i = 1, 2, \ldots, K_0), \tag{4.2.4}$$
$$(\phi^i)_{S_i=0} = 0,$$

$$\begin{aligned}(1 - \lambda_r/\lambda_i) \frac{\partial w_{ri}}{\partial T_i} &= \lambda_r \frac{\partial w_{ri}}{\partial S_i} + \gamma_{rh} \frac{\partial w_{hi}}{\partial S_i} + \alpha_{rh}{}^\rho \frac{\partial w_{hi}}{\partial Y_i^\rho} \\ &+ \bar{\gamma}_{rh}(w_r{}^i) \sum_{j=1}^{i-1} \left[\frac{\partial w_{hj}}{\partial S_j} + \frac{\partial w_{hj}}{\partial T_j} (\phi_S{}^j - \phi_S{}^i)\right] + \bar{\alpha}_{rh}{}^\rho (w_r{}^i) \\ &\times \sum_{j=1}^{i-1} \left[\frac{\partial w_{hj}}{\partial Y^\rho} + \frac{\partial w_{hj}}{\partial T_j} (\phi_Y{}^j - \phi_Y{}^i)\right] \\ &+ [\alpha_{rh}{}^\rho \phi_Y{}^i + \gamma_{rs}\, \phi_S{}^i] \frac{\partial w_{hi}}{\partial T_i} + \lambda_r \left[\phi_S{}^i - \frac{1}{\lambda_i}\right] \frac{\partial w_{ri}}{\partial T_i} \\ &+ \bar{\alpha}_{rh}{}^\rho (w_r{}^i) \left[\sum_{j=1}^{i-1} \phi_Y{}^j \frac{\partial w_{hj}}{\partial T_j}\right] + \bar{\gamma}_{rh}(w_r{}^i) \\ &\times \left[\sum_{j=1}^{i-1} \phi_S{}^j \frac{\partial w_{hi}}{\partial T_j}\right] + \bar{\beta}(w_r{}^i);\end{aligned} \tag{4.2.5}$$

$$(w_{ri})_{T_i=0} = 0 \qquad (r = 1, 2, \ldots, R;\ i = 1, 2, \ldots, K_0;\ r \neq i),$$

$$\frac{\partial w_{ii}}{\partial S_i} = -R_i + \left(\frac{1}{\lambda_i(0, S_i, Y_i^\rho, 0)} - \frac{1}{\lambda_i(0, 0, Y_i^\rho, 0)}\right) \frac{\partial w_{ii}}{\partial T_i} \quad (r = i), \tag{4.2.6}$$

$$w_{ii} = f_i\left(T_i, Y_i^\rho, \sum_{j=1}^{K_0} w_{sj}\right) - \sum_{r=1}^{K_0} w_{ir} \qquad (r \neq i). \tag{4.2.7}$$

Here R_i denotes the right-hand side of (4.2.5) when $r = i$ except for the term $\partial w_{ii}/\partial S_i$. Equation (4.2.4) comes from (4.1.6) while (4.2.5) and (4.2.6) can be deduced from (4.1.18) by using the following relations which hold for an arbitrary function $v = v(t, x, x^\rho)$:

$$\frac{\partial v}{\partial T_h} = \frac{\partial v}{\partial t},$$

$$\frac{\partial v}{\partial S_h} = \frac{\partial v}{\partial S_i} + \frac{\partial v}{\partial T_i} (\phi_S{}^i - \phi_S{}^h),$$

$$\frac{\partial v}{\partial Y_h^\rho} = \frac{\partial v}{\partial Y_i^\rho} + \frac{\partial v}{\partial T_i} (\phi_Y{}^i - \phi_Y{}^h).$$

These equations are obtained from the rules for transforming derivatives and (4.1.3). In particular for the function ϕ^i we obtain

$$\frac{\partial\phi^i}{\partial t} = 0, \qquad \frac{\partial\phi^i}{\partial x} = \frac{\partial\phi^i}{\partial S_h} = \frac{\partial\phi^i}{\partial S_i}, \qquad \frac{\partial\phi^i}{\partial x^\rho} = \frac{\partial\phi^i}{\partial Y_h^\rho} = \frac{\partial\phi^i}{\partial Y_i^\rho}.$$

Equation (4.2.7) comes from (4.2.2) and the fact that in the K_0th domain

$$w_r = \sum_{h=1}^{K_0} w_{rh}.$$

We shall show by induction on j and $m + n$ that for every i we can calculate $b_j{}^i$ and $a_{mn}{}^{ri}$. We begin with terms of order zero, that is, $j = 0$ and $m + n = 0$. These terms are all zero since the solution is to be continuous across the characteristic surfaces. Now assume that all terms of order less than $K + 1$ are known. Then we can calculate all terms of order $K + 1$.

We can easily calculate $b_{K+1}{}^i$ by differentiating (4.2.4) K times with respect to S_i. The calculation of the a_{mn} is more difficult. Suppose $r \neq i$. Then by differentiating the boundary condition, $(w_{ri})\ T_i = 0 = 0$, $K + 1$ times with respect to S_i, we can calculate $a^{ri}_{K+1,0}$. Then by induction on l we can show that we can calculate $a^{ri}_{K+1-l,l}$. For suppose that all a^{ri}_{mn} with $n \leqslant l$ and $m + n = K + 1$ and also a^{ri}_{mn} with $m + n \leqslant K$ are known. Then we can calculate $a^{ri}_{K-l,l+1}$ by an application of Leibnitz's rule. For differentiating (1.2.5) l times with respect to T_i and $K - l$ times with respect to S_i, we obtain an equation in which all the coefficients of terms of order $K + 1$ in T_i and S_i, except that of $\partial^{K+1}w_{ri}/\partial S_i^{K-l+1}\partial T_i{}^l$, vanish when $S_i = T_i = 0$. Since $a^{ri}_{K-l,l}$ is known, it follows by induction that we can calculate all the a^{ri}_{mn} for $r \neq i$, $m + n = K + 1$. Therefore if all the terms of order less than K are known, we can calculate a^{ri}_{mn} with $m + n = K + 1$ and $r \neq i$.

Now let $r = i$. Suppose all terms of order K are known; then we can calculate a^{ii}_{0K+1} from (1.2.7). For w_{ir} $(i \neq r)$ are known and only such terms appear on the right-hand side in (1.2.7) since $S \geqslant K_0 + 1$ for the w_{Sj} appearing in f_i. Then by differentiating (1.2.6) K times with respect to t, we can calculate a^{ii}_{1K}. By proceeding in the same way as in the case $r \neq i$, we can calculate a^{ii}_{mn} for $m + n = K + 1$. Thus all terms of order $K + 1$ can be calculated. By dominating series it can be shown that the solution converges in a small neighbourhood of the edge C. We omit the details. However, it is interesting to note that the radius of convergence of the solution depends on the radii of convergence of the boundary conditions.

THEOREM V. *Let non-characteristic surfaces S: $t = 0$ and T: $x = 0$ relative to the analytic system*

$$\frac{\partial u_r}{\partial t} = a_{rs}(t, x, x^\rho, u_i)\frac{\partial u_s}{\partial x} + a_{rs}{}^\rho(t, x, x^\rho, u_i)\frac{\partial u_s}{\partial x^\rho} + b_r(t, x, x^\rho, u_i)$$
$$(i, r, s = 1, \ldots, R;\ \rho = 3, \ldots, N),$$

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intersect in an edge C. Suppose the matrix $([a_{rs}]_{t=0})$ *has R distinct eigenvalues in a neighbourhood of C. Then there exists a solution, in the neighbourhood of C, which is continuous in the quadrant and analytic except across the* K_0 *selected characteristic surfaces through C.*

This solution takes Cauchy data on S and satisfies the following boundary conditions on T, where g_i *are analytic functions of the indicated arguments and give the same values of* u_r *on the edge C as given there by the prescribed initial values, provided that*

$$\det\left[\frac{\partial g_i}{\partial u_r}(n_{rk})\right] \neq 0 \qquad (r = 1, 2, \ldots, R; i, k = 1, 2, \ldots, K_0),$$

where n_{rk} *are the eigenvectors of* $(a_{rs})_{t=0}$.

5. Reduction of a non-linear system to a quasi-linear system. Consider the system of R non-linear partial differential equations of first order

$$\text{(5.1)} \qquad \frac{\partial u_r}{\partial t} = F_r\left(t, x, x^\rho, u_i, \frac{\partial u_i}{\partial x}, \frac{\partial u_i}{\partial x^\rho}\right) \qquad (i, r = 1, 2, \ldots, R; \rho = 3, \ldots, N)$$

in R dependent variables u_r and N independent variables t, x, and x^ρ. The F_r are analytic functions of the indicated variables.

A surface, $t + \phi^i(x, x^\rho) = 0$, is called characteristic for the system (5.1) and for a given solution $u_1, \ldots, u_R$ if the solution functions satisfy the equation

$$\text{(5.2)} \qquad \det\left(\delta_{rs} - \frac{\partial F_S}{\partial u_{rs}}\phi_x - \frac{F}{u_{rx^\rho}}\phi_{x^\rho}\right) = 0$$

at all points of the surface. The coefficients of $\partial\phi/\partial x$ and $\partial\phi/\partial x^\rho$ in (5.2) depend, in general, on u_i, u_{ix}, u_{ix^ρ} and so, as in (5.1), we cannot determine the characteristic surfaces until we know the solutions u_r of (4.4.1). However, as in §4.1, we can determine the slopes of the tangent planes to the characteristic surfaces along the edge C.

Suppose that the u_r satisfy the following initial conditions:

$$\text{(5.3)} \qquad u_r = 0 \qquad (r = 1, 2, \ldots, R),$$

and the following boundary conditions:

$$\text{(5.4)} \qquad g_i(t, x^\rho, u_r) = 0 \qquad (i = 1, \ldots, K_0),$$

where the g_i are analytic functions of the indicated arguments and $g_i(t, x^\rho, u_r) = 0$ when $t = 0$. Then we can show that under certain conditions there exists a piecewise analytic solution taking the prescribed auxiliary data. To prove this we shall reduce the system (5.1) to a quasi-linear system.

We define new variables as follows:

$$\text{(5.5)} \qquad u_{rt} = \frac{\partial u_r}{\partial t}, \qquad u_{r\rho} = \frac{\partial u_r}{\partial x^\rho}, \qquad u_{rx} = \frac{\partial u_r}{\partial x}.$$

If we differentiate (5.1) with respect to t and make the substitutions indicated in (5.5), we obtain

$$(5.6)\qquad \frac{\partial u_{rt}}{\partial t} = \frac{\partial F_r}{\partial u_{sx}}\frac{\partial u_{st}}{\partial x} + \frac{\partial F_r}{\partial u_{sx^\rho}}\frac{\partial u_{st}}{\partial x^\rho} + \frac{\partial F_r}{\partial u_s}u_{st} + \frac{\partial F_r}{\partial t}, \qquad (u_{rt})_{t=0} = (F_r)_{t=0},$$

where the initial condition we have added will ensure that we can deduce (5.1) from (5.6). In addition to (5.6) we have the following equations and initial conditions:

$$(5.7)\qquad \frac{\partial u_{rx}}{\partial t} = \frac{\partial u_{rt}}{\partial x}, \qquad (u_{rx})_{t=0} = \left(\frac{\partial u_r}{\partial x}\right)_{t=0},$$

$$(5.8)\qquad \frac{\partial u_{rx^\rho}}{\partial t} = \frac{\partial u_{rt}}{\partial x^\rho}, \qquad (u_{rx^\rho})_{t=0} = \left(\frac{\partial u_r}{\partial x^\rho}\right)_{t=0},$$

$$(5.9)\qquad \frac{\partial u_r}{\partial t} = u_{rt}, \qquad (u_r)_{t=0} = 0.$$

The boundary condition

$$(5.10)\qquad \frac{\partial g_r}{\partial t} + \frac{\partial g_r}{\partial u_s}u_{st} = 0$$

for the new system is deduced by differentiating (5.4) with respect to t. This new system is a quasi-linear system in the dependent variables u_r, u_{rt}, u_{rx}, and u_{rx^ρ}. Any solution of (5.1) which satisfies (5.3) and (5.4) is a solution of (5.6)–(5.9) and satisfies (5.10). Conversely, it can be shown, as in §3.2, that any solution of the new system is also a solution of the old system. Also the characteristic surfaces of the new system through the edge C are the same as for the old system except for the introduction of a new characteristic surface $x = 0$.

If we denote $(\partial F_r/\partial u_{sx})_{t=0}$ by $a_{rs}{}^0$, then (4.4.6) can be rewritten as follows:

$$(5.11)\qquad \frac{\partial u_{rt}}{\partial t} - a_{rs}{}^0\frac{\partial u_{st}}{\partial x} = \frac{\partial F_r}{\partial u_{sx^\rho}}\frac{\partial u_{st}}{\partial x^\rho} + \frac{\partial F_r}{\partial u_s}u_{st} + \frac{\partial F_r}{\partial t} + \left(\frac{\partial F_r}{\partial u_{sx}} - a_{rs}{}^0\right)\frac{\partial u_{st}}{\partial x}.$$

The left-hand side of (4.5.1) together with (5.7) can be rewritten, in matrix notation,

$$(5.12)\qquad \partial U/\partial t = B\partial U/\partial x,$$

where

$$(5.13)\qquad B = \begin{bmatrix} a_{rs}{}^0 & 0 \\ \delta_{rs} & 0 \end{bmatrix}, \qquad U = \begin{bmatrix} u_{rt} \\ u_{rx} \end{bmatrix} \qquad (r, s = 1, \dots, R).$$

A right-hand characteristic vector

$$(5.14)\qquad n = \begin{bmatrix} n_1 \\ n_2 \end{bmatrix}$$

satisfies the equation $Bn = \lambda n$. From (5.14) and (5.15) we see that

$$An_1 = n_1, \qquad n_1 = n_2 \qquad (\text{where } A = (a_{rs}{}^0)). \tag{5.15}$$

From the above equations the matrix N whose columns are the characteristic vectors of B is

$$N = \begin{bmatrix} N_{11} & 0 \\ N_{21} & I \end{bmatrix}$$

where

$$N_{11} = (n_{rs}) \qquad (r, s = 1, \ldots, R) \tag{5.16}$$

and

$$N_{21} = \left(\frac{1}{\lambda_s} n_{rs}\right). \tag{5.17}$$

Here the n_{rs} are the characteristic vectors of A. If we assume that all the characteristic roots of A are distinct, then the characteristic vectors of A are independent; thus $|N| \neq 0$ for it is the determinant of the characteristic vectors of A. Now

$$BN = N\Lambda \tag{5.18}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \lambda_2, \ldots, \lambda_R, 0, \ldots, 0)$. Therefore, if we set

$$U = NW, \tag{5.19}$$

from (4.4.18) we see that (4.4.13) becomes

$$\frac{\partial W}{\partial t} = \Lambda \frac{\partial W}{\partial x} + N^{-1} \frac{\partial N}{\partial x} W. \tag{5.20}$$

Thus (5.7) and (5.12) are of the form

$$\frac{\partial w_r}{\partial t} = \lambda_r \frac{\partial w_r}{\partial x} + \alpha_{rs} \frac{\partial w_s}{\partial x^\rho} + \beta_r + \gamma_{rs} \frac{\partial w_s}{\partial x} \qquad (r = 1, 2, \ldots, R), \tag{5.21}$$

$$\frac{\partial w_{R+r}}{\partial t} = \bar{\alpha}_{rs} \frac{\partial w_s}{\partial x^\rho} + \bar{\beta}_r + \bar{\gamma}_{rs} \frac{\partial w_s}{\partial x}, \tag{5.22}$$

where $\lambda_r = \lambda_r(x, x^\rho)$ and the other coefficients are functions of w_s, u_r, $u_r x^\rho$. Equations (5.8) and (5.9) become

$$\frac{\partial u_{rx^\rho}}{\partial t} = n_{rj} \frac{\partial w_j}{\partial x^\rho} + W_j \frac{\partial n_{rj}}{\partial x}, \tag{5.23}$$

$$\frac{\partial u_r}{\partial t} = n_{rj} w_j. \tag{5.24}$$

These equations form a quasi-linear system. A series solution can be found by expanding about the characteristic surfaces as in §§3.1 and 3.2. Equation (5.10) is of the same form as (5.2.1) and is to be treated exactly the same in finding the coefficients of the series expansion. The only modification required in the series expansion is due to the presence of (5.22)–(5.24). However, the coefficients in the series for the variables w_{R+r}, u_{rx^ρ}, and u_r can easily be calculated from (5.22)–(5.24) using only the initial conditions. Thus we have the following theorem:

THEOREM VI. *Let non-characteristic surfaces S: $t = 0$ and T: $x = 0$ relative to the analytic non-linear first-order system*

$$\frac{\partial u_r}{\partial t} = F_r\left(t, x, x^\rho, u_i, \frac{\partial u_i}{\partial x}, \frac{\partial u_i}{\partial x^\rho}\right) \qquad (r, i = 1, 2, \ldots, R;\ \rho = 3, \ldots, N)$$

intersect in an edge C. Suppose the matrix

$$[(\partial F_r/\partial u_{sx})_{t=0}]$$

has R distinct eigenvalues in a neighbourhood of the edge C. Then there exists a solution in the neighbourhood of the edge C which is continuous in the quadrant and analytic except across the K_0 selected characteristic surfaces through C. This solution takes given Cauchy data on S and satisfies the following boundary conditions on T:

$$g_i(t, x^\rho, u_r) = 0 \qquad (i = 1, \ldots, K_0),$$

where the g_i are analytic functions of the indicated variables, provided that

$$\det\left[\frac{\partial g_i}{\partial u_r}(n_{rk})\right] \neq 0 \qquad (r = 1, \ldots, R;\ i, k = 1, \ldots, K_0),$$

the n_{rk} being the eigenvectors of

$$[(\partial F_r/\partial u_{sx})_{t=0}].$$

6. Reduction of a high-order non-linear system to a first-order non-linear system. We now consider a system of R non-linear partial differential equations of mixed orders

$$\text{(6.1)} \qquad \frac{\partial^{k_r} u_r}{\partial t^{k_r}} = F_r(t, x, x^\rho, D^{i_1+i_2+\ldots+i_N}_{i_1, i_2, \ldots, i_N} u_s)$$
$$(i_1 + i_2 + \ldots + i_N = 0, \ldots, k_s;\ r, s = 1, 2, \ldots, N),$$

in R dependent variables u_r and N independent variables t, x, x^ρ. The F_r are analytic functions of the indicated variables. We also assume that no derivatives of higher order than k_r appear in any of the F_r and if a partial derivative of order k_r appears, it is of lower order than k_r with respect to differentiations which involve t. In addition, the above system is weakly hyperbolic—that is, in some neighbourhood of the edge C, the roots of

$$\text{(6.2)} \qquad A(\lambda) = \operatorname{diag}(\lambda^{k_1}, \lambda^{k_2}, \ldots, \lambda^{k_R}) - \sum_{j=0}^{k_s-1}\left[\frac{\partial^{k_s} F_r}{(\partial u_{st})^j (\partial u_{sx})^{k_s-j}}\right]_{t=0} \lambda^j$$

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are real and distinct. The initial conditions on $t = 0$ are

$$\partial^{k_r - s} u_r / \partial t^{k_r - s} = 0 \qquad (s = 1, \ldots, k_r;\ r = 1, \ldots, R), \tag{6.3}$$

while the boundary conditions are

$$g_i\left(t, x^\rho, \frac{\partial^{k_{r(ij)}-1} u_{r(ij)}}{\partial x^{s_{ij}-1} \partial t^{k_{r(ij)}-s_{ij}}}\right) = 0 \tag{6.4}$$

$$(i = 1, 2, \ldots, K_0;\ \rho = 3, \ldots, N;\ 1 \leqslant r(ij) \leqslant R;\ 1 \leqslant s_{ij} \leqslant r(ij)).$$

The boundary conditions are compatible with the Cauchy data on the edge C and the g_i are analytic functions of the indicated variables.

By the process employed in §3.2, we can reduce the system (6.1) to a first-order non-linear system. We now apply the results developed in §5 for first-order non-linear systems. These show us that we can prove, by the methods of §3.4, a series of theorems and corollaries similar to Theorem IV and its corollaries. In fact the statement of results for non-linear systems is identical with that given in §3.5 if we define $A(\lambda)$ by (6.2), replace the boundary conditions by (6.4), and replace C_{ij} in the condition determined by $\partial g_i / \partial V$ where

$$v = \frac{\partial^{k_{r(ij)}-1} u_{r(ij)}}{\partial x^{s_{ij}-1} \partial t^{k_{r(ij)}-1}}.$$

However, the radius of convergence of the solution will now depend on the radii of convergence of the boundary conditions (see 1.3).

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