Canad. Math. Bull. Vol. **51** (1), 2008 pp. 15–20

The Duality Problem for the Class of AM-Compact Operators on Banach Lattices

Belmesnaoui Aqzzouz, Redouane Nouira, and Larbi Zraoula

Abstract. We prove the converse of a theorem of Zaanen about the duality problem of positive AM-compact operators.

1 Introduction and Notations

A regular operator T from a vector lattice E into a Banach lattice F is said to be AM-compact if the image of each order bounded subset of E is relatively compact in F. This class of operators was introduced by Fremlin in [4]. It is easy to see that each regular compact operator between two Banach lattices is AM-compact, but the converse is false in general. In fact, the identity operator of the Banach lattice l^1 is AM-compact but not compact. Whenever E is an AM-space with unit, the class of AM-compact operators on E coincides with the class of regular compact operators on E.

As compact operators, the subspace of AM-compact operators forms a closed two sided ideal in the space of all operators on a Banach lattice. But in contrast to compact operators, there exist AM-compact operators whose dual operators are not AM-compact, and conversely, there exist operators which are not AM-compact but their dual operators are AM-compact. In fact, the identity operator of the Banach lattice l^1 is AM-compact, but its dual operator, which is the identity operator of l^∞, is not AM-compact. Conversely, the identity operator of the Banach lattice of all convergent sequences c is not AM-compact, but its dual operator, which is the identity operator of the Banach lattice c', is AM-compact where c' is the topological dual of c.

Zaanen [6, Theorem 125.6] studied the duality problem of AM-compact operators on Banach lattices. He gave sufficient conditions for which the AM-compactness of an operator implies the AM-compactness of its dual and conversely. More precisely, he proved that if E and F are two Banach lattices such that F has an order continuous norm, and if T is a regular operator from E into F, then the AM-compactness of the dual operator T' from F' into E' implies the AM-compactness of T. Conversely, he showed that if E' has an order continuous norm and T is AM-compact, then the dual operator T' from F' into E' is AM-compact. These results are natural analogues of Schauder's theorem for compact operators. The proofs in Zaanen's book are really from scratch, not using any other previous theory. We will start our paper with an

Received by the editors October 4, 2005; revised October 28, 2005.
AMS subject classification: Primary: 46A40; secondary: 46B40, 46B42.

alternative proof of these results. Although our proof seem to be shorter, it is based on some other non-trivial results.

The rest of the paper is devoted to studying the converse of Zaanen's theorem [6]. More precisely, we will prove that whenever $E = F$, if the dual operator of each AM-compact operator T from a Banach lattice E into itself is also AM-compact, then E' has an order continuous norm. Also, if E is order σ-complete, we will establish that if each regular operator T from E into E is AM-compact whenever its dual operator T' from E' into E' is AM-compact, then E has an order continuous norm. Finally, if the Banach lattice E is not necessary order σ-complete, we will prove that if E and F are two Banach lattices such that each regular operator T from E into F is AM-compact whenever its dual operator T' from F' into E' is AM-compact, then the norm of F is order continuous or E' is discrete.

To state our results, we need to fix some notations and recall some definitions. A *vector lattice* E is an ordered vector space in which $\sup(x, y)$ exists for every $x, y \in E$. A subspace F of a vector lattice E is said to be a *sublattice* if for every pair of elements a, b of F the supremum of a and b taken in E belongs to F. A subset B of a vector lattice E is said to be *solid* if it follows from $|y| \leq |x|$ with $x \in B$ and $y \in E$ that $y \in B$. An *order ideal* of E is a solid subspace. Let E be a vector lattice. For each $x, y \in E$ with $x \leq y$, the set $[x, y] = \{z \in E \; x \leq z \leq y\}$ is called an *order interval*. A subset of E is said to be *order bounded* if it is included in some order interval. A *Banach lattice* is a Banach space $(E, \|\cdot\|)$ such that E is a vector lattice and its norm satisfies the following property: for each $x, y \in E$ such that $|x| \leq |y|$, we have $\|x\| \leq \|y\|$. If E is a Banach lattice, its topological dual E', endowed with the dual norm, is also a Banach lattice. We refer to Zaanen [6] for unexplained terminology on Banach lattice theory.

Also, a vector lattice equipped with a vector topology is said to be a *locally convex solid lattice* if zero admits a fundamental system of convex and solid neighborhoods.

The topology τ of a locally convex solid vector lattice is said to be *Lebesgue* if each generalized sequence (x_α) such that $x_\alpha \downarrow 0$ in E, converges to 0 for the topology τ, where the notation $x_\alpha \downarrow 0$ means that the sequence (x_α) is decreasing and $\inf(x_\alpha) = 0$.

If E' is the topological dual of E, the absolute weak topology $|\sigma|(E, E')$ is the locally convex solid topology on E generated by the family of lattice seminorms $\{P_f : f \in E'\}$ where $P_f(x) = |f|(|x|)$ for each $x \in E$. Similarly, $|\sigma|(E', E)$ is the locally convex solid topology on E' generated by the family of lattice seminorms $\{P_x : x \in E\}$ where $P_x(f) = |f|(|x|)$ for each $f \in E'$. For more information about locally convex solid topologies, we refer the reader to the book by Aliprantis and Burkinshaw [1].

2 Main Results

Let us recall that if an operator $T \colon E \to F$ between two Banach lattices is positive (*i.e.,* $T(x) \geq 0$ in F whenever $x \geq 0$ in E), then its dual operator $T' \colon F' \to E'$ is likewise positive, where T' is defined by $T'(f)(x) = f(T(x))$ for each $f \in F'$ and for each $x \in E$. An operator $T \colon E \to F$ is *regular* if $T = T_1 - T_2$ where T_1 and T_2 are positive operators. We denote by $E^+ = \{x \in E : 0 \leq x\}$. It is well known that each positive linear mapping on a Banach lattice is continuous.

A norm $\|\cdot\|$ of a Banach lattice E is order continuous if the locally convex solid topology defined by this norm is Lebesgue. For example, the norm of the Banach lattice l^1 is order continuous but the norm of the Banach lattice l^∞ is not.

As we have established by examples in the introduction, the AM-compactness property of an operator is not inherited by dual operators, and conversely. In fact, there exist Banach lattices E and F and an operator T from E into F which is AM-compact but its dual T' from F' into E' is not AM-compact, and conversely. Zaanen [6, Theorem 125.6] studied this problem and he proved that with auxiliary conditions on Banach lattices E and F we obtain an analogue of Schauder's theorem for AM-compact operators.

The following theorem is essentially due to Zaanen [6, Theorem 125.6], but we find that its proof is long and very difficult. In the following, we give an easy and original proof of this theorem, using arguments which are different from those of Zaanen.

Theorem 2.1 *Let E and F be two Banach lattices and T a regular operator from E into F.*

(i) *If E' has an order continuous norm and T is AM-compact, then the dual operator T' is AM-compact from F' into E'.*
(ii) *If F has an order continuous norm, the oerator T is AM-compact whenever its dual operator T' from F' is AM-compact.*

Proof (i) Let $x \in E^+$ and $f \in F^+$. Since T is an AM-compact operator, the subset $T([0, x])$ is norm relatively compact in F, and then relatively compact for $|\sigma|(F, F')$. Now, by applying [2, Theorem 1.3], we obtain that $T'([0, f])$ is relatively compact for $|\sigma|(E', E)$. As the topology $|\sigma|(E', E)$ and the topology defined by the norm of E' are Lebesgue, it follows from [2, Theorem 1.4] that they are equal on order bounded subsets of E'. Hence, $T'([0, f])$ is norm relatively compact in E'. This proves that T' is AM-compact.

(ii) Since the topology defined by the norm of F is Lebesgue, it coincides with $|\sigma|(F, F')$ on order bounded subsets of F [2, Theorem 1.4]. Hence, by a similar proof, we prove that T is an AM-compact operator. ■

Recall that a Banach lattice E is said to be an AM-space if for each $x, y \in E$ such that $\inf(x, y) = 0$, we have $\|x + y\| = \max\{\|x\|, \|y\|\}$. A Banach lattice E is an AL-space if its topological dual E' is an AM-space. For example, the Banach lattice l^1 is an AL-space and the Banach lattice l^∞ is an AM-space.

Also, a vector lattice E is order σ-complete if every majorized countable nonempty subset of E has a supremum.

We observe that the converse of Zaanen's theorem is not always true whenever the Banach lattices E and F are different. In fact, we have:

(i) If we take E to be a Banach lattice such that the norm of E' is not order continuous (for example l^1) and F is a finite-dimensional space, then it is clear that each operator T from E into F is AM-compact and its dual operator T' from F' into E' is also AM-compact.

(ii) If we take F be a Banach lattice such that its norm is not order continuous (for example l^∞) and E is a finite-dimensional space, then each regular operator T from E into F is AM-compact and its dual operator T' from F' into E' is also AM-compact.

Now, we state the converse result of Zaanen's theorem when $E = F$. In fact, we have the following theorem.

Theorem 2.2 *Let E be a Banach lattice.*

(i) *If, for each AM-compact operator T from E into E, the dual operator T' from E' into E' is AM-compact, then E' has an order continuous norm.*

(ii) *If E is order σ-complete and if each regular operator T from E into E is AM-compact whenever its dual operator T' from E' into E' is AM-compact, then E has an order continuous norm.*

Proof (i) Assume that the norm of E' is not order continuous. Then it follows from outlining the proof of [5, Theorem 1] that E contains a sublattice isomorphic to l^1 and there exists a positive projection P from E into l^1.

Consider the operator product $i \circ P \colon E \to l^1 \to E$, where i is the inclusion operator of l^1 into E. Since $i \circ P = i \circ Id_{l^1} \circ P$, the operator $i \circ P$ is AM-compact. But its dual operator $P' \circ i' \colon E' \to l^\infty \to E'$ is not AM-compact. If not, the operator $P' \circ i' \circ P' \colon l^\infty \to E'$ will be also AM-compact. Since l^∞ is an AM-space with unit, $P' \circ i' \circ P'$ is compact and hence the operator $P \circ i \circ P \colon E \to l^1$ is also compact. But its restriction to l^1, which is the identity operator of l^1, will be compact. This gives a contradiction.

(ii) Assume that the norm of E is not order continuous. Since E is an order σ-complete Banach lattice, it follows from (outlining) the proof of [5, Theorem 1] that E contains a sublattice which is isomorphic to l^∞ and there exists a positive projection P from E into l^∞. We know that the topological dual of l^∞ is not discrete [3, Corollary3.5]. Then by applying [5, Theorem 1], we obtain the existence of two operators S and T from l^∞ into E such that $0 \leq S \leq T$ with T compact and S not compact. This implies that $0 \leq S' \leq T'$ with $S', T' \colon E' \to (l^\infty)'$ and T' compact. Since the Banach lattice $(l^\infty)'$ has an order continuous norm and T' is AM-compact, it follows from [2, Theorem 1.2] that S' is AM-compact.

On the other hand, the operator product $P' \circ S' \colon E' \to (l^\infty)' \to E'$ is AM-compact, but the operator $S \circ P \colon E \to l^\infty \to E$ is not AM-compact. Otherwise, its restriction to the Banach lattice l^∞, which is S, would be AM-compact and since l^∞ is an AM-space, the operator S would be compact. This is a contradiction. ■

Recall that a nonzero element x of a vector lattice E is *discrete* if the order ideal generated by x equals the subspace generated by x. The vector lattice E is discrete if it admits a complete disjoint system of discrete elements. For example, the Banach lattice l^1 is discrete, but $C([0, 1])$ is not discrete.

Now, if in Theorem 2.2(ii) the Banach lattice E is not necessarily order σ-complete, we obtain the following result.

https://doi.org/10.4153/CMB-2008-002-0 Published online by Cambridge University Press

Theorem 2.3 *Let E and F be two Banach lattices and T be a regular operator from E into F. If T is AM-compact whenever its dual operator T' from F' into E' is AM-compact, then one of the following statements holds:*

(i) *the norm of F is order continuous,*
(ii) *E' is discrete.*

Proof Assume by the way of contradiction that conditions (i) and (ii) fail. Since the the norm of F is not order continuous, there exist some $z \in F^+$ and a disjoint sequence (z_n) in $[0, z]$ which does not admit any subsequence converging to 0 for the norm [1, Theorem 10.1]. Also, there exist some $\Phi \in (E')^+$ and a sequence (Φ_n) in $[0, \Phi]$ which converges to 0 for the weak topology $\sigma(E', E)$, but does not converge to 0 for the absolute weak topology $|\sigma|(E', E)$ [1, Corollary 21.13]. This implies that there exists some $y \in E^+$ and there exists a sequence (y_n) in $[0, y]$ such that $\Phi_n(y_n) = 1$ for each $n \in \mathbb{N}$.

Let $\widehat{E}$ be the completion of E for the absolute weak topology $|\sigma|(E, E')$ and let P_n be the principal projection on the band B_n generated by y_n in $\widehat{E}$. We can assume that $\Phi_n(y_m) = 0$ if $n \neq m$ (if not, we replace Φ_n by $\Phi_n \circ P_n$).

Let T be the positive operator defined by $T(x) = \left(\sum_{n=1}^{+\infty} \Phi_n(x) z_n\right) + \Phi(x) z$. Since (z_n) is a disjoint sequence and (Φ_n) converges to 0 weakly, the operator T is well defined.

We claim that T is not AM-compact. If not, the sequence $(T(y_n)) = (\Phi(y_n)z + z_n)$ admits a convergent subsequence that we design also by $(\Phi(y_n)z + z_n)$. But since the sequence $(\Phi(y_n))$ admits a convergent subsequence, it follows that (z_n) admits a convergent subsequence. This presents a contradiction, and hence T is not AM-compact as claimed.

Now we prove that T' is AM-compact. First, the operator T' is defined by the following formula:

$$T'(f)(x) = \Big(\sum_{n=1}^{+\infty} \Phi_n(x) f(z_n)\Big) + \Phi(x) f(z)$$

for each $f \in E'$ and for each $x \in E$.

Let $f \in (E')^+$ and $(f_n) \subset [0, f]$. Since

$$\sum_{k=1}^{+\infty} f_k(z_n) \leq \sum_{k=1}^{+\infty} f(z_n) \leq f(z),$$

$(f_k(z_n)) \in l^1$, $(f(z_n))_n \in l^1$, and $0 \leq (f_k(z_n))_n \leq (f(z_n))_n$ for each $k \in \mathbb{N}^*$.

On the other hand, since l^1 is discrete and its norm is order continuous, it follows from [1, Corollary 21.13] that the order interval $[0, (f(z_n))_n]$ is compact in l^1. Hence, the sequence $((f_k(z_n))_n)_k$ admits a convergent subsequence in l^1 that we design also by $((f_k(z_n))_n)_k$.

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for each p and $q \in \mathbb{N}$, with $p, q \geq N$, we have $\|\Phi\| \sum_n |f_p(z_n) - f_q(z_n)| + \|\Phi\| |f_p(z) - f_q(z)| < \varepsilon$.

Then if $p, q \geq N$ and $x \in E$ such that $\|x\| \leq 1$, we have

$$|T'(f_p)(x) - T'(f_q)(x)| \leq \|\Phi\| \sum_n |f_p(z_n) - f_q(z_n)| + \|\Phi\| |f_p(z) - f_q(z)| < \varepsilon.$$

This proves that $(T'(f_p))$ is a Cauchy sequence in E', and hence it is convergent. This proves that T' is an AM-compact operator. ■

Remark In Theorem 2.3, the second necessary condition is not sufficent. In fact, if we take $E = F = c$, the Banach lattice of all convergent sequences, it is clear that the identity operator of E is not AM-compact but its dual operator, which is the identity of the dual topological c', is AM-compact. However the Banach lattice c' is discrete.

References

[1] C. D. Aliprantis and O. Burkinshaw, *Locally Solid Riesz Spaces.* Pure and Applied Mathematics 76, Academic Press, New York, 1978.

[2] ———, *Positive compact operators on Banach lattices.* Math. Z. **174**(1980), no. 3, 289–298.

[3] B. Aqzzouz and R. Nouira, *Sur les opérateurs précompacts positifs.* C. R. Math. Acad. Sci. Paris **337**(2003), no. 8, 527–530.

[4] D. H. Fremlin, *Riesz spaces with the order continuity property. I.* Math. Proc. Cambridge Philos. Soc. **81**(1977), no. 1, 31–42.

[5] A. W. Wickstead, *Converses for the Dodds-Fremlin and Kalton-Saab theorems.* Math. Proc. Cambridge. Philos. Soc. **120**(1996), 175–179.

[6] A. C. Zaanen, *Riesz Spaces. II.* North-Holland Mathematical Library 30, North-Holland, Amsterdam, 1983.

(Aqzzouz)
Université Mohammed V-Souissi
Faculté des Sciences Economiques, Juridiques et Sociales
Départment d'Economie
B.P. 5295, Sala Eljadida
Morocco
e-mail: baqzzouz@hotmail.com

(Nouira, Zraoula)
Université Ibn Tofail
Département de Mathématiques
B.P. 133, Kenitra
Morocco