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Sabidussi-type theorems for stability

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In this note we give details of a method by which we can produce an index-0 graph from any unstable graph and use it to show that given any finite group there exists an index-0 graph whose automorphism group is isomorphic, as an abstract group, to the given group. We proceed to construct two infinite families of connected index-0 graphs with connected complements whose automorphism group contains a transposition. This enables us to produce, for any finite group G, an index-0 graph whose automorphism group, isomorphic as an abstract group to $C_2 \times G$, contains a transposition.

1. Index-0 graphs with given automorphism group

Throughout this note all graphs G are undirected, have no loops or multiple edges, and have finite vertex set $V(G)$ with $|V(G)| = p$. The basic terminology and notation is that of Behzad and Chartrand, [1].

If $v \in V(G)$, then by G_v we mean the induced subgraph $\langle V(G)-\{v\}\rangle$ of G, and by $G_{v_1v_2,\ldots,v_k}$ we mean $\left(\left(\cdots\left(\left(G_{v_1}\right)_{v_2}\right)\cdots\right)_{v_{k-1}}\right)_{v_k}$. G is said to be *semi-stable* (at $v \in V(G)$) if $\Gamma\left(G_v\right) = \Gamma(G)_v$, where $\Gamma(G)_v$ is the subgroup of $\Gamma(G)$ which fixes vertex v. If G is a graph which is not semi-stable we call G an *index*-0 graph (see [3]). If there exists a sequence $\{v_1, \ldots, v_p\}$ of all the vertices of G such that G

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is semi-stable at v_1 and $G_{v_1,\ldots,v_k}$ is semi-stable at v_{k+1} for $1 \le k \le p-1$ we say that G is *stable*.

We define the subgraph G_v of G to be *1-admissible* if and only if G is semi-stable at v. We denote by $A_1(G)$ the collection $\{G_v : G \text{ is semi-stable at } v\}$, by $B_1(G)$ the set of distinct components of the graphs in $A_1(G)$ and by $C_1(G)$ the unstable graphs in $B_1(G)$ which are not components of G. If M is a set of graphs define $A_1(M)$ as $\{A_1(G) : G \in M\}$, and $B_1(M)$ as the set of distinct components of the graphs in $A_1(M)$. If G is a graph, define $C_0(G)$ to be G if G is unstable and the empty graph if G is stable. For $k = 1, 2, \ldots$, define $B_k(G)$ to be $B_1(C_{k-1}(G))$ and $C_k(G)$ to be the set of unstable graphs in $B_k(G)$ which are not in $\bigcup_{i=0}^{k-1} C_i(G)$. Let $c(G) = \max\{k : C_k(G) \ne \emptyset\}$. Thus either each graph in $C_{c(G)}(G)$ is an index-0 graph or each unstable component of each 1-admissible subgraph of each graph in $C_{c(G)}(G)$ is in $\bigcup_{i=0}^{c(G)} C_i(G)$. Finally, we let $G^0 = \bigcup_{i=0}^{c(G)} C_i(G)$.

In Figure 1 we illustrate the process of obtaining G^0 from G.

LEMMA 1 ([4], Theorem 5). *If $D = \{G^1, \ldots, G^n\}$, then $\bigcup_{i=1}^{n} G^i$ is stable if and only if each G^i is stable.*

LEMMA 2. *Let G be a graph. If $C_k(G) \ne \emptyset$, then some component of each graph in $A_1(C_{k-1}(G))$ is in $\bigcup_{i=0}^{k} C_i(G)$.*

Proof. By definition, no graph in $C_{k-1}(G)$ is stable. Thus no 1-admissible subgraph of such a graph can be stable, whence at least one component of each 1-admissible subgraph of each graph in $C_{k-1}(G)$ is

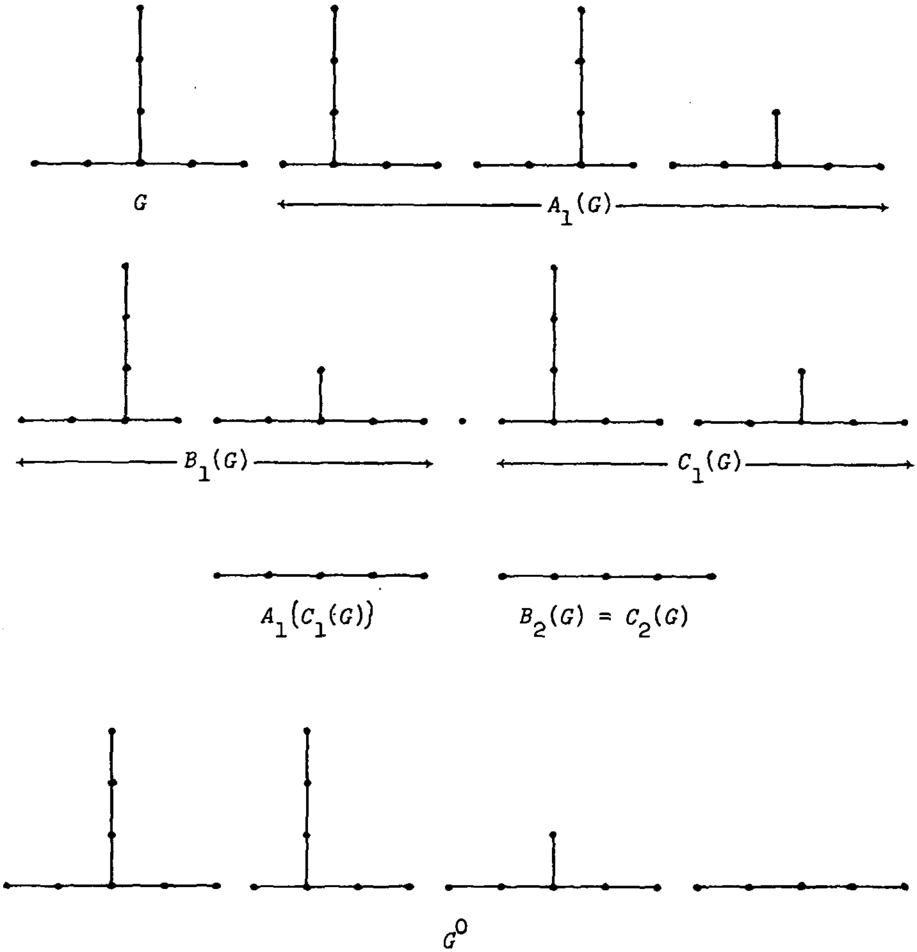


Figure 1

unstable. Such a component is, by definition, in $C_k(G)$ or $\bigcup_{i=0}^{k-1} C_i(G)$, whence the lemma follows.

The next result is Theorem 5 of [3]. To enable us to state it, we first need to introduce some notation.

If D is a collection of graphs (not necessarily distinct) at least one of which is semi-stable, then $A_1(D) \nmid D$ means that no component of some graph in $A_1(D)$ is a component of a graph in D .

LEMMA 3. *Let* $D = \{G^1, \ldots, G^n\}$ *be a collection of connected graphs at least one of which is semi-stable. Then* $\bigcup_{i=1}^{n} G^i$ *is semi-stable if and only if* $A_1(D) \nsubseteq D$.

COROLLARY. *Let* D *as defined above be a collection of graphs (not necessarily connected) at least one of which is semi-stable. Let* D_1 *be the collection of components of the graphs in* D *. Then* $\bigcup_{i=1}^{n} G^i$ *is semi-stable if and only if* $A_1(D_1) \nsubseteq D_1$.

We may now deduce:

THEOREM 1. *If* G *is an unstable graph, then* G^0 *is an index-0 graph.*

Proof. We remark first of all that G^0 is empty if G is stable. If G is unstable as hypothesised, then either G is an index-0 graph or G is semi-stable. In the former case $G^0 = G$ and the theorem is proved. If G is semi-stable then it follows from Lemmas 2 and 3 and the definition of G^0 that G^0 is an index-0 graph.

In [2], Frucht proved:

LEMMA 4. *Given any finite group* G *there exists a graph* $F(G)$ *(the* Frucht graph *of* G *), whose automorphism group is isomorphic, as an abstract group, to* G .

Frucht based his construction of $F(G)$ on the Cayley colour graph of G . From the construction also follows:

LEMMA 5. *The automorphism group* $\Gamma = \Gamma(F)$ *of the* Frucht *graph of a group acts semi-regularly on* $V(F)$ *(that is, each non-identity element of* Γ *is fixed-point free).*

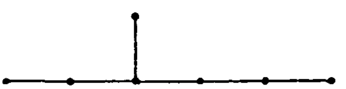
Sabidussi, [9], has extended Frucht's results, showing that, given any finite group, there exists a graph possessing various given graph-theoretical properties whose automorphism group is isomorphic, as an abstract group, to the given group. We now prove a Sabidussi-type theorem in which we prescribe our graphs to be index-0 graphs.

THEOREM 2. *Let* G *be any finite group of order* > 1 . *Then if* F *is the* Frucht *graph of* G , *the graph* F^0 *is an index*-0 *graph whose automorphism group is isomorphic, as an abstract group, to* G .

Proof. By Lemma 5, if v is any vertex of F , $\Gamma(F)_v$, the set of elements in $\Gamma(F)$ which fix v , is the identity. Thus $\Gamma(F)$ contains no transpositions, for certainly $|V(F)| > 2$. By Lemma 3 of [7], F is unstable. From Theorem 1 we deduce that F^0 is non-empty and is an index-0 graph. If F is not semi-stable, then $F^0 = F$ and $\Gamma(F^0) = \Gamma(F) \cong G$. Thus assume that F is semi-stable and let v be an arbitrary vertex at which F is semi-stable. It follows that $\Gamma(F_v) = \Gamma(F)_v$, and is the identity. Thus F_v is asymmetric, whence each component of F_v is asymmetric. We deduce that each graph in $A_1(F)$, $B_1(F)$ and $C_1(F)$ is asymmetric. By continuing the above argument, noting that the automorphism group of each graph in $C_1(F)$, being the identity, is semi-regular, we conclude that each graph in $C_k(F)$, for $k = 0, \ldots, c(F)$, is asymmetric. It then follows, since the components of F^0 are all different, that $\Gamma(F^0) \cong \Gamma(F) \cong G$.

COROLLARY. *Given any finite group* G *there exists an index*-0 *graph whose automorphism group is isomorphic, as an abstract group, to* G .

Proof. If the order of G is > 1 , the graph F^0 defined above is a suitable graph. If G is the identity group, the tree E_7 of Figure 2 is a suitable graph.



: E_7

Figure 2

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2. Index-0 graphs whose automorphism group contains a transposition

Lemma 3 of [7] states that if G is a stable graph then either G is K_1 or $\Gamma(G)$ contains a transposition. A great deal of research (see [5], [6] and [8]) has been done on the problem of finding those graphs whose automorphism group contains a transposition which are stable. Not all such graphs are stable, as witnessed by the graph G of Figure 3. $\Gamma(G)$ contains the transposition (12) but G is unstable.

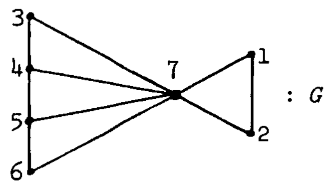

Figure 3

Here we show that in fact there exist graphs whose automorphism group contains a transposition which are not even semi-stable.

Let G be a graph. Denote by $K'(G)$ the graph $G + G = \overline{\bar{G} \cup \bar{G}}$ and by $D'(G)$ the graph $G + G + G = \overline{\bar{G} \cup \bar{G} \cup \bar{G}}$. Define $K(G)$ and $D(G)$ to be the graphs shown in Figure 4, where the various symbols used have the indicated meanings. In Figure 5 we show $K(P_4)$ and $D(P_4)$ in full detail.

We now obtain, as abstract groups, the automorphism groups of $K(G)$ and $D(G)$.

THEOREM 3. *For any graph* G ,

$$\Gamma(K(G)) \cong \Gamma(D(G)) \cong C_2 \times \Gamma(G) \times \Gamma(G) \times \Gamma(G) \times \Gamma(G) \times \Gamma(G) \times \Gamma(G) .$$

Proof. It is easy to see that the latter group is a subgroup of each of the automorphism groups. That there are no other automorphisms is proved by exhaustion; we omit the details.

For all G , $\Gamma(K(G))$ and $\Gamma(D(G))$ both contain a transposition. We now show that neither of these graphs is semi-stable if G is not semi-stable.

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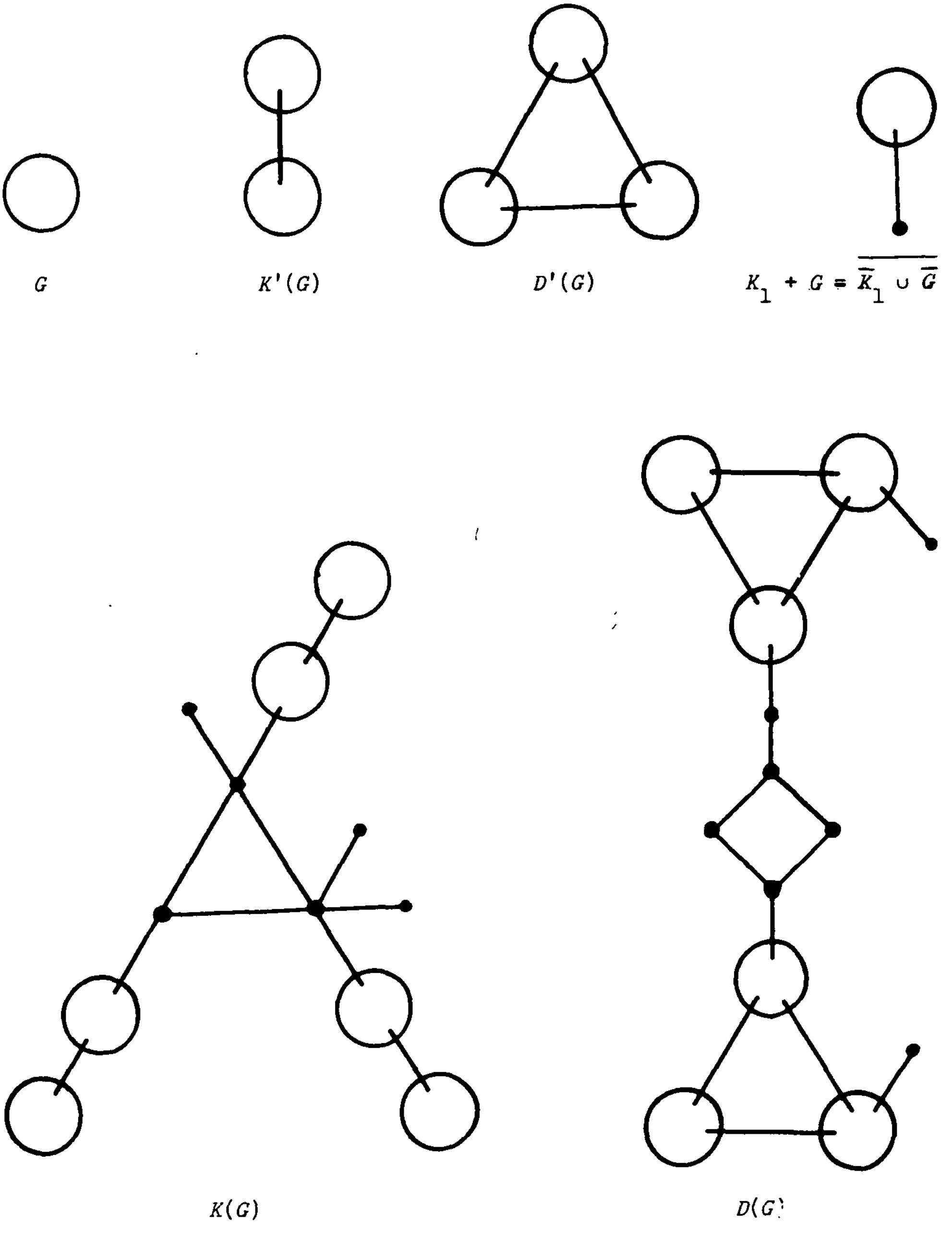
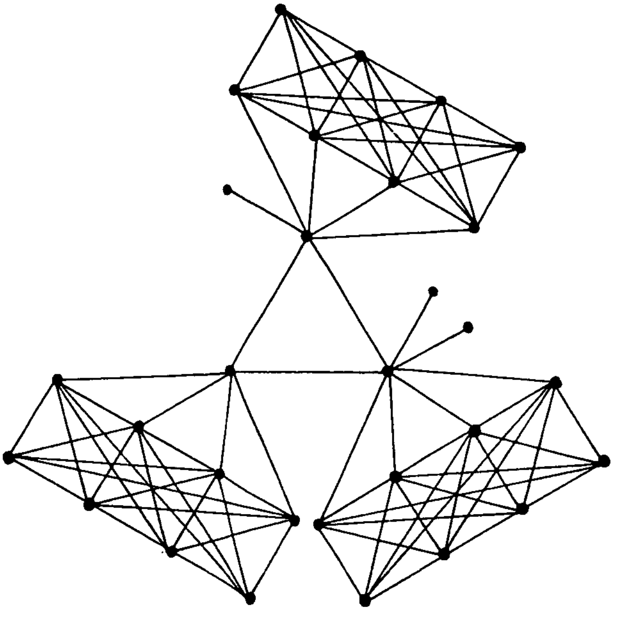
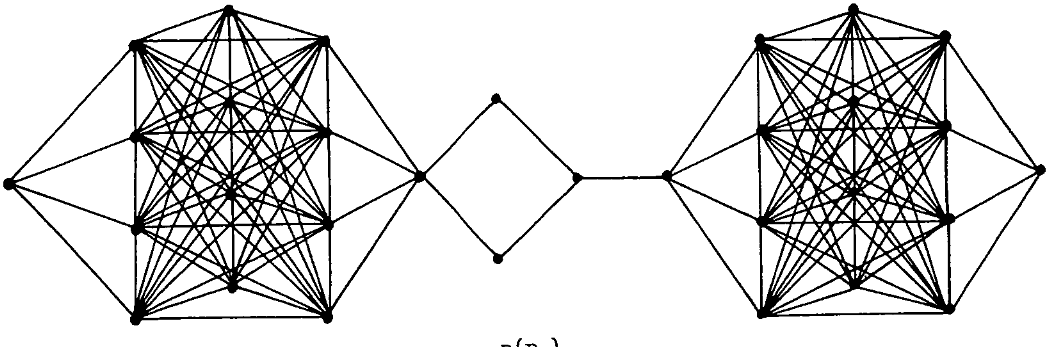


Figure 4



$K(P_4)$



$D(P_4)$

Figure 5

THEOREM 4. *If G is an index-0 graph, then $K(G)$ and $D(G)$ are both index-0 graphs.*

Proof. We give here a proof for $K(G)$; that for $D(G)$ is similar. Let the vertices of G be labelled $w_1, \ldots, w_r$, and those of $K(G)$ according to the scheme illustrated in Figure 6, where the vertices in copy i of G are labelled v_{ij} , $j = 1, \ldots, r$, corresponding respectively to w_j , $j = 1, \ldots, r$.

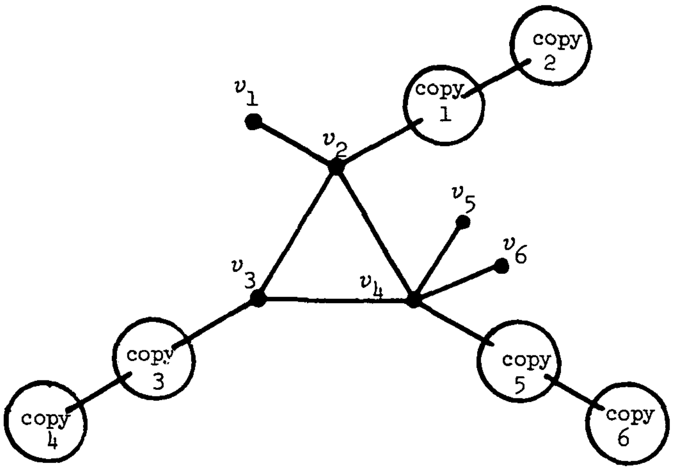


Figure 6

Removal of v_1 from $K = K(G)$ introduces automorphisms mapping v_2 onto v_3 and vertices in copies 1 and 2 of G onto vertices in copies 3 and 4 respectively. Removal of v_5 or v_6 introduces automorphisms mapping v_2 onto v_4 and vertices in copies 1 and 2 of G onto vertices in copies 5 and 6 respectively. Removal of v_2, v_3 or v_4 introduce respectively automorphisms mapping vertices in copy 1 of G onto vertices in copy 2, vertices in copy 3 onto vertices in copy 4 or vertices in copy 5 onto vertices in copy 6. Removal of vertex v_{ij} introduces the automorphism a given by $a(v_{ik}) = v_{is(k)}$ for $k \neq j$, which is generated

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by the automorphism a' which is introduced in $\Gamma\left(G_{w_j}\right)$ by removing vertex w_j from G, this automorphism being given by $a'\left(w_k\right) = w_{s(k)}$ for $k \neq j$. Thus removal of any vertex from K introduces new automorphisms, whence $\Gamma\left(K_v\right) \neq \Gamma(K)_v$ for any $v \in V(K)$, whence K is an index-0 graph.

Theorem 4 thus provides us with two infinite families of index-0 graphs whose automorphism group contains a transposition. We remark also that for all G, both $K(G)$ and $D(G)$ are complement connected (that is, $K(G)$, $\overline{K(G)}$, $D(G)$ and $\overline{D(G)}$ are all connected).

The smallest (in the sense of having the least number of vertices) index-0 graph whose automorphism group contains a transposition, which is given by Theorem 4, is $K\left(P_4\right)$, this graph having 30 vertices. We conjecture that in fact this graph is the smallest index-0 graph whose automorphism group contains a transposition.

We conclude by observing:

THEOREM 5. *Given any finite group G there exists an index-0 graph whose automorphism group contains a transposition and which is isomorphic, as an abstract group, to $C_2 \times G$.*

Proof. Let F be the Frucht graph of G. Then the graph $F' = F^0 \cup K\left(E_7\right)$ is an index-0 graph, being the union of two index-0 graphs, provided that G is of order > 1. If G is the identity group, the graph $F' = E_7 \cup K\left(E_7\right)$ is an index-0 graph. In each case F' has automorphism group isomorphic to $C_2 \times G$, as each component of F' is different.

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