Can. J. Math., Vol. XXXIV, No. 5, 1982, pp. 1091–1096

LOCALLY COMPACT NORMAL SPACES IN THE CONSTRUCTIBLE UNIVERSE

W. STEPHEN WATSON

Arhangel'skiĭ proved around 1959 [**1**] that, for the class of perfectly normal locally compact spaces, metacompactness and paracompactness are equivalent. It is shown to be consistent that this equivalence holds for the (larger) class of normal locally compact spaces (answering a question of Tall [**8**], [**9**]).

The consistency of the existence of locally compact normal non-collectionwise Hausdorff spaces has been known since 1967. It is shown that the existence of such spaces is independent of the axioms of set theory, thus establishing that Bing's example G cannot be modified under ZFC to be locally compact.

All topological spaces are assumed to be Hausdorff.

First, a definition and three standard lemmata are needed.

Definition. A family $\{A_i\}_{i\in I}$ is *separated* (by $\{B_i\}_{i\in I}$) if $\{B_i\}_{i\in I}$ is a disjoint family of open sets such that for each $i \in I$, A_i is contained in B_i.

LEMMA 1. *If X is normal, K is a subset of X and V is an open set containing K, then there is an open F_σ-set U which contains K and is contained in V.*

Proof. Applying normality, define inductively V_i for $i \geqq 0$ as follows: Let V_0 be an open set containing K such that the closure of V_0 is contained in V. Let V_{n+1} be an open set containing the closure of V_n such that the closure of V_{n+1} is contained in V. $U = \{V_i: i \geqq 0\}$ contains K and is contained in V. Since $U = \{\bar{V}_i: i \geqq 0\}$, U is an open F_σ-set as required.

LEMMA 2. *If X is a compact space and A is a closed subset of X, then the pseudocharacter of A in X equals the character of A in X.*

Proof. Suppose that A is the intersection of a family of open sets $\{U_\alpha: \alpha < \kappa\}$. By applying normality, assume without loss of generality that A is the intersection of $\{\bar{U}_\alpha: \alpha < \kappa\}$. Claim that the family of all finite intersections of elements of $\{U_\alpha: \alpha < \kappa\}$ is a neighbourhood base for A in X. Let W be an open set containing A. $X - W$ is a compact

Received February 3, 1981 and in revised form January 20, 1982 and February 25, 1982. This work was supported by the Natural Sciences and Engineering Research Council.

https://doi.org/10.4153/CJM-1982-078-8 Published online by Cambridge University Press

set which is covered by $\{X - \bar{U}_\alpha: \alpha < \kappa\}$. If

$$(X - \bar{U}_{\alpha 1}) \cup (X - \bar{U}_{\alpha 2}) \cup \ldots \cup (X - \bar{U}_{\alpha n})$$

contains $X - W$, then $\bar{U}_{\alpha_1} \cap \bar{U}_{\alpha_2} \cap \ldots \cap \bar{U}_{\alpha_n}$ is contained in W and thus a finite intersection of elements of $\{U_\alpha: \alpha < \kappa\}$ is contained in W as required.

LEMMA 3. *If X is a locally compact space and G is a compact subset of X, then there is a compact set K of countable character which contains G.*

Proof. Applying Theorem 3.3.2 of [**4**], let U be an open set with compact closure such that U contains G. Applying normality in the compact set $\bar{U}$, define inductively V_i for $i \geqq 0$ as follows:

Let V_0 be an open set which contains G such that $\overline{V_0}$ is contained in U. Let V_{n+1} be an open set which contains G such that $\overline{V_{n+1}}$ is contained in V_n. Let $K = \cup \{V_n: n \geqq 0\}$. $K = \cup \{\overline{V_n}: n \geqq 0\}$ and so K is a compact set containing G. K is a G_δ-set and so, since K is a closed set contained in the open set U which has compact closure, applying Lemma 2 shows that K has countable character in the closure of U and thus also in X.

Second, the main lemma is proved.

LEMMA 4. *Let X be a locally compact normal space. If there is a discrete unseparated family of compact sets $\{G_\alpha: \alpha < \kappa\}$ and if κ is minimal with respect to the existence of such a family, then there is a discrete unseparated family of compact sets $\{L_\alpha: \alpha < \kappa\}$ such that each L_α has character at most* $\mathrm{cf}(\kappa)$.

Proof. Let $\{S_\gamma: \gamma < \mathrm{cf}(\kappa)\}$ be a partition of κ into subsets of cardinality less than κ. Since κ is minimal, for each $\gamma < \mathrm{cf}(\kappa)$, we may choose a family $\{O_\alpha': \alpha \in S_\gamma\}$ of disjoint open sets such that, for each $\alpha \in S_\gamma$, O_α' contains G_α. By Lemma 5.1.17 of [**4**], each such family may be assumed discrete. For each $\alpha < \kappa$, by normality, let O_α be an open set containing G_α whose closure is disjoint from the closed set $\cup \{G_\delta: \beta \neq \alpha\}$. Apply Lemma 1 to assume, without loss of generality, that each O_α is an open F_σ-set contained in O_α'. $\cup \{O_\alpha: \alpha < \kappa\}$ contains $\cup \{G_\alpha: \alpha < \kappa\}$, so, by normality, let A be an open set containing $\cup \{G_\alpha: \alpha < \kappa\}$ whose closure is contained in $\cup \{O_\alpha: \alpha < \kappa\}$. By Lemma 3, for each $\alpha < \kappa$, let K_α be a compact set of countable character which contains G_α. Without loss of generality, assume that each K_α is contained in $\bar{A}$. For each $\alpha < \kappa$, define

$$L_\alpha = K_\alpha - \cup \{O_\beta: \beta \neq \alpha\}.$$

We claim that each $K_\alpha - L_\alpha$ is covered by a subfamily of $\{O_\beta: \beta \neq \alpha\}$ of cardinality at most $\mathrm{cf}(\kappa)$. For each $\alpha < \kappa$, $K_\alpha - L_\alpha$ is contained in $\cup \{O_\beta: \beta \neq \alpha\}$, each K_α intersects at most finitely-many elements of each

https://doi.org/10.4153/CJM-1982-078-8 Published online by Cambridge University Press

$\{O_\beta: \beta \in S_\gamma\}$ (since K is compact and each $\{O_\beta: \beta \in S_\gamma\}$ is a discrete family of sets) and so each K_α intersects at most $\mathrm{cf}(\kappa)$-many elements of $\{O_\beta: \beta \neq \alpha\}$. Since each $K_\alpha - L_\alpha$ is contained in $\cup \{O_\beta: \beta \neq \alpha\}$, while intersecting at most $\mathrm{cf}(\kappa)$-many elements of $\{O_\beta: \beta \neq \alpha\}$, the claim is proven. Each O_β is an F_σ-set, so each $K_\alpha - L_\alpha$ is contained in the union of at most $\mathrm{cf}(\kappa)$-many closed sets. Each of these closed sets is contained in O_β for some $\beta \neq \alpha$. For each $\beta \neq \alpha$, O_β is disjoint from L_α so that $K_\alpha - L_\alpha$ is the intersection of K_α with the union of, at most, $\mathrm{cf}(\kappa)$-many closed sets. That is, the pseudocharacter of L_α in K_α is at most $\mathrm{cf}(\kappa)$. K_α is compact, so applying Lemma 2, the character of each L_α in K_α is at most $\mathrm{cf}(\kappa)$. The character of each L_α in X is at most the product of the character of L_α in K_α and the character of K_α in X (see 3.1.E of [**4**]) which is at most $\mathrm{cf}(\kappa) \cdot \omega = \mathrm{cf}(\kappa)$ as required.

Third, two results of Fleissner are stated in the general form needed here.

THEOREM 1. ($V = L$) *If κ is regular and X is normal and λ-collectionwise Hausdorff for each $\alpha < \kappa$, then any closed discrete set of cardinality κ whose points have character at most κ is separated.*

THEOREM 2. (GCH) *If κ is singular and X is normal and λ-collectionwise Hausdorff for each $\lambda < \kappa$, then any closed discrete set of cardinality κ whose points have character bounded in κ is separated.*

Proofs. The proofs for first-countable spaces are in [**5**]. Those proofs work in the general case.

Fourth, these two results of Fleissner are combined with the main lemma:

THEOREM 3. ($V = L$) *Locally compact normal spaces are collectionwise Hausdorff (and collectionwise normal for compact sets).*

Proof. If not, let X be a counterexample. Apply Lemma 4 to X to obtain κ. If κ is regular, apply Theorem 1 to obtain a contradiction. If κ is singular, apply Theorem 2 to obtain a contradiction. These theorems apply when we identify each element of a discrete family of compact sets to a point since this identification is a perfect mapping and the hypotheses of these theorems are preserved by perfect mappings.

THEOREM 4. ($V = L$) *Normal spaces of countable type are collectionwise Hausdorff (the spaces of countable type include the Čech-complete spaces and the p-spaces of Arhangel'skiĭ* [**2**]).

Proof. The proof of Lemma 4 incorporates a strengthening due to T. Przymusiński which allows us to avoid the use of local compactness except in Lemma 3 which states that locally compact spaces are of countable type.

LEMMA 5. *Locally compact metacompact spaces which are collectionwise normal for compact sets are paracompact.*

Proof. The Michael–Nagami theorem (Theorem 5.3.3 of [**4**]) that metacompact collectionwise normal spaces are paracompact requires only collectionwise normality for compact sets in a locally compact space (refine any cover to consist of open sets with compact closure).

THEOREM 5. ($V = L$) *Locally compact normal metacompact spaces are paracompact.*

Proof. Combine Theorem 3 and Lemma 5.

THEOREM 6. ($V = L$) *Locally compact normal submetacompact spaces (submetacompact = θ-refinable) are paracompact.*

Proof. The proof of Lemma 5 holds for submetacompact spaces.

These results may be put into perspective by considering the following theorem of Arhangel'skiĭ:

THEOREM 7. *Perfectly normal locally compact metacompact spaces are paracompact.*

Proof. See [**1**] or [**8**].

Local compactness is a necessary hypothesis in Theorem 7 since without it, the Bing–Michael space (see exercise 5.5.3 of [**4**]) is a counter-example.

Metacompactness may not be a necessary hypothesis in Theorem 7.

Question 1. Is there a perfectly normal locally compact nonpara-compact space under ZFC?

The Kunen line [**7**] under CH and the rational sequence topology over a Q-set [**8**] under MA + ¬ CH are consistent examples of such a space. If there were such a space, it would be collectionwise normal for compact sets under $V = L$ by Theorem 3 but would fail to be collection-wise normal for compact sets under MA + ¬ CH by a result of Gruen-hage [**6**].

Perfectness is not a necessary hypothesis in Theorem 7 under $V = L$ by Theorem 5.

Question 2. Does ZFC imply that normal locally compact meta-compact spaces are paracompact?

P. Daniels [**3**] has shown, under ZFC, that if X is a normal locally compact space which is boundedly metacompact (that is, such that each open cover has an open refinement $\mathscr{U}$ and an integer n such that any element of X lies in at most n elements of $\mathscr{U}$) then X is paracompact.

THEOREM 8. *It is independent of the axioms of set theory whether there is a locally compact normal space which is not collectionwise Hausdorff.*

Proof. Theorem 3 shows that such a space cannot exist under $V = L$. Under MA + $\neg$ CH, the rational sequence topology over a Q-set is such a space.

THEOREM 9. *It is independent of the axioms of set theory whether there is a compact hereditarily normal space which is not hereditarily collectionwise Hausdorff.*

Proof. The Alexandroff compactification of the (hereditarily normal) rational sequence topology over a Q-set under MA + $\neg$ CH is hereditarily normal. Any space which is not hereditarily collectionwise Hausdorff has an open subspace which is not collectionwise Hausdorff and open subspaces of a compact space are locally compact.

Question 3. Is it consistent with the axioms of set theory that locally compact normal spaces are collectionwise normal?

THEOREM 10. ($V = L$) *Locally compact normal spaces are collectionwise normal for paracompact sets (and thus for metrizable sets).*

Proof. Let $\{A_\alpha: \alpha < \kappa\}$ be a discrete family of closed sets such that each A_α is paracompact. As a closed subspace, each A_α is locally compact (Theorem 3.3.8 of [**4**]). By Theorem 5.1.27 of [**3**], as a space, A_α is the free union of Lindelof subspaces and thus, by 3.8.C of [**4**], σ-compact subspaces. As a closed set each A_α is the union of a discrete family of σ-compact closed sets and so, without loss of generality, we assume that each A_α is σ-compact. Let

$$A_\alpha = \bigcup \{A_\alpha^n: n \geqq 0\}$$

where each A_α^n is compact. Define $\{O_\alpha^n: \alpha < \kappa\}$ inductively on $n \geqq 0$ such that:

1. $\{O_\alpha^n: n \geqq 0\}$ is an increasing sequence of open sets for each $\alpha < \kappa$.
2. $\{O_\alpha^n: \alpha < \kappa\}$ is a discrete family of open sets for each $n \geqq 0$.
3. O_α^n contains A_α^n for each $n \geqq 0$ and $\alpha < \kappa$.
4. The closure of O_α^n does not intersect A_β for any $\beta \neq \alpha$ and $n \geqq 0$.
5. O_α^n does not intersect O_β^{n-1} for any $n \geqq 1$ and $\beta \neq \alpha$.

Let

$$O_\alpha = \{O_\alpha^n: n \geqq 0\}.$$

$\{O_\alpha: \alpha < \kappa\}$ is a disjoint family of open sets such that, for each $\alpha < \kappa$, O_α contains A_α and the proof is complete.

Question 4. Does ZFC imply that locally compact normal collectionwise Hausdorff spaces are collectionwise normal?

Alas and Tall have observed that Lemma 5 and the results of [**10**] are sufficient to show:

THEOREM 11. *If the continuum function is one-one, and X is a locally compact normal space, then* $e(X) \leqq c(X)$.

REFERENCES

1. A. Arhangel'skiĭ, *The property of paracompactness in the class of perfectly normal local bicompact spaces*, Soviet Math. Doklady *13* (1972), 517–520.

2. ——— *On a class of spaces containing all metric spaces and all locally bicompact spaces*, Amer. Math. Soc. Transl. *92* (1965), 1–62.

3. P. Daniels, *Normal locally compact boundedly metacompact spaces are paracompact*, Can. J. Math., to appear.

4. R. Engelking, *General topology* (Polish Scientific Publishers, Warszawa, 1977).

5. W. G. Fleissner, *Normal Moore spaces in the constructible universe*, Proc. Amer. Math. Soc. *46* (1974), 294–298.

6. G. Gruenhage, *Paracompactness and subparacompactness in perfectly normal locally compact spaces*, preprint.

7. I. Juhász, K. Kunen and M. E. Rudin, *Two more hereditarily separable non-Lindelof spaces*, Can. J. Math. *28* (1976), 998–1005.

8. F. D. Tall, *On the existence of normal metacompact Moore spaces which are not metrizable*, Can. J. Math. *26* (1974), 1–6.

9. ——— *The countable chain condition versus separability*, Gen. Top. & Appl. *4* (1974), 315–340.

10. ——— *Weakly collectionwise Hausdorff spaces*, Top. Proc. *1* (1975), 295–304.

University of Toronto,
Toronto, Ontario