POSITIVE FUNCTIONALS AND REPRESENTATIONS OF TENSOR PRODUCTS OF SYMMETRIC BANACH ALGEBRAS

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1. Introduction. Except for using "algebra" rather than "ring" and "compact" rather than "bicompact", we adopt the terminology used in (**6**). Every symmetric algebra, A_i, will have an identity, e_i. All representations will be cyclic and symmetric. Sets of functionals will carry the relative weak* topology.

Gelbaum (**1**) and Tomiyama (**10**) studied Banach algebras which are tensor products of commutative Banach algebras completed in a cross-norm (see **7**). A result of their work provides conditions under which the maximal ideal space of such a Banach algebra is homeomorphic with the direct product of those of the two factors. This result has been generalized in various ways. Replacing maximal ideals by multiplicative functionals, Gil de Lamadrid (**3**) extended the theorem to non-commutative Banach algebras. A partial extension to maximal ideals with hull-kernel topology is given in (**2**). In (**9**), Gil de Lamadrid's theorem was generalized to certain locally convex algebras.

If the commutative Banach algebras are symmetric, the tensor product having the inherited involution $(x \otimes y)^* = x^* \otimes y^*$, maximal ideals can be identified with indecomposable normalized positive functionals. There is a one-to-one correspondence between these indecomposable normalized positive functionals and equivalence classes of irreducible representations, therefore these equivalence classes can also be identified with maximal ideals. Since all these irreducible representations are one-dimensional, it is possible to identify maximal ideals only with equivalence classes of finite-dimensional representations.

In this paper we investigate the extension of the Gelbaum-Tomiyama result to non-commutative symmetric Banach algebras by replacing maximal ideals with these various objects. An appropriate generalization exists when maximal ideals are replaced by equivalence classes of irreducible finite-dimensional representations, but no generalization of this type exists when maximal ideals are replaced by indecomposable positive functionals or simply by equivalence classes of irreducible representations.

2. Tensor products of positive functionals.

LEMMA 1. *Let A_3 be the completion, with respect to a cross-norm, ν, of the tensor*

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product, $A_1 \otimes A_2$, of symmetric Banach algebras A_1, A_2 and let A_3 be a symmetric Banach algebra with the inherited involution. Every tensor product, $p_1 \otimes p_2$, of two normalized positive functionals is extendible to a normalized positive functional on A_3 if and only if

$$(*) \qquad \nu(a) \geqq \sup |p_1 \otimes p_2(a)|$$

for all a in $A_1 \otimes A_2$, where the supremum is taken over all normalized positive functionals p_i on A_i, $i = 1, 2$.

Proof. Normalized positive functionals have bound unity; thus the condition is necessary. The functional $p_1 \otimes p_2$ is positive on $A_1 \otimes A_2$ for if $x = \sum_{i=1}^n f_i \otimes g_i$, then

$$p_1 \otimes p_2(x^*x) = \sum_{i,j=1}^{n} p_1(f_i^* f_j) p_2(g_i^* g_j).$$

The arrays $p_1(f_i^*f_j)$ and $p_2(g_i^*g_j)$ are positive hermitian matrices. Schur (**8**) showed the Hadamard product of positive hermitian matrices to be positive, thus $p_1 \otimes p_2(x^*x) \geqq 0$. If $p_1 \otimes p_2$ is ν-bounded on $A_1 \otimes A_2$, it can be extended to a positive functional on A_3. Normalization is immediate.

Henceforth, the conditions of Lemma 1 are assumed to hold. The continuous extension will be written simply as $p_1 \otimes p_2$.

LEMMA 2. *The mapping, $\otimes$: $(p_1, p_2) \to p_1 \otimes p_2$, of the direct product of the spaces of normalized positive functionals on A_1 and A_2 is a homeomorphism into the space of those on A_3. Moreover, $p_1 \otimes p_2$ is indecomposable if and only if p_1 and p_2 are.*

Proof. Given p, a positive functional on A_3, define $p_1(f) = p(f \otimes e_2)$, $p_2(g) = p(e_1 \otimes g)$, f and g being arbitrary elements of A_1 and A_2, respectively. The mapping of p to (p_1, p_2) is an inverse for the mapping $\otimes$ when restricted to its range, therefore $\otimes$ is one-to-one. Since the spaces of normalized positive functionals are compact Hausdorff spaces, it suffices to show that $\otimes$ is continuous. For any $a \in A_3$, there is, for any $\delta > 0$, $\sum_{i=1}^n f_i \otimes g_i = b \in A_1 \otimes A_2$ such that $\nu(a - b) < \delta$. This implies that $|p(a - b)| < \delta$ for any normalized positive functional, p, on A_3. Let $M = \max\{||f_j||, ||g_j||\}$ and let p_i, p_i' denote normalized positive functionals on A_i. If $|p_1'(f_i) - p_1(f_i)| < \gamma$ and $|p_2'(g_i) - p_2(g_i)| < \gamma$, then $|p_1' \otimes p_2'(b) - p_1 \otimes p_2(b)| < 2nM\gamma$, thus

$$|p_1' \otimes p_2'(a) - p_1 \otimes p_2(a)| < 2nM\gamma + \delta.$$

By choosing first δ and then γ, we can make this last quantity arbitrarily small. As a runs over a finite set in A_3 and defines a basic weak* neighbourhood in the dual, one obtains (for given δ) corresponding finite sets in A_1 and A_2. Using these sets and an appropriate γ to define neighbourhoods in the dual spaces of A_1 and A_2, one obtains in the product of the dual spaces a neighbourhood such that every pair (p_1, p_2) in this neighbourhood has its image in the originally chosen basic neighbourhood in the dual of A_3.

Suppose that p_1 and p_2 are indecomposable, and that

$$\lambda p_1 \otimes p_2(x^*x) \geqq F(x^*x) \geqq 0$$

for all x in A_3. For all u in A_2, define $(F_{1u}(f) = F(f \otimes u^*u)$, positive functionals on A_1. When these functionals do not vanish identically, they can be normalized. We denote the normalized functionals by F_{1u}'.

For any f in A_1 and u in A_2, if F_{1u}' exists, then

$$\lambda p_1(f^*f)p_2(u^*u) = \lambda p_1 \otimes p_2(f^*f \otimes u^*u) \geqq F(f^*f \otimes u^*u) = F_{1u}'(f^*f)F_{1u}(e_1) \geqq 0$$

which is $[\lambda p_2(u^*u)/F(e_1 \otimes u^*u)]p_1(f^*f) \geqq F_{1u}'(f^*f) \geqq 0$. Since p_1 is indecomposable, $F_{1u}' = cp_1$. By normalization, $F_{1u}' = p_1$. Thus, if F_{1u} does not vanish identically, we have that $F(f \otimes u^*u) = p_1(f)F(e_1 \otimes u^*u)$, and if F_{1u} does vanish identically, the equation holds trivially. Using the polarization identity, we conclude that $F(f \otimes g) = p_1(f)F(e_1 \otimes g)$ for all f and g. A similar calculation yields $F(f \otimes g) = F(f \otimes e_2)p_2(g)$, thus

$$F(f \otimes g) = F(e_1 \otimes e_2)p_1(f)p_2(g).$$

By linearity and continuity, $F = F(e_1 \otimes e_2)p_1 \otimes p_2$ on A_3. Thus $p_1 \otimes p_2$ is indecomposable if both p_1 and p_2 are.

Suppose that $p_1 \otimes p_2$ is indecomposable and that $\lambda p_1(f^*f) \geqq p(f^*f) \geqq 0$ for all f in A_1. By Lemma 1, $(\lambda p_1 - p) \otimes p_2$ is positive, thus

$$\lambda p_1 \otimes p_2(a^*a) \geqq p \otimes p_2(a^*a) \geqq 0$$

for all a in A_3. Since $p_1 \otimes p_2$ is indecomposable, $p \otimes p_2 = Cp_1 \otimes p_2$ for some C. Since p_2 is normalized,

$$p(f) = p \otimes p_2(f \otimes e_2) = Cp_1 \otimes p_2(f \otimes e_2) = Cp_1(f)$$

for all f in A_1. Thus p_1 is indecomposable. Similar arguments show p_2 indecomposable.

Lemma 2 provides two homeomorphisms of interest, $\otimes$ and its restriction to pairs of indecomposable elements. No indecomposable element of the range is excluded from the range of the restriction. Elementary counter-examples show neither of these homeomorphisms is, in general, onto. The range of $\otimes$ need not be all normalized positive functionals, and if A_3 is not commutative, the range of the restriction is not necessarily all indecomposable normalized positive functionals on A_3. (We can conclude from the previously mentioned fact, that no indecomposable element of the range of $\otimes$ is excluded from the range of the restriction, that those indecomposable normalized positive functionals not in the range of the restrictions are also not in the range of $\otimes$.)

3. Tensor products of representations. Let P_i be the set of indecomposable normalized positive functionals and R_i the set of equivalence classes of irreducible representations of the symmetric Banach algebra A_i. There is a canonical mapping Π_i: $P_i \to R_i$ of P_i onto R_i (**6**, §17, Theorem 2 and §19, Theorem 2). We give R_i the quotient topology induced by the topology of P_i and the mapping Π_i, thus Π_i becomes continuous.

LEMMA 3. *The mapping* Π_i *is open.*

Proof. This is equivalent to showing the inverse image of the image of an open set to be open. For p in P_i, consider a neighbourhood of the form

$$U = \{q \in P_i: |q(a_j) - p(a_j)| < \epsilon, j = 1, 2, \ldots, n\}.$$

We recall that all representations in this paper are to be cyclic. For some representation ϕ and cyclic vector ξ, $p(a) = \langle \phi(a)\xi, \xi\rangle$ for all a in A_i. Any p' is in the inverse image of the equivalence class of ϕ if and only if

$$p'(a) = \langle \phi(a)\xi', \xi'\rangle$$

for some cyclic vector ξ'. About any such p', we exhibit a neighbourhood, V, contained in the inverse image of the image of U. Since ξ' is cyclic, there is x in A_i such that

$$||\xi - \phi(x)\xi'|| < \min\{\epsilon/8||\phi(a_j)||, (\epsilon/4||\phi(a_j)||)^{1/2}, \epsilon/48||a_j||, (\epsilon/48||a_j||)^{1/2}, 1/8\},$$

where the minimum of the five quantities for $j = 1, 2, \ldots, n$ is meant, zero denominators excluded. Define

$$V = \{q' \in P_i: |q'(x^*a_jx) - p'(x^*a_jx)| < \epsilon/8, |q'(x^*x) - p'(x^*x) < 1/8, |q'(x^*x) - p'(x^*x)|\,||a_j|| < \epsilon/48;\ j = 1, 2, \ldots, n\}.$$

Each q' in V can be expressed as $q'(a) = \langle \psi(a)\eta', \eta'\rangle$ for some irreducible representation ψ and cyclic vector η'. We can define the normalized functional q by $q(a) = \langle \psi(a)\psi(x)\eta', \psi(x)\eta'\rangle/\langle \psi(x)\eta', \psi(x)\eta'\rangle = q'(x^*ax)/q'(x^*x)$ since

$$|1 - q'(x^*x)| \leqq |1 - p'(x^*x)| + |p'(x^*x) - q'(x^*x)| \leqq$$
$$|\,||\xi||^2 - ||\phi(x)\xi'||^2| + 1/8 \leqq 2||\xi - \phi(x)\xi'|| + ||\xi - \phi(x)\xi'||^2 + 1/8 < 1/2,$$

thus $q'(x^*x) \neq 0$. Variants of the preceding computation yield

$$|1 - q'(x^*x)|\,||a_j|| < \epsilon/12,$$

$j = 1, 2, \ldots, n$, and $|1 - p'(x^*x)| < 1/2$. We then have that

$$\begin{aligned}|q(a_j) - p(a_j)| &\leqq |q(a_j) - p'(x^*a_jx)| + |p'(x^*a_jx) - p(a_j)|\\ &\leqq |[q'(x^*a_jx)/q'(x^*x)] - p'(x^*a_jx)|\\ &\qquad + |\langle \phi(x^*a_jx)\xi', \xi'\rangle - \langle \phi(a_j)\xi, \xi\rangle|\\ &\leqq \frac{|q'(x^*a_jx) - p'(x^*a_jx)| + |1 - q'(x^*x)|\,|p'(x^*a_jx)|}{1 - |1 - q'(x^*x)|}\\ &\qquad + |\langle \phi(a_j)\phi(x)\xi', \phi(x)\xi'\rangle - \langle \phi(a_j)\xi, \xi\rangle|\\ &< \epsilon/2 + ||\phi(a_j)||(2||\phi(x)\xi' - \xi|| + ||\phi(x)\xi' - \xi||^2)\\ &< \epsilon\end{aligned}$$

which shows q to be in U. Since the representations are all irreducible, any non-zero vector is cyclic and q and q' have the same image under Π_i.

Suppose that p belongs to an open subset G of P_i and p' to have the same image under Π_i. Then p has a neighbourhood of the form U contained in G and there is a neighbourhood of p' of the form V such that V is in the inverse image of the image of U, hence of G. Thus, every point of the inverse image of the image of an open set has a neighbourhood in this inverse image, i.e., the inverse image of the image of an open set is open.

An inner product is defined on the tensor product of two Hilbert spaces by $\langle h_1 \otimes h_2, \; k_1 \otimes k_2 \rangle = \langle h_1, k_1 \rangle \langle h_2, k_2 \rangle$. The resulting norm, σ, is a uniform cross-norm and the σ-completion of $H_1 \otimes H_2$ is a Hilbert space (see **5** and **7**). If ϕ_i is a representation of the symmetric Banach algebra A_i by operators on H_i, $i = 1, 2$, then $\phi_1 \otimes \phi_2$ is a representation of $A_1 \otimes A_2$ on the completion of $H_1 \otimes H_2$. (A cyclic element is exhibited in the proof of Lemma 4.)

LEMMA 4. *Under the conditions of Lemma* 1, (*) *is necessary and sufficient for every tensor product* $\phi_1 \otimes \phi_2$ *of representations* ϕ_i *of* A_i *on* H_i, $i = 1, 2$, *to be extendible to a representation of* A_3 *on* H_3, *the* σ*-completion of* $H_1 \otimes H_2$.

Proof. We first show that if ϕ_i has a cyclic vector ξ_i, then $\xi_1 \otimes \xi_2$ is a cyclic vector of $\phi_1 \otimes \phi_2$ as a representation of $A_1 \otimes A_2$, and, consequently, of any extension. It suffices to show that the orbit of $\xi_1 \otimes \xi_2$ has an element arbitrarily close to every element $\sum_{j=1}^{n} h_{1j} \otimes h_{2j}$ of $H_1 \otimes H_2$. For each j and every $\epsilon > 0$ there are a_{1j}, a_{2j} in A_1 and A_2, respectively, such that

$$2n||h_{2j}|| \, ||\phi_1(a_{1j})\xi_1 - h_{1j}|| < \epsilon \quad \text{and} \quad 2n||\phi_1(a_{1j})\xi_1|| \, ||\phi_2(a_{2j})\xi_2 - h_{2j}|| < \epsilon,$$

therefore

$$\left|\left|\phi_1 \otimes \phi_2\left(\sum_{j=1}^{n} a_{1j} \otimes a_{2j}\right)\xi_1 \otimes \xi_2 - \sum_{j=1}^{n} h_{1j} \otimes h_{2j}\right|\right| \leqq$$

$$\sum_{j=1}^{n} ||\phi_1(a_{1j})\xi_1 \otimes [\phi_2(a_{2j})\xi_2 - h_{2j}]|| + ||[\phi_1(a_{1j})\xi_1 - h_{1j}] \otimes h_{2j}|| < \epsilon.$$

Let $p_i(a_i) = \langle \phi_i(a_i)\xi_i, \xi_i \rangle$, $i = 1, 2$. Then

$$p_1 \otimes p_2(a_1 \otimes a_2) = \langle \phi_1 \otimes \phi_2(a_1 \otimes a_2)\xi_1 \otimes \xi_2, \xi_1 \otimes \xi_2 \rangle,$$

thus, if $\phi_1 \otimes \phi_2$ is extendible to A_3, then $p_1 \otimes p_2$ is also. Thus if every $\phi_1 \otimes \phi_2$ is extendible, every $p_1 \otimes p_2$ is extendible, which implies (*).

Assuming (*), we show that every tensor product of positive functionals can be extended. To show that any $\phi_1 \otimes \phi_2$ can be extended, it suffices to show it is of bounded norm as an operator from $A_1 \otimes A_2$ into the bounded operators on H_3. (The range of $\phi_1 \otimes \phi_2$ consists of bounded operators because σ is a uniform cross-norm, thus $||\phi_1 \otimes \phi_2(a_1 \otimes a_2)|| = ||\phi_1(a_1)|| \, ||\phi_2(a_2||)$.) Consider any representation ϕ, with cyclic vector ξ_0, of a symmetric normed algebra, A, on a Hilbert space H. Define the positive functional f on A by

$$f(a) = \langle \phi(a)\xi_0, \xi_0 \rangle.$$

Then

$$\begin{aligned} ||\phi(a)|| &= \sup_{\xi \in H} |\langle \phi(a)\xi, \xi\rangle / \langle \xi, \xi \rangle| \\ &= \sup_{b \in A, \phi(b)\xi_0 \neq 0} |\langle \phi(a)\phi(b)\xi_0, \phi(b)\xi_0\rangle / \langle \phi(b)\xi_0, \phi(b)\xi_0\rangle| \\ &= \sup_{b \in A, \phi(b)\xi_0 \neq 0} |\langle \phi(b^*ab)\xi_0, \xi_0\rangle / \langle \phi(b^*b)\xi_0, \xi_0\rangle| \\ &= \sup_{b \in A, f(b^*b) \neq 0} |f(b^*ab)/f(b^*b)|. \end{aligned}$$

If all positive functionals are extendible to the completion of A, f can be extended so that $f(b^*ab)$ is a positive functional on a symmetric Banach algebra, whence $|f(b^*ab)| \leqq ||a|| f(b^*b)$ for all b. From this, $||\phi(a)||/||a|| \leqq 1$, thus ϕ can also be extended. Applying this argument to $\phi_1 \otimes \phi_2$, (*) guarantees the extendibility of $\phi_1 \otimes \phi_2$ to A_3.

As with the positive functionals, we denote the extension simply by $\phi_1 \otimes \phi_2$. We thus have a mapping $\otimes$: $(\phi_1, \phi_2) \to \phi_1 \otimes \phi_2$ of pairs of representations of A_1 and A_2 into the representations of A_3. If ϕ_i is equivalent to ϕ_i', $i = 1, 2$, $\phi_1 \otimes \phi_2$ is equivalent to $\phi_1' \otimes \phi_2'$, thus the mapping $\otimes$ defined on pairs of representations induces a mapping $\otimes^{\#}$ of pairs of equivalence classes of representations of A_1 and A_2 to equivalence classes of those of A_3.

LEMMA 5. *The mapping $\otimes^{\#}$ of pairs of equivalence classes of representations is induced by the mapping $\otimes$ of pairs of positive functionals. The representation $\phi_1 \otimes \phi_2$ of A_3 is irreducible if and only if both ϕ_1 and ϕ_2 are.*

Proof. Suppose that the p_i's are normalized positive functionals associated with representations ϕ_i, $i = 1, 2$. We show that $p_1 \otimes p_2$ is associated with $\phi_1 \otimes \phi_2$, i.e., $\otimes$ induces $\otimes^{\#}$. We have $p_i(a_i) = \langle \phi_i(a_i)\xi_i, \xi_i\rangle$ for all a_i in A_i, $i = 1, 2$, and for ξ_i a cyclic vector associated with ϕ_i. Then

$$\begin{aligned} p_1 \otimes p_2(a_1 \otimes a_2) = p_1(a_1)p_2(a_2) = \langle \phi_1(a_1)\xi_1, \xi_1\rangle\langle \phi_2(a_2)\xi_2, \xi_2\rangle = \\ \langle \phi_1(a_1)\xi_1 \otimes \phi_2(a_2)\xi_2, \xi_1 \otimes \xi_2\rangle = \langle [\phi_1(a_1) \otimes \phi_2(a_2)]\xi_1 \otimes \xi_2, \xi_1 \otimes \xi_2\rangle = \\ \langle [\phi_1 \otimes \phi_2(a_1 \otimes a_2)]\xi_1 \otimes \xi_2, \xi_1 \otimes \xi_2\rangle, \end{aligned}$$

and the equality holds on all A_3 by linearity and continuity. This, the fact that a positive functional is indecomposable if and only if the associated representation is irreducible, and Lemma 2 yield immediately the result on the irreducibility of $\phi_1 \otimes \phi_2$.

From Lemma 5 it follows that the diagram

$$\begin{array}{ccc} P_1 \times P_2 & \xrightarrow{\otimes} & P_3 \\ \Pi_1 \times \Pi_2 \big\downarrow & & \big\downarrow \Pi_3 \\ R_1 \times R_2 & \xrightarrow{\otimes^{\#}} & R_3 \end{array}$$

commutes. In Lemma 2, $\otimes$ was shown to be a homeomorphism into P_3. The mappings Π_i are continuous by definition of the topology of R_i and in Lemma 3 they were shown to be open. The product mapping $\Pi_1 \times \Pi_2$ also is continuous and open.

LEMMA 6. *The mapping $\otimes^{\#}$ is one-to-one and continuous.*

Proof. By continuity of Π_3, the inverse image in P_3 of an open set in R_3 is open. Since $\otimes$ is a homeomorphism into P_3, the inverse image of this open set is open in $P_1 \times P_2$. As $\Pi_1 \times \Pi_2$ is open, its image is open in $R_1 \times R_2$. This is just the inverse image under $\otimes^{\#}$ of the original open set in R_3, therefore $\otimes^{\#}$ is continuous. (This type of argument does not show that $\otimes^{\#}$ is open unless if the image of $P_1 \otimes P_2$ is open in P_3.) To show that $\otimes^{\#}$ is one-to-one, for any representation ϕ of A_3 construct the following representations of A_1 and A_2, namely, $\phi_1(a_1) = \phi(a_1 \otimes e_2)$ and $\phi_2(a_2) = \phi(e_1 \otimes a_2)$. The mapping of the equivalence class of ϕ to the pair of equivalence classes of ϕ_1 and ϕ_2 is an inverse for $\otimes^{\#}$ when restricted to its range.

4. Tensor products of finite-dimensional representations. The mapping $\otimes^{\#}$ is not onto R_3 in general. Mackey (**4**) has exhibited two locally compact groups such that not every irreducible unitary representation of their direct product is equivalent to a tensor product of representations of the two groups. Using the corresponding group algebras, one can construct a counter-example to the supposition that $\otimes^{\#}$ is necessarily onto R_3. We now restrict consideration to the finite-dimensional representations. Denote the set of equivalence classes of finite-dimensional irreducible representations of A_i by R_i', the set of associated indecomposable positive functionals by P_i'. Restrict the previous diagram to these finite-dimensional parts by attaching primes to each symbol. Since the range of $\otimes^{\#'}$ is clearly in R_3', that of $\otimes'$ is in P_3' and $\otimes'$ is a homeomorphism into P_3'. The mappings Π_i' are continuous. Since the open mappings Π_i carry the complements of P_i' onto the complements of R_i', the restrictions Π_i' are also open, and $\Pi_1' \times \Pi_2'$ is continuous and open.

Before proceeding, we consider briefly the representations of finite-dimensional reduced symmetric algebras and their tensor products. A reduced algebra is semi-simple. If it is finite-dimensional it is the direct sum of simple ideals each isomorphic to the algebra of all linear transformations on some finite-dimensional vector space. (It is easily seen that a simple ideal of a reduced symmetric algebra is symmetric and an inner-product can be assigned to the finite-dimensional vector space so that the isomorphism of the simple ideal with the algebra of all linear transformations is symmetric.) Any irreducible representation of a finite-dimensional reduced symmetric algebra vanishes except on a single simple ideal. Its restriction to that ideal is a symmetric isomorphism with the algebra of all linear transformations on a finite-dimensional inner-product space. Each simple ideal induces an irreducible representation and representations induced by the same simple ideal are equivalent.

Consider the tensor product of two finite-dimensional reduced symmetric algebras. Decomposing each factor as a direct sum of simple ideals yields a decomposition of the tensor product as a direct sum of tensor products of simple ideals. Each of the simple ideals being isomorphic to the algebra of all linear transformations on a finite-dimensional vector space, the tensor product of two of them is isomorphic to the algebra of all linear transformations on the tensor product of the two spaces and is therefore simple. The decomposition of the tensor product as a direct sum of tensor products of simple ideals is thus also the decomposition as a direct sum of simple ideals. Each simple ideal induces an irreducible representation, and any non-zero element has a non-zero component in some direct summand, therefore the tensor product of finite-dimensional reduced symmetric algebras is reduced. Each simple ideal of the tensor product is a tensor product of simple ideals and the representation associated with it is equivalent to the tensor product of those associated with its factors.

THEOREM. *The mapping* $\otimes^{\#'}$ *is a homeomorphism of* $R_1' \times R_2'$ *onto* R_3'.

Proof. By Lemma 6, $\otimes^{\#}$ is one-to-one and continuous, therefore $\otimes^{\#'}$ is also. We must show $\otimes^{\#'}$ to be onto R_3' and open.

Let $\phi \in r_3 \in R_3'$. To show that $\otimes^{\#'}$ is onto R_3' we must show that there exist $\phi_i \in r_i \in R_i'$ $(i = 1, 2)$ such that $\phi_1 \otimes \phi_2 \in r_3$. To this end define M_i, closed symmetric ideals of A_i $(i = 1, 2, 3)$ by $M_1 = \{a_1: \phi(a_1 \otimes e_2) = 0\}$, $M_2 = \{a_2: \phi(e_1 \otimes a_2) = 0\}$, $M_3 = \{a_3: \phi(a_3) = 0\}$. Each M_i is the kernel of a representation of the corresponding A_i and thus contains its reducing ideal. Since ϕ is finite-dimensional, all the M_i are of finite co-dimension. Thus A_i/M_i is a finite-dimensional reduced symmetric algebra and, by the remarks preceding the theorem, so is $(A_1/M_1) \otimes (A_2/M_2)$.

It is easily seen that $M_3 \supset A_1 \otimes M_2 + M_1 \otimes A_2$, thus we can define a representation, $\phi^{\#}$, of $(A_1/M_1) \otimes (A_2/M_2)$ by

$$\phi^{\#}((a_1 + M_1) \otimes (a_2 + M_2)) = \phi(a_1 \otimes a_2).$$

The image of $\phi^{\#}$ is the image of ϕ, since ϕ is finite-dimensional, therefore $\phi^{\#}$ is irreducible. Again by the remarks preceding the theorem, there are irreducible representations $\phi^{\#}_i$ on A_i/M_i $(i = 1, 2)$ such that $\phi^{\#}$ is equivalent to $\phi^{\#}_1 \otimes \phi^{\#}_2$. This means that there is an isometry U such that

$$\phi^{\#} U = U(\phi^{\#}_1 \otimes \phi^{\#}_2).$$

Defining irreducible representations ϕ_i on A_i $(i = 1, 2)$ by

$$\phi_i(a_i) = \phi^{\#}_i(a_i + M_i)$$

we have that $\phi U = \phi^{\#} U = U(\phi^{\#}_1 \otimes \phi^{\#}_2) = U(\phi_1 \otimes \phi_2)$ on $A_1 \otimes A_2$. By the standing assumption, (*) guaranteeing a continuous extension of $\phi_1 \otimes \phi_2$, these relations hold on A_3. Thus ϕ is equivalent to $\phi_1 \otimes \phi_2$ and the mapping $\otimes^{\#'}$ is onto R_3'.

The images under Π_i' of any basis for the topology of P_i' form a basis for the topology of R_i'. If $r \in W$, an open set of R_i', the pre-image of W under Π_i' is open in P_i' and any p in the pre-image of r is in some U, a member of a basis for the topology of P_i', contained in the pre-image of W. Then r is in $\Pi_i'(U)$ which is open and contained in W.

It follows that any open set in $R_1' \times R_2'$ containing the pair of equivalence classes of ψ_1 and ψ_2 contains a product $\Pi_1'(U_1) \times \Pi_2'(U_2)$, U_i being subsets of P_i' of the form $U_i = \{q_i: |q_i(a_{ij}) - p_i(a_{ij})| < \epsilon_i, j = 1, 2, \ldots, n_i\}$, where $p_i(a_i) = \langle \psi_i(a_i)\xi_i, \xi_i\rangle$, $i = 1, 2$. Define a neighbourhood of $p_1 \otimes p_2$ in P_3' by

$$V = \{s: |s(a_{1j} \otimes e_2) - p_1 \otimes p_2(a_{1j} \otimes e_2)| < \epsilon_1, \quad j = 1, 2, \ldots, n_1\}$$

$$\cap \{s: |s(e_1 \otimes a_{2j}) - p_1 \otimes p_2(e_1 \otimes a_{2j}) < \epsilon_2, \quad j = 1, 2, \ldots, n_2\}.$$

For s in P_3' define $s_1(a_1) = s(a_1 \otimes e_2)$, $s_2(a_2) = s(e_1 \otimes a_2)$. Assume for the moment that the s_i are indecomposable. For s in V, each s_i is in U_i, thus $s_1 \otimes s_2 \in U_1 \otimes U_2$.

Let θ be a member of the equivalence class $\Pi_3'(s)$. By the fact that $\otimes^{\#'}$ is onto R_3', θ is equivalent to $\theta_1 \otimes \theta_2$ with θ_i an irreducible representation of A_i on a finite-dimensional inner-product space H_i, $i = 1, 2$. Therefore

$$s(x) = \langle \theta_1 \otimes \theta_2(x)\eta, \eta\rangle,$$

and $s_1(a_1) = \langle \theta_1(a_1) \otimes I_2\eta, \eta\rangle$, $s_2(a_2) = \langle I_1 \otimes \theta_2(a_2)\eta, \eta\rangle$, where $I_i = \theta_i(e_i)$ is the identity operator on H_i and η is a vector of $H_1 \otimes H_2$ (which has the usual inner-product structure). The orbits of η under the action of the images of $\theta_1 \otimes I_2$ and $I_1 \otimes \theta_2$ are subspaces, H_3 and H_4, respectively, of $H_1 \otimes H_2$. The representations $\theta_1 \otimes I_2$ on H_3 and $I_1 \otimes \theta_2$ on H_4 are equivalent to θ_1 on H_1 and θ_2 on H_2, respectively; thus $(\theta_1 \otimes I_2) \otimes (I_1 \otimes \theta_2)$, $\theta_1 \otimes \theta_2$, and θ all belong to the same class in R_3'. The positive functionals s_i are indecomposable, as assumed, and

$$s_1 \otimes s_2(a_1 \otimes a_2) = s_1(a_1)s_2(a_2) = \langle \theta_1(a_1) \otimes I_2\eta, \eta\rangle\langle I_1 \otimes \theta_2(a_2)\eta, \eta\rangle =$$
$$\langle (\theta_1 \otimes I_2) \otimes (I_1 \otimes \theta_2)(a_1 \otimes a_2)\eta \otimes \eta, \eta \otimes \eta\rangle$$

thus θ is an element of $\Pi_3'(s_1 \otimes s_2)$.

We have shown that if an equivalence class of representations is a member of $\Pi_3'(V)$, then it is also a member of $\Pi_3'(U_1 \otimes U_2)$. But

$$\Pi_3'(U_1 \otimes U_2) = \Pi_1'(U_1) \otimes \Pi_2'(U_2),$$

thus there is a neighbourhood, $\Pi_3'(V)$, of $\Pi_3'(s_1 \otimes s_2)$, contained in

$$\Pi_1'(U_1) \otimes \Pi_2'(U_2)$$

and thus also contained in the image of the original open set containing $(\Pi_1'(s_1), \Pi_2'(s_2))$, the pair of equivalence classes of ψ_1 and ψ_2. Thus $\otimes^{\#'}$ is open.

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It is of interest to note the existence of important examples in which all irreducible representations are finite-dimensional, such as the commutative symmetric Banach algebras and the group algebras of compact groups. In such cases, of course, the theorem characterizes the space of equivalence classes of all irreducible representations. One might study, for instance, the algebra of integrable functions on compact groups having values in a symmetric commutative Banach algebra much as such algebras were studied for locally compact Abelian groups in (**1**).

There is another natural topology for R_i'. Kernels of finite-dimensional irreducible representations are the primitive symmetric ideals of finite codimension. These representations are equivalent if and only if they have the same kernel, therefore we can identify points of R_i' with ideals and introduce the hull-kernel topology. Our topology need not agree with this hull-kernel topology.

Remark 1. The given topology for R_i' is not weaker than the hull-kernel topology.

Proof. We show that any point in the closure of a set is in the hull of its kernel. Let S be a set in R_i', F its pre-image under Π_i. We must show that if p is in the closure of F in P_i', then the ideal $\Pi_i(p)$ contains the intersection of the ideals S. The ideal $\Pi_i(f)$ for any positive functional f is the set of x such that $f(uxv) = 0$ for all u and v in A_i. The ideal $\Pi_i(p)$ contains the intersection of all the ideals of S if and only if $f(uxv) = 0$ for all u, v, and all f in F implies $p(uxv) = 0$ for all u, v. Let $\{U_k\}$ be a basis for a (finite-dimensional) complement of the ideal $\Pi_i(p)$ in A_i. Since p is in the closure of F, for any x and any $\epsilon > 0$ there is f in F such that $|p(u_j x u_k) - f(u_j x u_k)| < \epsilon$ for all j and k. But if x is in the intersection of the ideals of S, $f(u_j x u_k) = 0$ for all f in F; thus $p(u_j x u_k) = 0$. Since p vanishes on the ideal $\Pi_i(p)$, we have $p(uxv) = 0$ for all u, v.

Remark 2. The spaces R_i' are T_1-spaces.

Proof. Showing a point is closed is equivalent to showing that the inverse image under Π_i' is closed. Let $\phi \in r$, the point being considered. Every element of the pre-image of the point is of the form $p(a) = \langle \phi(a)\xi, \xi\rangle$ for some ξ, where $\langle \xi, \xi \rangle = p(e_i) = 1$. Suppose that p_0 is a limit point of the pre-image in P_i'. Then each neighbourhood $U = \{q\colon |q(a_j) - p_0(a_j)| < \epsilon,\ j = 1, 2, \ldots, n\}$ contains a point of the pre-image, thus for each $\epsilon > 0$ and every finite set $\{a_j\}$, there is a vector ξ of unit norm such that $|\langle \phi(a_j)\xi, \xi\rangle - p_0(a_j)| < \epsilon$.

Since ϕ is in a member of R_i' and p_0 is in P_i', each is determined by its value on some finite set of points of A_i. Choose the set $\{a_j\}$ to be the union of such a set for ϕ with such a set for p_0, then for any positive integer n, if we set $\epsilon = 1/n$, there is a vector ξ_n of unit norm such that

$$|\langle \phi(a_j)\xi_n, \xi_n\rangle - p_0(a_j)| < 1/n.$$

The vectors of unit norm in a finite-dimensional space form a compact set, thus there is a subsequence of ξ_n converging to a vector ξ_0 of unit norm. But

$$|\langle\phi(a_j)\xi_0, \xi_0\rangle - p_0(a_j)| \leqq |\langle\phi(a_j)\xi_0, \xi_0\rangle - \langle\phi(a_j)\xi_n, \xi_n\rangle| + |\langle\phi(a_j)\xi_n, \xi_n\rangle - p_0(a_j)|$$

$$\leqq 2\|\phi(a_j)\| \|\xi_n - \xi_0\| + 1/n.$$

Since a subsequence of ξ_n converges to ξ_0, the bound becomes arbitrarily small. Since ϕ and p_0 are both determined by their values on the finite set $\{a_j\}$, we have $p_0(a) = \langle\phi(a)\xi_0, \xi_0\rangle$, therefore p_0 is in the inverse image of r.

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