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CYLINDRICAL REPRESENTATIONS OF SOME INFINITE DIMENSIONAL NUCLEAR LIE GROUPS

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Let $\Gamma.\mathcal{A}$ be the semi-direct product group of a nuclear Lie group Γ with the additive group $\mathcal{A}$ of a real nuclear vector space. We give an explicit description of all the continuous representations of $\Gamma.\mathcal{A}$ the restriction of which to $\mathcal{A}$ is a cyclic unitary representation, and a necessary and sufficient condition for the unitarity of such cylindrical representations is stated. This general result is successfully used to obtain irreducible unitary representations of the nuclear Lie groups of Riemannian motions, and, in the setting of the theory of multiplicative distributions initiated by I.M. Gelfand, it is proved that for any connected real finite dimensional Lie group G and for any strictly positive integer k there exist non located and non trivially decomposable representations of order k of the nuclear Lie group $C_0^\infty(M;G)$ of all the G-valued test-functions on a given paracompact manifold M.

I. INTRODUCTION

The concept of nuclear Lie group initiated by Gelfand and Vilenkin [**8**, Chapter 4, Section 5] is the group-theoretical version of the concept of nuclear vector space. A large class of nuclear Lie groups is constituted by the groups of compactly supported smooth sections of smooth bundles of finite dimensional connected real Lie groups arising mainly from mathematical physics ([**4**, **5**, **17**], and references therein).

A basic example is the group $C_0^\infty(M,G)$ of all the compactly supported smooth mappings from a smooth Riemannian manifold M into a finite dimensional connected real Lie group G, often called a current group or a gauge group. From the point of view of the functional analysis, this group is the space of all the G-valued test-functions on M. The main motivation which, initially, led Gelfand, Graev and Vershik to be interested in some continuous representations of $C_0^\infty(M,G)$ is the following (for example, [**7**, **13**, **21**, **23**]): in the case $G = \mathbb{R}$ there is a one-to-one assignment $T \mapsto \chi_T = e^{i\langle T,.\rangle}$ from the space of real distributions on M onto the set of continuous unitary

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characters of the additive group of the nuclear vector space $C_0^\infty(M,\mathbb{R})$; consequently, replacing $\mathbb{R}$ by any Lie group G, the continuous irreducible unitary representations of $C_0^\infty(M,G)$ must be taken as G-distributions on M.

As early as the seventies it appeared that it was very difficult, at least in the case G semi-simple, to find non located (that is, with continuous support), unitary representations of $C_0^\infty(M,G)$; for the definitions of support and order of a G-distribution, basic properties and main results on such objects we refer to [**13**] and [**15**]. For instance, in the case G compact semi-simple, it can be proved that the only non located unitary representations of $C_0^\infty(M,G)$ non trivially decomposable into sums of representations of finite dimensional Lie groups are of order 1 and are mainly constituted by the so-called energy representations [**1, 24, 17**].

We have to notice that, except in the case where T is a real distribution on the manifold M, generally $\chi_T = e^{i\langle T,\cdot\rangle}$ is not a unitary character, but remains a continuous character of $C_0^\infty(M,\mathbb{C})$; my thesis is that a relevant generalisation, in the G-multiplicative sense, of the concept of distribution on M, leads to considering all the (non equivalent) continuous and irreducible representations of $C_0^\infty(M,G)$, without prescribing a unitary condition. An important remark is the following: when we look at the list of non trivially decomposable non located continuous representations of $C_0^\infty(M,G)$ known at the present time for various non abelian Lie groups G, it appears that these representations are restrictions of representations of semi-direct products of $C_0^\infty(M,G)$ with some abelian nuclear Lie group to the image of $C_0^\infty(M,G)$, in this semi-direct product, by a twisted injection coming from a 1-cocycle (see for example, [**2, 19, 20, 22**]). This remark has been the starting point of the present work.

In Section II we study the so-called nuclear Lie groups of class (S) which is composed of the semi-direct products $\Gamma.\mathcal{A}$, where Γ is a nuclear Lie group and $\mathcal{A}$ the additive group of a real countably hilbertian nuclear vector space; in Section III the most important examples of such groups are given: the semi-direct products of nuclear Lie groups of compactly supported smooth sections of bundles of Lie groups by nuclear vector spaces of compactly supported smooth sections of vector bundles.

In Section IV we give an explicit description of all the so-called cylindrical representations of the semi-direct products $\Gamma.\mathcal{A}$, that is, of the continuous representations of $\Gamma.\mathcal{A}$ the restriction of which to $\mathcal{A}$ is a unitary representation that, for sake of simplicity, we assume to be cyclic; a necessary and sufficient condition in order that a cylindrical representation be unitary is also given.

In Section V, following up this general result, it is proved that, a manifold M with a smooth Riemannian structure R being given, each smooth strictly positive function ρ on M gives rise to an irreducible unitary cylindrical representation of the nuclear Lie group of Riemannian motions of the space of all the compactly supported smooth vector fields of M.

In Section VI it is proved that the nuclear Lie group $\mathcal{D}_0(F)$ of all the compactly supported smooth sections of a bundle F over a given manifold M, of finite dimensional connected real Lie groups, has many non located and non trivially decomposable continuous representations of order 1, which are the restrictions of cylindrical representations of the semi-direct product $\mathcal{D}_0(F).\mathcal{D}_0(Hom(TM,\mathcal{F}))$, where $\mathcal{F}$ is the bundle over M of Lie algebras associated with F, to a twisted embedding of $\mathcal{D}_0(F)$ via its generalised Maurer-Cartan cocycle. When one takes for F the trivial bundle $M \times G$, G being a compact semi-simple Lie group, some of these representations are the so-called energy representations (for example, [**23, 1, 2,14**]); this is the reason for which we have called such representations the generalised energy representations of $\mathcal{D}_0(F)$.

Let M be a connected smooth manifold, and let G be a connected real Lie group; coming back to the initial motivation of the present work, in Section VII we apply the results of Section VI to the case where F is the jet bundle of k-jets of the elements of $\mathcal{D}_0(M \times G) = C_0^\infty(M,G)$, $k \geqslant 0$. One proves then, for the first time to our knowledge when G is a semi-simple Lie group, that for any integer $k \geqslant 1$ there exist non located and non trivially decomposable continuous representations of order k of $C_0^\infty(M,G)$. The problem of the irreducibility (or of the decomposition into irreducible components) of these representations shall be taken up in a subsequent paper; however one can already hope that this result will provide a new contribution to the theory of the so-called multiplicative distributions.

II. Nuclear Lie groups of class (S).

Definition 1: A nuclear Lie group is of class (S) if it is the semi-direct product of a nuclear Lie group Γ by the additive group $\mathcal{A}$ of a real countably hilbertian nuclear vector space.

The above definition implies the existence of a homomorphism τ of nuclear Lie groups from Γ into the group Aut $(\mathcal{A})$ of automorphisms of $\mathcal{A}$ such that the mapping $(\gamma,\omega) \mapsto \tau(\gamma)\omega$ is a continuous mapping from $\Gamma \times \mathcal{A}$ onto $\mathcal{A}$. We shall denote by $(\Gamma.\mathcal{A},\tau)$ such a nuclear Lie group of class (S), the product of which being given, for any pair $((\gamma,\omega),(\gamma',\omega'))$ of elements in $\Gamma \times \mathcal{A}$, by:

$$(\gamma,\omega)\cdot(\gamma',\omega') = (\gamma.\gamma',\omega + \tau(\gamma)\omega') \ . \tag{1}$$

Let ε be the unit element in Γ, and let $\underline{0}$ be the null vector in $\mathcal{A}$; we shall identify the subgroup $\Gamma \times \{\underline{0}\}$ of $\Gamma.\mathcal{A}$ with Γ, and the abelian normal subgroup $\{\varepsilon\} \times \mathcal{A}$ with $\mathcal{A}$, so that for any (γ,ω) in $\Gamma \times \mathcal{A}$:

$$(\gamma,\omega) = (\varepsilon,\omega)\cdot(\gamma,\underline{0}) = (\gamma,\underline{0})\cdot\big(\varepsilon,\tau(\gamma^{-1})\omega\big) \ , \tag{2}$$

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that we can write as the commutation relation:

$$(\gamma,\omega)=\omega\cdot\gamma=\gamma\cdot\tau(\gamma^{-1})\omega\ . \tag{2'}$$

A nuclear Lie group $(\Gamma.\mathcal{A},\tau)$ of class (S) being given, we equip the dual space $\mathcal{A}'$ of $\mathcal{A}$ with the weak topology, and with the structure of measurable space given by the σ-field generated by the cylinders with Borel base, that is to say by the subsets of $\mathcal{A}'$ of the form:

$$\mathcal{C}^B_{\omega_1,\dots,\omega_q}=\{\chi\in\mathcal{A}'/(\langle\chi,\omega_1\rangle,\dots,\langle\chi,\omega_q\rangle)\in B\}\ , \tag{3}$$

where q is some positive integer, B a Borel subset of $\mathbb{R}^q$, and $(\omega_1,\dots,\omega_q)$ a finite sequence of q elements in $\mathcal{A}$ [**8**].

We associate with τ the representation τ^* of Γ into $\mathcal{A}'$, defined for any γ in Γ and any χ in $\mathcal{A}'$ by:

$$\langle\tau^*(\gamma)\chi,\omega\rangle=\langle\chi,\tau(\gamma^{-1})\omega\rangle\ ,\quad \omega\in\mathcal{A}\ . \tag{4}$$

The continuity of the mapping $(\gamma,\omega)\mapsto\tau(\gamma)\omega$ from $\Gamma\times\mathcal{A}$ onto $\mathcal{A}$ implies:

- for any γ in Γ, the continuity of the mapping $\chi\mapsto\tau^*(\gamma)\chi$ from $\mathcal{A}'$ into $\mathcal{A}'$;
- for any χ in $\mathcal{A}'$, the continuity of the mapping $\gamma\mapsto\tau^*(\gamma)\chi$ from Γ into $\mathcal{A}'$.

Moreover:

LEMMA 1. *For any γ, in Γ, the mapping $\chi\mapsto\tau^*(\gamma)\chi$ is measurable with respect to any cylindrical measure on $\mathcal{A}'$.*

PROOF: Let $\overline{\gamma}$ be the mapping from $\mathcal{A}'$ into $\mathcal{A}'$ defined by $\overline{\gamma}(\chi)=\tau^*(\gamma)\chi$, and let $\mathcal{C}^B_{\omega_1,\dots,\omega_q}$ be a cylinder with Borel base (for example, (3)). The assertion follows from the fact that $\overline{\gamma}\left(\mathcal{C}^B_{\omega_1,\dots,\omega_q}\right)=\mathcal{C}^B_{\tau(\gamma)\omega_1,\dots,\tau(\gamma)\omega_q}$. ∎

III. THE NUCLEAR LIE GROUPS $\mathcal{D}_0(F).\mathcal{D}_0(\mathcal{E})$.

In this section we want to provide the most important — and interesting — class of nuclear Lie groups of class (S); we select once and for all a smooth and connected paracompact manifold M of finite dimension.

(a) Let $F=\bigcup\limits_{x\in M}F^x$ be a smooth bundle over M of connected and finite dimensional real Lie groups; for each x in M, ε_x denotes the unit element and $\mathcal{F}^x$ the Lie algebra of the Lie group F^x so that $\mathcal{F}=\bigcup\limits_{x\in M}\mathcal{F}^x$ is the corresponding smooth bundle of Lie algebras.

We consider the set $\mathcal{D}_0(F)$ of all the compactly supported smooth sections of F, that is, the set of all the smooth mappings σ from M into F such that:

(1) for any x in M, $\sigma(x)$ belongs to F^x;

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(2) there exists a compact subset K of M such that if x is any element in $M-K$, then $\sigma(x)=\varepsilon_x$.

Equipped with the pointwise defined product of sections and with the so-called Schwartz topology, that is, the topology induced from the C^∞-topology of Whitney of $C^\infty(F)$ (for such a topology we refer to [**10**]), it is well-known that $\mathcal{D}_0(F)$ is a nuclear Lie group (for example, [**4, 16, 3**]). Let us recall that a sequence $(\sigma_n)_n$ converges to σ in $\mathcal{D}_0(F)$ with respect to the Schwartz topology if there exists a compact subset K of M such that for any index n, $\sigma_n(x)=\sigma(x)=\varepsilon_x$, $\forall x\in M-K$, and $(\sigma_n)_n$ converges to σ on K with respect to the C^∞-uniform convergence.

In mathematical physics $\mathcal{D}_0(F)$ is often called a nuclear current group, and in the special case where F is the associated bundle of some principal bundle, a nuclear gauge group. Let us consider now a smooth real vector bundle $\mathcal{E}=\bigcup_{x\in M}\mathcal{E}^x$ over M such that for each x in M there exists a homomorphism τ_x from F^x into the group $\mathrm{Aut}(\mathcal{E}^x)$ of automorphisms of $\mathcal{E}^x$ so that the mapping $(\gamma_x,\omega_x)\mapsto\tau_x(\gamma_x)\omega_x$ is a continuous mapping from $F^x\times\mathcal{E}^x$ onto $\mathcal{E}^x$, and such that the assignment $x\mapsto\tau_x$ is a smooth section of the bundle:

$$\mathrm{Hom}\,(F,\,\mathrm{Aut}\,(\mathcal{E}))=\bigcup_{x\in M}\mathrm{Hom}\,(F^x,\;\mathrm{Aut}\,(\mathcal{E}^x)). \tag{5}$$

One gets a homomorphism τ from $\mathcal{D}_0(F)$ into $\mathrm{Aut}\,(\mathcal{D}_0(\mathcal{E}))$ such that for any σ in $\mathcal{D}_0(F)$, for any ω in $\mathcal{D}_0(\mathcal{E})$, $\tau(\sigma)\omega$ is the section $x\mapsto\tau_x(\sigma(x))(\omega(x))$, $x\in M$.

It follows then from Definition 1 that:

Lemma 2. *If the mapping $(\sigma,\omega)\mapsto\tau(\sigma)\omega$ is a continuous mapping from $\mathcal{D}_0(F)\times\mathcal{D}_0(\mathcal{E})$ onto $\mathcal{D}_0(\mathcal{E})$, then $(\mathcal{D}_0(F).\mathcal{D}_0(\mathcal{E}),\tau)$ is a nuclear Lie group of class (S).*

(b) An important example of such a nuclear Lie group of class (S) constructed with the factor $\mathcal{D}_0(F)$ is the following. Let us consider the smooth real vector bundle over M:

$$\mathrm{Hom}\,(TM,\mathcal{F})=\bigcup_{x\in M}\mathrm{Hom}\,(T_xM,\mathcal{F}^x)\;, \tag{6}$$

where for each x in M, T_xM is the tangent space of M at x, $\mathcal{F}^x$ is the Lie algebra of F^x, and where $TM=\bigcup_{x\in M}T_xM$ is the tangent bundle of M, and $\mathcal{F}=\bigcup_{x\in M}\mathcal{F}^x$ is the Lie algebra-bundle associated with F. $\mathcal{D}_1(\mathcal{F})=\mathcal{D}_0(\,\mathrm{Hom}\,(TM,\mathcal{F}))$ is the countably hilbertian real nuclear vector space of all the compactly supported $\mathcal{F}$-valued smooth 1-forms on M.

Let Ad_x be the adjoint representation of F^x into its Lie algebra $\mathcal{F}^x$, $x\in M$; one gets a continuous representation V of the nuclear Lie group $\mathcal{D}_0(F)$ into $\mathcal{D}_1(\mathcal{F})$ such that for any σ in $\mathcal{D}_0(F)$ and any ω in $\mathcal{D}_1(\mathcal{F})$:

$$(V(\sigma)\omega)(x) = Ad_x(\sigma(x)) \circ \omega(x) \ , \ x \in M \ . \tag{7}$$

Lemma 3. *$(\mathcal{D}_0(F).\mathcal{D}_1(\mathcal{F}), V)$ is a nuclear Lie group of class (S).*

Proof: We have to prove that the assignment $(\sigma,\omega) \mapsto V(\sigma)\omega$ is a continuous mapping from $\mathcal{D}_0(F) \times \mathcal{D}_1(\mathcal{F})$ onto $\mathcal{D}_1(\mathcal{F})$. For any integer $k \geqslant 0$, and any smooth section f of a smooth bundle over M, let us denote by $j_x^k(f)$ the jet of order k of f at the source x, $x \in M$, and by $j^k(f)$ the assignment $x \mapsto j_x^k(f)$.

For each x in M let $J_x^k(F)$ be the set $\{j_x^k(\sigma) \mid \sigma \in \mathcal{D}_0(F)\}$, and let $J_x^k(\mathcal{F})^1$ be the set $\{j_x^k(\omega) \mid \omega \in \mathcal{D}_1(\mathcal{F})\}$. It is well-known that the jet bundle $J^k(F) = \bigcup_{x\in M} J_x^k(F)$ is a smooth bundle of finite dimensional real connected Lie groups (for example, [**19, 12**]) and that $J^k(\mathcal{F})^1 = \bigcup_{x\in M} J_x^k(\mathcal{F})^1$ is a smooth vector bundle over M [**18**].

Let Ad_x^k be the adjoint representation of the finite dimensional Lie group $J_x^k(F)$, $x \in M$; one easily sees that for any σ in $\mathcal{D}_0(F)$ and any ω in $\mathcal{D}_1(\mathcal{F})$:

$$j_x^k(V(\sigma)\omega) = Ad_x^k\big(j_x^k(\sigma)\big) \circ j_x^k(\omega), \ x \in M \ . \tag{8}$$

Now let $(\sigma_n, \omega_n)_n$ be a sequence converging to some element (σ,ω) in $\mathcal{D}_0(F) \times \mathcal{D}_1(\mathcal{F})$ with respect to the Schwartz topology; there exists a compact subset K of M such that:

- $(*)$ for any index n and for all x in $M - K$: $\sigma_n(x) = \sigma(x) = \varepsilon_x$ and $\omega_n(x) = \omega(x) = \underline{0}$;
- $(**)$ on K, for any integer $k \geqslant 0$ the sequences $\big(j^k(\sigma_n)\big)_n$ and $\big(j^k(\omega_n)\big)_n$ converge uniformly to $j^k(\sigma)$ and $j^k(\omega)$ respectively.

From (8) it follows that $(V(\sigma_n)\omega_n)_n$ converges to $V(\sigma)\omega$ in $\mathcal{D}_1(\mathcal{F})$ with respect to the Schwartz topology, and therefore the mapping $(\sigma,\omega) \mapsto V(\sigma)\omega$ is a continuous mapping from $\mathcal{D}_0(F) \times \mathcal{D}_1(\mathcal{F})$ onto $\mathcal{D}_1(\mathcal{F})$. ∎

IV. Cylindrical Representations of $\Gamma.\mathcal{A}$.

(a) Let $(\Gamma.\mathcal{A}, \tau)$ be a nuclear Lie group of class (S), and let $M(\mathcal{A}')$ be the set of all the normed cylindrical positive measures on $\mathcal{A}'$; for the construction and basic properties of such measures of the dual space of a countably hilbertian nuclear vector space we refer to [**8**, Chapter 4]. In particular it is known (see also [**11**, Chapters 7–8]) that any element μ in $M(\mathcal{A}')$ is the spectral measure of a continuous unitary representation θ_μ of the abelian nuclear Lie group $\mathcal{A}$ which can be realised in $L^2(\mathcal{A}';\mu)$ so that for any ω in $\mathcal{A}$ and any Φ in $L^2(\mathcal{A}';\mu)$:

$$(\theta_\mu(\omega)\Phi)(\chi) = e^{i\langle\chi,\omega\rangle} \cdot \Phi(\chi) \ , \ \chi \in \mathcal{A}' \ . \tag{9}$$

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Moreover it is known that the constant function $\underline{1} : \mathcal{A}' \to \mathbb{C}$ defined for any χ in $\mathcal{A}'$ by $\underline{1}(\chi) = 1$ is a cyclic vector for θ_μ.

DEFINITION 2: Let μ be in $M(\mathcal{A}')$; a continuous representation T of the nuclear Lie group $(\Gamma.\mathcal{A}, \tau)$ is called a μ-cylindrical representation if T may be realised in $L^2(\mathcal{A}'; \mu)$ and if, in this realisation, the restriction of T to $\mathcal{A}$ is exactly θ_μ.

We notice that any continuous unitary representation of $\Gamma.\mathcal{A}$, the restriction to $\mathcal{A}$ of which is cyclic, is a μ-cylindrical representation where the normed cylindrical measure μ is the spectral measure of this restriction.

In order to get an explicit description of the μ-cylindrical representations of $\Gamma.\mathcal{A}$, $\mu \in M(\mathcal{A}')$ we have to consider the set $\mathcal{C}(\Gamma, \mathcal{A}, \tau, \mu)$ consisting of all the continuous mappings c from Γ into $L^2(\mathcal{A}'; \mu)$ fulfilling the following cocycle condition for any pair (γ, γ') of elements in Γ:

$$\text{(10)} \qquad \begin{cases} c(\varepsilon)(\chi) = \underline{1}, & \forall \chi \in \mathcal{A}' \\ c(\gamma.\gamma')(\chi) = c(\gamma)(\chi).c(\gamma')\big(\tau^*(\gamma^{-1})\chi\big), & \forall \chi \in \mathcal{A}' . \end{cases}$$

Let $Z^1(\Gamma, \tau)$ be the vector space of all the continuous 1-cocycles of Γ with respect to τ, that is, the space of all the continuous mappings β from Γ into $\mathcal{A}$ such that for any pair (γ, γ') of elements in Γ:

$$\text{(11)} \qquad \beta(\gamma.\gamma') = \beta(\gamma) + \tau(\gamma)\beta(\gamma') .$$

LEMMA 4. *Let μ be in $M(\mathcal{A}')$.*

(1) *If μ is quasi-invariant by the operators $\tau^*(\gamma)$, $\gamma \in \Gamma$, the corresponding square root of the Radon-Nikodym derivative function, that is, the mapping $\gamma \mapsto \big[\big(d\mu\big(\tau^*(\gamma^{-1})(\cdot)\big)\big)/(d\mu(\cdot))\big]^{1/2}$, belongs to $\mathcal{C}(\Gamma, \mathcal{A}, \tau, \mu)$.*

(2) *For any real number r and any 1-cocycle β in $Z^1(\Gamma, \tau)$ the mapping $c_{r,\beta} : \gamma \mapsto \exp\{ir\langle \cdot, \beta(\gamma)\rangle\}$ belongs to $\mathcal{C}(\Gamma, \mathcal{A}, \tau, \mu)$.*

PROOF: (1) is trivial; in order to prove (2) let us notice that $c_{r,\beta}(\gamma) = \theta_\mu(r\beta(\gamma))\underline{1}$, where θ_μ is the unitary cyclic representation of $\mathcal{A}$ the spectral measure of which is μ, which proves that $c_{r,\beta}$ takes its values in $L^2(\mathcal{A}'; \mu)$, moreover the continuity of $c_{r,\beta}$ follows from the continuity of θ_μ and of β; lastly; an easy computation proves that $c_{r,\beta}$ satisfies the condition (10). $\square$

We have the following general result:

THEOREM 1. *Let $(\Gamma.\mathcal{A}, \tau)$ be a nuclear Lie group of class (S) and let μ be an element in $M(\mathcal{A}')$. Any element c in $\mathcal{C}(\Gamma, \mathcal{A}, \tau, \mu)$ provides a μ-cyclindrical representation T_c of $\Gamma.\mathcal{A}$ into $L^2(\mathcal{A}'; \mu)$ such that for any element γ in Γ, any element ω in $\mathcal{A}$, and for any element Φ in $L^2(\mathcal{A}'; \mu)$:*

$$\text{(12)} \qquad (T_c(\gamma, \omega)\Phi)(\chi) = e^{i\langle \chi, \omega\rangle}.c(\gamma)(\chi).\Phi\big(\tau^*(\gamma^{-1})\chi\big) , \quad \chi \in \mathcal{A}' ,$$

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and conversely any μ-cylindrical representation of $\Gamma.\mathcal{A}$ is, up to a unitary equivalence, of the form T_c for some c in $C(\Gamma, \mathcal{A}, \tau, \mu)$. Moreover, T_c is unitary if and only if μ is quasi-invariant by the operators $\tau^*(\gamma)$, $\gamma \in \Gamma$, c being the positive square root of the Radon-Nikodym derivative function of μ with respect to the operators $\tau^*(\gamma^{-1})$, $\gamma \in \Gamma$, that is, such that $d\mu(\tau^*(\gamma^{-1})\chi) = |c(\gamma)(\chi)|^2 d\mu(\chi)$.

PROOF: Let T be a continuous representation of $\Gamma.\mathcal{A}$; by restriction, T provides a continuous representation Π of Γ by taking $\Pi(\gamma) = T(\gamma, \underline{0})$, $\gamma \in \Gamma$, and a continuous representation θ of $\mathcal{A}$ by taking $\theta(\omega) = T(\varepsilon, \omega)$, $\omega \in \mathcal{A}$; moreover for a given μ in $M(\mathcal{A}')$, T is μ-cylindrical if and only if we can realise T in $L^2(\mathcal{A}'; \mu)$ so that $\theta = \theta_\mu$ (see (9)). Let us notice that the commutation relation (2') implies that for any (γ, ω) in $\Gamma \times \mathcal{A}$:

$$T(\gamma, \omega) = \theta_\mu(\omega).\Pi(\gamma) = \Pi(\gamma).\theta_\mu(\tau(\gamma^{-1})\omega). \tag{13}$$

For each element ω in $\mathcal{A}$ let us consider the element $\overline{\omega}$ in $L^2(\mathcal{A}'; \mu)$ defined by $\overline{\omega}(\chi) = e^{i\langle \chi, \omega \rangle} = (\theta_\mu(\omega)\underline{1})(\chi)$; for any element γ in Γ one gets:

$$\Pi(\gamma)\overline{\omega} = \Pi(\gamma).\theta_\mu(\omega)\underline{1} = \Pi(\gamma).\theta_\mu(\tau(\gamma^{-1}).\tau(\gamma).\omega)\underline{1},$$

and then, from (13), for any χ in $\mathcal{A}'$:

$$\begin{aligned}(\Pi(\gamma)\overline{\omega})(\chi) &= \theta_\mu(\tau(\gamma)\omega).\Pi(\gamma)\underline{1}(\chi) = e^{i\langle \chi, \tau(\gamma)\omega \rangle}.\Pi(\gamma)\underline{1}(\chi) \\ &= e^{i\langle \tau^*(\gamma^{-1})\chi, \omega \rangle}.\Pi(\gamma)\underline{1}(\chi),\end{aligned}$$

so that:

$$(\Pi(\gamma)\overline{\omega})(\chi) = \overline{\omega}(\tau^*(\gamma^{-1})\chi).\Pi(\gamma)\underline{1}(\chi). \tag{14}$$

The space of all the finite linear combinations of elements of the type $\overline{\omega}$, $\omega \in \mathcal{A}$ is densely contained in $L^2(\mathcal{A}'; \mu)$ (for example, [**8**, Chapter 4]), it follows from (14) and from the continuity of Π that for any γ in Γ and any Φ in $L^2(\mathcal{A}'; \mu)$:

$$(\Pi(\gamma)\Phi)(\chi) = (\Pi(\gamma)\underline{1})(\chi).\Phi(\tau^*(\gamma^{-1})\chi), \quad \chi \in \mathcal{A}'. \tag{15}$$

We note that $\Pi(\gamma)\underline{1}$ belongs to $L^2(\mathcal{A}'; \mu)$, and that the mapping $c : \gamma \mapsto c(\gamma) = \Pi(\gamma)\underline{1}$ is a continuous mapping from Γ into $L^2(\mathcal{A}'; \mu)$; moreover let γ, γ' be in Γ; from (15) one gets $(\Pi(\gamma.\gamma')\Phi)(\chi) = c(\gamma.\gamma')(\chi).\Phi(\tau^*(\gamma'^{-1}.\gamma^{-1})\chi)$; but taking into account that Π is a representation one gets:

$$\begin{aligned}(\Pi(\gamma).\Pi(\gamma')\Phi)(\chi) &= c(\gamma)(\chi).\Pi(\gamma')\Phi(\tau^*(\gamma^{-1})\chi) \\ &= c(\gamma)(\chi).c(\gamma')(\tau^*(\gamma^{-1})\chi).\Phi(\tau^*(\gamma'^{-1}.\gamma^{-1})\chi),\end{aligned}$$

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so that, from (10), it follows that c belongs to $\mathcal{C}(\Gamma,\mathcal{A},\tau,\mu)$.

Conversely, selecting any element c in $\mathcal{C}(\Gamma,\mathcal{A},\tau,\mu)$, an easy computation proves that Π_c, defined for any element γ in Γ so that for any element Φ in $L^2(\mathcal{A}';\mu)$:

$$(\Pi_c(\gamma)\Phi)(\chi) = c(\gamma)(\chi).\Phi(\tau^*(\gamma^{-1})\chi), \quad \chi \in \mathcal{A}', \tag{16}$$

is a continuous representation of Γ into $L^2(\mathcal{A}';\mu)$ satisfying the commutation relation (13) with θ_μ. The first part of the theorem is then proved by taking for any (γ,ω) in $\Gamma\times\mathcal{A}$:

$$T_c(\gamma,\omega) = \theta_\mu(\omega).\Pi_c(\gamma)\ . \tag{17}$$

We note that the unitarity of θ_μ implies that T_c is unitary if and only if Π_c is unitary too.

Let us assume that Π_c is unitary; from (16) one gets, for any γ in Γ and any Φ in $L^2(\mathcal{A}';\mu)$,

$$\int_{\mathcal{A}'} |c(\gamma)(\chi)|^2\cdot|\Phi(\tau^*(\gamma^{-1})\chi)|^2 d\mu(\chi) = \int_{\mathcal{A}'} |\Phi(\chi)|^2.d\mu(\chi);$$

noticing that $\int_{\mathcal{A}'} |\Phi(\chi)|^2 d\mu(\chi) = \int_{\mathcal{A}'} |\Phi(\tau^*(\gamma^{-1})\chi)|^2.d\mu(\tau^*(\gamma^{-1})\chi)$, one proves then that μ is quasi-invariant by the operators $\tau^*(\gamma)$, $\gamma\in\Gamma$, and that $d\mu(\tau^*(\gamma^{-1})\chi) = |c(\gamma)(\chi)|^2 d\mu(\chi)$.

Conversely, if μ is quasi-invariant by the operators $\tau^*(\gamma)$, $\gamma\in\Gamma$, one gets a continuous unitary representation Π of Γ into $L^2(\mathcal{A}';\mu)$ by taking for any γ in Γ, and any Φ in $L^2(\mathcal{A}';\mu)$:

$$(\Pi(\gamma)\Phi)(\chi) = \left[\frac{d\mu(\tau^*(\gamma^{-1})\chi)}{d\mu(\chi)}\right]^{1/2}\cdot\Phi(\tau^*(\gamma^{-1})\chi), \quad \chi\in\mathcal{A}',$$

and therefore, by Lemma 4, Π is of the form Π_c by taking for relevant element c in $\mathcal{C}(\Gamma,\mathcal{A},\tau,\mu)$ the square root of the Radon-Nikodym of the image of μ by the operators $\tau^*(\gamma^{-1})$, $\gamma\in\Gamma$; this achieves the proof of the theorem. ∎

V. Cylindrical irreducible unitary representations of nuclear Lie groups of Riemannian motions

(a) Let M be a finite dimensional connected and paracompact smooth manifold and let R be a smooth Riemannian structure on M; we select also an orientation on M. R induces a smooth volume measure v on M, and for each element x in M, a positive definite inner product $R^x(.,.)$ on the tangent space T_xM, and a real connected compact

Lie group $SO(T_xM)$, isomorphic to the rotation group of $\mathbb{R}^{\dim(M)}$ consisting of all the automorphisms of T_xM preserving the inner product $R^x(.,.)$ and the orientation of T_xM. Let τ_x be the natural faithful irreducible orthogonal representation of $SO(T_xM)$ into T_xM; one gets the motion group $\Delta(T_xM)$ as the semi-direct product $\Delta(T_xM) = SO(T_xM) \cdot T_xM$, the product being given for any pair $((g_x, h_x), (g'_x, h'_x))$ of elements of $SO(T_xM) \times T_xM$ by:

$$(g_x, h_x) \cdot (g'_x, h'_x) = (g_x . g'_x, h_x + \tau_x(g_x)h'_x). \tag{17'}$$

We notice that $SO(TM) = \bigcup_{x \in M} SO(T_xM)$ and $\Delta(M, R) = \bigcup_{x \in M} \Delta(T_xM)$ are smooth bundles of finite dimensional real connected Lie groups. The notations being the ones that have been initiated in Section III, we notice that the nuclear Lie group $\mathcal{D}_0(SO(TM))$ acts continuously on the nuclear vector space $\mathcal{D}_0(TM)$ of all the compactly supported smooth vectorfields on M via the continuous representation τ defined for any g in $\mathcal{D}_0(SO(TM))$ and any vectorfield ξ in $\mathcal{D}_0(TM)$:

$$(\tau(g)\xi)(x) = \tau_x(g(x))(\xi(x)), \quad x \in M. \tag{18}$$

Consequently one can form the semi-direct product of nuclear Lie groups $\mathcal{D}_0(SO(TM)) \cdot \mathcal{D}_0(TM)$, the product being given for any pair $((g, \xi), (g', \xi'))$ of elements in $\mathcal{D}_0(SO(TM)) \times \mathcal{D}_0(TM)$ by:

$$(g, \xi) \cdot (g', \xi') = (g.g', \xi + \tau(g)\xi') . \tag{19}$$

From the above discussion it follows that $\mathcal{D}_0(\Delta(M, R))$ is exactly the nuclear Lie group of class (S):

$$\mathcal{D}_0(\Delta(M, R)) = \mathcal{D}_0(SO(TM)) \cdot \mathcal{D}_0(TM) . \tag{20}$$

DEFINITION 3: $\mathcal{D}_0(\Delta(M, R))$ will be called the nuclear Lie group of Riemannian motions associated with the Riemannian structure R on M.

(b) Let $C^+(M)$ be the cone of all the continuous mappings ρ from M into the set $\mathbb{R}$ of real numbers such that $\rho(x) > 0$, $\forall x \in M$. Each element ρ in $C^+(M)$ allows one to get a positive definite inner product $(\ ,\ )_\rho$ on $\mathcal{D}_0(TM)$ by taking for any pair (ξ, ξ') of elements in $\mathcal{D}_0(TM)$,

$$(\xi, \xi')_\rho = \int_M R^x(\xi(x), \xi'(x))\rho(x)dv(x). \tag{21}$$

From the fact that, for each x in M, τ_x is a continuous orthogonal representation in T_xM with respect to $R^x(.,.)$ we have the obvious following result.

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LEMMA 5. *τ is a continuous orthogonal representation of $\mathcal{D}_0(SO(TM))$ into the real prehilbertian space $\left(\mathcal{D}_0(TM), (\ ,\ )_\rho\right)$ for any ρ in $C^+(M)$.*

(c) Let us select a density ρ in $C^+(M)$, and among the normed cylindrical positive measures on $\mathcal{D}'_0(TM)$ let us select the gaussian measure m_ρ, the Fourier transform of which is given for any ξ in $\mathcal{D}_0(TM)$ by:

$$\widehat{m}_\rho(\xi) = \int_{\mathcal{D}'_0(TM)} e^{i\langle \chi, \xi\rangle} dm_\rho(\chi) = e^{-(\xi,\xi)_\rho/2} . \tag{22}$$

The orthogonality of τ with respect to $(\ ,\ )_\rho$ implies that m_ρ is invariant under the operators $\tau^*(g)$, $g \in \mathcal{D}_0(SO(TM))$; as a result of Theorem 1 it follows that the representation T_ρ of $\mathcal{D}_0(\Delta(M,R)) = \mathcal{D}_0(SO(TM))\cdot\mathcal{D}_0(TM)$ into $L^2(\mathcal{D}'_0(TM); m_\rho)$ defined, for any (g,ξ) in $\mathcal{D}_0(SO(TM))\cdot\mathcal{D}_0(TM)$ and for any Φ in $L^2(\mathcal{D}'_0(TM); m_\rho)$, by:

$$(T_\rho(g,\xi)\Phi)(\chi) = e^{i\langle\chi,\xi\rangle} \cdot \Phi\left(\tau^*(g^{-1})\chi\right) , \quad \chi \in \mathcal{D}'_0(TM) , \tag{23}$$

is a continuous and non located unitary cylindrical representation of $\mathcal{D}_0(\Delta(M,R))$.

THEOREM 2. *For any density ρ in $C^+(M)$ the unitary cylindrical representation T_ρ of $\mathcal{D}_0(\Delta(M,R))$ is irreducible.*

PROOF: For each x in M let b_x be the 1-cocycle of the motion group $SO(T_xM).T_xM$ with respect to the irreducible unitary representation λ_x of this group in T_xM such that for any element (g_x,h_x) in $SO(T_xM)\times T_xM$:

$$\lambda_x(g_x,h_x) = \tau_x(g_x) \quad \text{and} \quad b_x(g_x,h_x) = h_x . \tag{24}$$

The pair (λ_x, b_x) provides a new unitary representation, often called an exponential representation and denoted $\mathrm{EXP}_{\lambda_x,b_x}$, of $SO(T_xM).T_xM$ which can be realised in the symmetric Fock space constructed on the base T_xM, and such that on the coherent states:

$$\mathrm{EXP}\ k_x = \sum_{n\geqslant 0}^{\oplus} (n!)^{-1/2} k_x^{\otimes n} , \quad k_x \in T_xM :$$

$$\mathrm{EXP}_{\lambda_x,b_x}(g_x,h_x)\ \mathrm{EXP}\ k_x = \exp\{-\frac{1}{2}\|h_x\|^2 - \langle\tau_x(g_x)k_x, h_x\rangle\} \ .\ \mathrm{EXP}\ (\tau_x(g_x)k_x + h_x).$$

(For such construction see for example, [**11**].) Theorem 7.2 of [**11**] allows us to assert that T_ρ is unitarily equivalent to the continuous tensor product $\bigotimes_{x\in M} \mathrm{EXP}_{\lambda_x,b_x}$ with respect to the non-atomic measure ρdv on M. As the representations τ_x are irreducible and as the b_x are non-trivial 1-cocycles vanishing on the compact subgroup $SO(T_xM)$ of $SO(T_xM).T_xM$ which generates the space T_xM of the representation λ_x, it follows from [**6**, Theorem 3], that the continuous tensor product $\bigotimes_{x\in M} \mathrm{EXP}_{\lambda_x,b_x}$ is irreducible. ∎

VI. Generalised energy representations of $\mathcal{D}_0(F)$.

Let us come back to Section III-(b), that is: one considers a smooth bundle $F = \bigcup_{x \in M} F^x$, over a connected paracompact smooth manifold M, of finite dimensional real Lie groups, the corresponding smooth bundle $\mathcal{F} = \bigcup_{x \in M} \mathcal{F}^x$ of Lie algebras, and the corresponding nuclear Lie group of class (S) (see Lemma 5):

$$(\mathcal{D}_0(F).\mathcal{D}_1(\mathcal{F}), V).$$

For each x in M and each γ in F^x let R^x_γ be the right translation by γ in F^x, and for any element σ in $\mathcal{D}_0(F)$ let $\beta(\sigma)$ be the element of $\mathcal{D}_1(\mathcal{F})$ defined by:

$$\beta(\sigma)(x) = \left(dR^x_{\sigma(x)}\right)_{\sigma^{-1}(x)} . \left(d\sigma^{-1}\right)_x, \quad x \in M, \tag{25}$$

where d denotes the exterior differential operator.

In the case where F is a trivial bundle, that is, of the form $F = M \times G$, G being some real connected Lie group, the assignment $\sigma \mapsto \beta(\sigma)$ is known by the name of Maurer-Cartan cocycle [**23, 24**]; we shall give the same name for the continuous mapping $\beta : \sigma \mapsto \beta(\sigma)$ in the general case. We point out that for any (σ, ω) in $\mathcal{D}_0(F) \times \mathcal{D}_1(\mathcal{F})$ one can formally write $\beta(\sigma) = \sigma.d\sigma^{-1}$ and $V(\sigma)\omega = \sigma.\omega.\sigma^{-1}$, from which one easily deduces the 1-cocycle relation:

$$\beta(\sigma.\sigma') = \beta(\sigma) + V(\sigma)\beta(\sigma') \, . \tag{26}$$

Consequently β belongs to $Z^1(\mathcal{D}_0(F), V)$; moreover, while $V(\sigma)$ depends only on σ, $\beta(\sigma)$ depends on the derivative of σ, so that β cannot be a 1-coboundary.

Theorem 3. *Let μ be a normed cylindrical positive measure on $\mathcal{D}'_1(\mathcal{F})$ and let c be any element of $\mathcal{C}(\mathcal{D}_0(F), \mathcal{D}_1(\mathcal{F}), V, \mu)$. One gets a continuous non-located representation U^c_μ of $\mathcal{D}_0(F)$ into $L^2(\mathcal{D}'_1(\mathcal{F}); \mu)$ by taking, for any σ in $\mathcal{D}_0(F)$ and any Φ in $L^2(\mathcal{D}'_1(\mathcal{F}); \mu)$:*

$$\left(U^c_\mu(\sigma)\Phi\right)(\chi) = e^{i\langle \chi, \beta(\sigma)\rangle}.c(\sigma)(\chi).\Phi\left(V^*(\sigma^{-1})\chi\right), \quad \chi \in \mathcal{D}'_1(\mathcal{F}).$$

Proof: Let μ be in $M(\mathcal{D}'_1(\mathcal{F}))$ and let c be in $\mathcal{C}(\mathcal{D}_0(F), \mathcal{D}_1(\mathcal{F}), V, \mu)$; from Theorem 1 one gets a μ-cylindrical representation T_c of $\mathcal{D}_0(F).\mathcal{D}_1(\mathcal{F})$ by taking for any (σ, ω) in $\mathcal{D}_0(F) \times \mathcal{D}_1(\mathcal{F})$, and for any Φ in $L^2(\mathcal{D}'_1(\mathcal{F}); \mu)$:

$$(T_c(\sigma, \omega)\Phi)(\chi) = e^{i\langle \chi, \omega\rangle}.c(\sigma)(\chi).\Phi\left(V^*(\sigma^{-1})\chi\right), \quad \chi \in \mathcal{D}'_1(\mathcal{F}).$$

Now let j be the mapping from $\mathcal{D}_0(F)$ into $\mathcal{D}_0(F).\mathcal{D}_1(\mathcal{F})$ defined for any σ in $\mathcal{D}_0(F)$ by $j(\sigma) = (\sigma, \beta(\sigma))$. From (26) and Lemma 3 it follows that j is a one-to-one continuous homomorphism of nuclear Lie groups; the assertion is then proved by taking $U^c_\mu = T_c \circ j$. ∎

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REMARKS. (1) If, in Theorem 3, one takes c of the form $c_{r,b}$, $r \in \mathbb{R}$, $b \in Z^1(\mathcal{D}_0(F), V)$ (see Lemma 4) such that $\beta + rb$ is a 1-coboundary, one easily sees that U_μ^c is equivalent to the representation Λ^0 defined by:

$$\big(\Lambda^0(\sigma)\Phi\big)(\chi) = \Phi\big(V^*(\sigma^{-1})\chi\big),$$

which is highly decomposable and reducible, and then if we want to have a chance that U_λ^c is irreducible we have to avoid choosing c of this form.

(2) If μ is quasi-invariant under the operators $V^*(\sigma)$, $\sigma \in \mathcal{D}_0(F)$, by taking for c the positive square root of the Radon-Nikodym derivative of the image of μ by the operators $V^*(\sigma^{-1})$, then U_μ^c is unitary by Theorem 1.

As a particular case of the situation described in the above remark 2 let us consider the case where $F = M \times G$, where G is a compact semi-simple Lie group. Selecting a smooth Riemannian structure R on M we can associate in a natural way a positive definite inner product $(\ ,\ )_R$ on $\mathcal{D}_1(\mathcal{F})$ bi-invariant under the operators $V(g)$, $g \in \mathcal{D}_0(M \times G) = C_0^\infty(M, G)$, and then a cylindrical Gaussian measure μ_R on $\mathcal{D}_1'(\mathcal{F})$ invariant under the operators $V^*(g)$. The corresponding unitary representation, by taking here $c \equiv 1$, $U_R = U_\mu^1$ is exactly the energy representation associated with R of the gauge group $C_0^\infty(M, G)$[**2**, **23**, **17**]. The same situation occurs in the case where, more generally, F is a smooth bundle of compact semi-simple Lie groups, for instance in the case $F = SO(TM)$ studied in Section V, and in the case where F is the associated bundle of some G-principal bundle P, with G compact semi-simple; in this last case $\mathcal{D}_0(F)$ is the so-called nuclear gauge group associated with P (see for example, [**9**]).

We are then led to bring in the following definition.

DEFINITION 4: The representations U_μ^c provided by Theorem 3 will be called the generalised energy representations of $\mathcal{D}_0(F)$.

VII. APPLYING THE THEORY OF G-DISTRIBUTIONS.

(a) Let M be a finite dimensional and connected smooth paracompact manifold, and let G be a finite dimensional connected real Lie group; $C_0^\infty(M, G) = \mathcal{D}_0(M \times G)$ may be considered as the nuclear space of the G-valued test-functions on M. As has been recalled in the introduction, in extension of the ideas of Gelfand *et al.*, one is led to look the (classes of) continuous and irreducible representations of the nuclear Lie group $C_0^\infty(M, G)$ as the multiplicative G-distributions on M, without the obligation to be unitary. As a matter of fact, when we consider solely the irreducible unitary representations of $C_0^\infty(M, G)$ we are obliged to observe that, generally, except in the case G nilpotent (see for example, [**13**]) there are very few non-located (that is, with continuous support) unitary G-distributions which are known, on account of various

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obstructions; we refer to [**15**] for a survey of what is known at the present time. For instance, in the case G non-compact simple Lie group different from $SO_0(n,1)$ or $SU(n,1)$, we are unable to find a non-located irreducible unitary representation of $C_0^\infty(M,G)$.

Even dropping the unitary stipulation we have nevertheless to observe that the determination of all (the classes of) non located continuous irreducible representations of $C_0^\infty(M,G)$ seems to be a terrific problem. In order to progress in this way we have to be sure that there exist non located continuous representations of any order having a chance to be irreducible or to be a direct sum of non located irreducible representations, that is to say, which, in particular, cannot be decomposable into a continuous sum of representations with finite support. We want to give here a first element of response in this direction.

(b) For any integer $m \geqslant 0$ the jet manifold $J^m(M \times G) = J^m(M,G)$ consisting of all the jets $j_x^m(g)$, $x \in M$, $g \in C_0^\infty(M,G) = \mathcal{D}_0(M \times G)$, (of order m) is a smooth bundle over M of finite dimensional Lie groups [**7, 19**], the fiber in x being the group $J_x^m(M,G) = \{j_x^k(g), g \in C_0^\infty(M,G)\}$, and for any g in $C_0^\infty(M,G)$ the mapping $j^m(g)$: $x \mapsto j_x^m(g)$ belongs to $\mathcal{D}_0(J^m(M,G))$. Moreover the assignment $j^m : g \mapsto j^m(g)$ is a one-to-one continuous homomorphism from $C_0^\infty(M,G)$ into $\mathcal{D}_0(J^m(M,G))$.

Let $\mathcal{G}$ be the Lie algebra of G, let $J^m(M,\mathcal{G})$ be the bundle of all the m-jets of elements in $C_0^\infty(M,\mathcal{G})$, and as in Section III-(b), let V be the adjoint representation of $\mathcal{D}_0(J^m(M,G))$ into the space $\mathcal{D}_1(J^m(M,\mathcal{G}))$, and β the Maurer-Cartan cocycle of $\mathcal{D}_0(J^m(M,G))$.

For any integer $k \geqslant 1$ and μ in $M\big(\mathcal{D}_1'\big(J^{k-1}(M,\mathcal{G})\big)\big)$ let us define the set $S(M,G,k,\mu)$ as the (non-empty) subset of

$$\mathcal{C}\left(\mathcal{D}_0\left(J^{k-1}(M,G)\right),\mathcal{D}_1\left(J^{k-1}(M,\mathcal{G})\right),V,\mu\right)$$

consisting of the elements which are not of the form $c_{r,b}$, $r \in \mathbb{R}$, $b \in Z^1\big(\mathcal{D}_0\big(J^{k-1}(M,G)\big),V\big)$, defined by $c_{r,b} = \exp\{ir\langle ., b(\sigma)\rangle\}$ (that is, see Lemma 4 (2)), and such that $\beta + rb$ is a 1-coboundary.

Taking into account the above discussion, Theorem 3 and its Remark 1, one has the following result.

Theorem 4. *Let k be a strictly positive integer, let μ be a normed cylindrical positive measure on $\mathcal{D}_1'\big(J^{k-1}(M,\mathcal{G})\big)$, and let c be in $S(M,G,k,\mu)$. One gets a non trivially decomposable non located continuous representation $\pi_{\mu,c}^{(k)}$ of order k of $C_0^\infty(M,G)$ by taking for any g in $C_0^\infty(M,G)$ and any Φ in $L^2\big(\mathcal{D}_1'\big(J^{k-1}(M,\mathcal{G});\mu\big)\big)$:*

$$\left(\pi_{\mu,c}^{(k)}(g)\Phi\right)(\chi) = e^{i\langle\chi,\beta(j^{k-1}(g))\rangle}.c\big(j^{k-1}(g)\big)(\chi).\Phi\big(V^*\big(j^{k-1}(g^{-1}\big)\chi\big)\big)\ .$$

REMARK. Let G be compact semi-simple, and let M be equipped with a sufficiently regular Riemannian structure R; by taking $k = 1$, $c \equiv 1$, and $\mu = \mu_R$, the Gaussian measure on $\mathcal{D}_1'(M \times \mathcal{G})$ associated with R, then the corresponding representation $\pi_{\mu,1}^{(1)}$, is exactly the energy representation of $C_0^\infty(M, G)$ associated with the Riemannian structure R (for example, [**17**]), which is known to be irreducible or decomposable into non located irreducible components of order 1 according to the dimension of M (for example, [**2, 17**]).

As things are, in the general case we can only assert that the representations $\pi_{\mu,c}^{(k)}$ are good candidates in order to provide, whatever are the properties of G, G-multiplicative distributions of order k on M. In order to complete the present contribution to the theory of multiplicative distributions it remains to study the problem of their irreducibility, and, in the reducible case, of their decomposition into irreducible non located components.

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