COMPUTATION OF THE GENERALISED FACTORIAL FUNCTION

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1. Introduction

The generalised factorial function $(z;\ k)!$ has been defined by Smith-White and Buchwald [1] in terms of an infinite product which converges very slowly, about 10^5 terms being required for four figure accuracy if $|z| = 10$. A method is given for the computation of $(z;\ k)!$ for $0 < |z| \leqq 10$ to four figure accuracy.

It will be seen that the method is easily adaptable to any value of $|z|$ and any desired order of accuracy. This paper deals only with the particular case $k = 1$.

2. The remainder term

$(z;\ 1)!$ is defined by

$$\frac{1}{(z;\ 1)!} = \sqrt{2} \lim_{2N\to\infty} \left\{(2N)^{-z} \prod_{0<R(\zeta)<2N} \left(1+\frac{z}{\zeta}\right)\right\} \tag{1}$$

where the ζ are the roots of the integral function $\sin \pi\zeta + \pi\zeta$.

We may rewrite (1) to define $(z;\ 1)!$ by an equivalent relation

$$\frac{1}{(z;\ 1)!} = \sqrt{2}\, 2^{-z} e^{\gamma z} \prod_{n=1}^{\infty} \left\{\left(1+\frac{z}{\zeta_n}\right)\left(1+\frac{z}{\bar{\zeta}_n}\right) e^{-z/n}\right\} \tag{2}$$

where

$$\begin{aligned} \zeta_n &= \lambda_n - \frac{\log 2\pi\lambda_n}{\pi^2\lambda_n} + \frac{i \log 2\pi\lambda_n}{\pi} + \varepsilon(n), \\ \lambda_n &= \tfrac{1}{2}(4n-1). \end{aligned} \tag{3}$$

The numerical value of $\varepsilon(n)$ is less than 2×10^{-7} when $n = 100$. The ζ_n can then easily be determined as accurately as necessary by an iterative technique such as the Newton-Raphson method.

Taking the logarithm of (2), it can be seen that

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$$\frac{1}{(z;1)!} = \sqrt{2}\, 2^{-z} e^{\gamma z + R_N(z)} \prod_{n=1}^{N} \left\{ \left(1+\frac{z}{\zeta_n}\right)\left(1+\frac{z}{\bar{\zeta}_n}\right) e^{-z/n} \right\}$$

where

$$R_N(z) = \sum_{n=N+1}^{\infty} \left\{ \log\left(1+\frac{z}{\zeta_n}\right) + \log\left(1+\frac{z}{\bar{\zeta}_n}\right) - \frac{z}{n} \right\}. \tag{4}$$

Expanding (4)

$$R_N(z) = z \sum_{n=N+1}^{\infty} \left\{ \frac{1}{\zeta_n} + \frac{1}{\bar{\zeta}_n} - \frac{1}{n} \right\} + \sum_{r=2}^{\infty} S_r(z)$$

where

$$S_r(z) = \sum_{n=N+1}^{\infty} \left\{ (-1)^{r+1} \frac{z^r}{r} \left(\frac{1}{\zeta_n^r} + \frac{1}{\bar{\zeta}_n^r} \right) \right\}.$$

It can be shown that if $|z| \leqq 10$, and N is chosen so that $|z|/N \leqq 1/10$, for $r_0 = 5$, $|\sum_{r=r_0}^{\infty} S_r(z)|$ is less than 3×10^{-6}. Thus

$$R_N(z) = z \sum_{n=N+1}^{\infty} \left\{ \frac{1}{\zeta_n} + \frac{1}{\bar{\zeta}_n} - \frac{1}{n} \right\} + \sum_{r=2}^{4} S_r(z) \tag{5}$$

with an error which is less than 3×10^{-6}. Using the Euler-Maclaurin Formula [2], with (3), we have that

$$\sum_{n=N+1}^{\infty} \left\{ \frac{1}{\zeta_n} + \frac{1}{\bar{\zeta}_n} - \frac{1}{n} \right\} = -\frac{1}{2} \left\{ \frac{2\left(\lambda_N - \dfrac{\log 2\pi\lambda_N}{\pi^2 \lambda_N}\right)}{\left(\lambda_N - \dfrac{\log 2\pi\lambda_N}{\pi^2\lambda_N}\right)^2 + \left(\dfrac{\log 2\pi\lambda_N}{\pi}\right)^2} - \frac{1}{N} \right\}$$
$$+ \int_{2\lambda_N}^{\infty} \left\{ \frac{1}{u} \frac{\left(1 - \dfrac{4 \log \pi u}{\pi^2 u^2}\right)}{\left(1 - \dfrac{4\log \pi u}{\pi^2 u^2}\right)^2 + 4\left(\dfrac{\log \pi u}{\pi u}\right)^2} - \frac{1}{u+1} \right\} du,$$

the error involved here being less than 2×10^{-7}.

Expansion of the integrand by the Binomial Theorem gives, after integration, that

$$\sum_{n=N+1}^{\infty} \left\{ \frac{1}{\zeta_n} + \frac{1}{\bar{\zeta}_n} - \frac{1}{n} \right\} = -\frac{1}{2\lambda_N} \left\{ 1 + \frac{1}{4\lambda_N} + \frac{(\log 2\pi\lambda_N)^2}{\pi^2 \lambda_N} \right\} + \frac{1}{2N}$$

with an error which is less than 1×10^{-6}. Applying the same technique to the other terms of (5) we obtain finally that

$$R_N(z) = z\left\{\frac{1}{2N} - \frac{1}{2\lambda_N}\left[1+\frac{1}{4\lambda_N}+\frac{(\log 2\pi\lambda_N)^2}{\pi^2\lambda_N}\right]\right\}$$
$$-\frac{z^2}{2}\left\{\frac{1}{\lambda_N} - \frac{3}{\pi^2\lambda_N^3}(\log 2\pi\lambda_N)^2\right\}$$
$$+\frac{z^3}{3}\left\{\frac{1}{\lambda_N^2} - \frac{1}{\lambda_N^3}\right\} - \frac{z^4}{12\lambda_N^3}$$

with an error of less than 4×10^{-5}. It can be noted that if $10 \leqq |z| \leqq 100$ and $N = 1{,}000$ the above formula still holds with an error of less than 3×10^{-5}.

3. Computations of particular cases

In connection with a problem of the infinite strip with mixed boundary conditions, some values of $(z;\, 1)!$ were computed.

These values were checked by use of the formula

$$(z;\, 1)!\,(-z;\, 1)! = \frac{\pi z}{\sin \pi z + \pi z},$$

it being found that by taking $N = 100$ for $0 < |z| \leqq 10$, and $N = 1{,}000$ for $10 < |z| \leqq 60$ there was agreement to at least four figures.

For the case $|z| = 10$, $(z;\, 1)!$ was calculated using $N = 100$ and $N = 1{,}000$ and here again agreement to four figures was obtained.

TABLE 1

$(n;\, 1)!$ $(n = 1, 2, \cdots, 10)$

n	$(n;\, 1)!$	n	$(n;\, 1)!$
1	1.728	6	1.394×10^2
2	3.666	7	9.806×10^3
3	1.128×10^1	8	7.872×10^4
4	4.579×10^1	9	7.102×10^5
5	3.209×10^2	10	7.119×10^6

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TABLE 2

$(2n-1;\ 1)!$ $(n = 6, 7, \cdots, 30)$.

n	$(2n-1;\ 1)!$	n	$(2n-1;\ 1)!$
6	7.844×10^{7}	19	2.737×10^{43}
7	1.226×10^{10}	20	4.058×10^{46}
8	2.580×10^{12}	21	6.656×10^{49}
9	7.032×10^{14}	22	1.202×10^{53}
10	2.406×10^{17}	23	2.380×10^{56}
11	1.012×10^{20}	24	5.147×10^{59}
12	5.126×10^{22}	25	1.210×10^{63}
13	3.077×10^{25}	26	3.085×10^{66}
14	2.160×10^{28}	27	8.499×10^{69}
15	1.756×10^{31}	28	2.521×10^{73}
16	1.633×10^{34}	29	8.036×10^{76}
17	1.725×10^{37}	30	2.743×10^{80}
18	2.054×10^{40}		

TABLE 3

$(\zeta_n;\ 1)!$ $(n = 1, 2, \cdots, 30)$

n	Re $(\zeta_n;\ 1)!$	Im $(\zeta_n;\ 1)!$
1	1.651	.9093
2	3.379	1.692×10^{1}
3	-1.962×10^{2}	3.993×10^{2}
4	-1.862×10^{4}	1.234×10^{4}
5	-1.826×10^{6}	2.573×10^{5}
6	-2.199×10^{8}	-5.557×10^{7}
7	-3.221×10^{10}	-2.165×10^{10}
8	-5.472×10^{12}	-6.926×10^{12}
9	-9.673×10^{14}	-2.384×10^{15}
10	-1.169×10^{17}	-9.294×10^{17}
11	4.836×10^{19}	-4.130×10^{20}
12	7.424×10^{22}	-2.084×10^{23}
13	7.318×10^{25}	-1.183×10^{26}
14	6.974×10^{28}	-7.430×10^{28}
15	6.966×10^{31}	-5.057×10^{31}
16	7.483×10^{34}	-3.600×10^{34}
17	8.731×10^{37}	-2.486×10^{37}
18	1.108×10^{41}	-1.269×10^{40}
19	1.530×10^{44}	6.36×10^{42}
20	2.293×10^{47}	4.468×10^{46}
21	3.717×10^{50}	1.301×10^{50}
22	6.489×10^{53}	3.343×10^{53}
23	1.214×10^{57}	8.486×10^{56}
24	2.422×10^{60}	2.208×10^{60}
25	5.117×10^{63}	6.000×10^{63}
26	1.134×10^{67}	1.712×10^{67}
27	2.603×10^{70}	5.156×10^{70}
28	6.052×10^{73}	1.639×10^{74}
29	1.367×10^{77}	5.512×10^{77}
30	2.672×10^{80}	1.957×10^{81}

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References

[1] W. B. Smith-White and V. T. Buchwald, A generalization of $z!$, This Journal 4 (1964), 327—341.

[2] E. T. Whittaker and G. N. Watson, A Course of Modern Analysis, Fourth Edition, Cambridge (1927), p. 127.

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