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PACKING DIMENSION AND MEASURE OF HOMOGENEOUS CANTOR SETS

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For a class of homogeneous Cantor sets, we find an explicit formula for their packing dimensions. We then turn our attention to the value of packing measures. The exact value of packing measure for homogeneous Cantor sets has not yet been calculated even though that of Hausdorff measures was evaluated by Qu, Rao and Su in (2001). We give a reasonable lower bound for the packing measures of homogeneous Cantor sets. Our results indicate that duality does not hold between Hausdorff and packing measures.

1. INTRODUCTION

From the wide variety of fractal dimensions in use, the definition of Hausdorff measures, based on a construction of Caratheodory, is the oldest and probably the most important. The Hausdorff dimension has the advantage of being defined for any set and is mathematically convenient as it is based on measures which are relatively easy to manipulate. Packing measures and dimensions, introduced by Tricot[**8**], are much more recent. Their similarities to and differences from Hausdorff measures and dimensions are providing an important theoretical tool. The introduction of packing measures has led to a greater understanding of the geometric measure theory of fractals ([**7**]), with packing measures behaving in a way that is dual to Hausdorff measures in many respects.

One of the disadvantages of studying fractal measures and dimensions (Hausdorff and packing measures) is that in many cases they are hard to calculate or estimate computationally. However, many researchers[**1, 3, 4, 5, 6, 9, 10**] have found exact values or estimated the lower and upper bounds of measures and dimensions for some fractal sets. In these papers and the references therein the interested reader can find an analysis of the exact dimension and measure of some particular fractal sets. In particular, explicit formulae for Hausdorff dimensions of specific Cantor sets were found in [**4, 5, 6**], while in [**1, 3, 9, 10**] exact values were calculated for Hausdorff measures for some Cantor sets, including homogeneous Cantor sets.

This paper analyses the behaviour of the packing dimensions and measures on a homogeneous Cantor set E, which is a generalised form of self - similar Cantor sets. We

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need Baire category theory and the mass distribution principle to get the exact value of the packing dimension of a set E. To obtain a lower bound of packing measure, we use the pseudo-packing measure first adopted by Raymond and Tricot [7].

We define the packing measure and dimension as follows:

A *δ-packing* of $E \subset \mathbb{R}^n$ is a family of pairwise disjoint balls with centre in $E \subset \mathbb{R}^n$ and diameter less than or equal to δ. The *s-dimensional pre-packing measure* of $E \subset \mathbb{R}^n$ is

$$P^s(E) = \lim_{\delta\to 0}\sup\Big\{\sum_{i=1}^{\infty}|U_i|^s : \{U_i\}_{i=1}^{\infty} \quad \text{is} \quad \text{a} \quad \delta-\text{packing} \quad \text{of } E\Big\},$$

where $|U|$ is the diameter of a subset U in $\mathbb{R}^n$. The *s-dimensional packing measure* of $E \subset \mathbb{R}^n$ is

$$p^s(E) = \inf\Big\{\sum_{i=1}^{\infty}P^s(E_i) : E \subset \cup E_i\Big\}.$$

We define the *packing dimension* in the usual way:

$$\dim_{\mathrm{p}}(E) = \sup\{s : p^s(E) = \infty\} = \inf\{s : p^s(E) = 0\}.$$

Homogeneous Cantor sets are nowhere dense perfect subsets of $[0,1]$ constructed in the following manner([**4, 9**]). Suppose $I = [0,1]$ and let$\{n_k\}_{k\geqslant 1}$ be a sequence of positive integers and $\{r_k\}_{k\geqslant 1}$ a real number sequence satisfying $n_k \geqslant 2$, $0 < n_k r_k < 1 (k \geqslant 1)$. For any $k \geqslant 1$, let

$$D_k = \{(i_1,\dots,i_k) : 1 \leqslant i_j \leqslant n_j, 1 \leqslant j \leqslant k\}, \quad D = \bigcup_{k\geqslant 0} D_k,$$

where $D_0 = \phi$. If

$$\sigma = (\sigma_1,\dots,\sigma_k) \in D_k, \tau = (\tau_1,\dots,\tau_m) \in D_m.$$

Let

$$\sigma * \tau = (\sigma_1,\dots,\sigma_k,\tau_1,\dots,\tau_m).$$

Let $\mathcal{F} = \{I_\sigma : \sigma \in D\}$ be the collection of closed sub-intervals of I which satisfy

(i) $I_\phi = I$;

(ii) For any $k \geqslant 1$ and $\sigma \in D_k$, $I_{\sigma*i} (1 \leqslant i \leqslant n_k)$ are sub-intervals of I_σ. $I_{\sigma*1},\dots,I_{\sigma*n_k}$ are arranged from the left to the right, $I_{\sigma*i}$ and I_σ have the same left endpoint, $I_{\sigma*n_k}$ and I_σ have the same right endpoint, and the lengths of the gaps between any two consecutive sub-intervals are equal. We denote the length of one of the gaps of d_k.

(iii) For any $k \geqslant 1$ and $\sigma \in D_k$, $1 \leqslant j \leqslant n_k$, we have

$$\frac{|I_{\sigma*j}|}{|I_\sigma|} = r_k,$$

where $|A|$ denote the diameter of A.

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Let $E_k = \bigcup_{\sigma \in D_k} I_\sigma$, $E = \bigcap_{k \geqslant 0} E_k$. We call E *the homogeneous Cantor set* determined by $\{n_k\}_{k\geqslant 1}\{r_k\}_{k\geqslant 1}$ and call $\mathcal{F}_k = \{I_\sigma : \sigma \in D_k\}$ the *kth - order basic intervals* of E. The middle-third Cantor set and the symmetric perfect set ([**5, 10**]) are well-known examples of homogeneous Cantor sets.

2. Packing dimension of homogeneous Cantor sets

In this section, we express the packing dimension of a homogeneous Cantor set as the explicit form with n_k and r_k. First, we define an auxiliary dimension set function by $q^s(E) = \varlimsup_{m\to\infty} \prod_{k=1}^{m} n_k r_k^s$ for $s \geqslant 0$. This function gives a number

$$\dim_{\mathrm{q}}(E) = \inf\{s \geqslant 0 \mid q^s(E) = 0\} = \sup\{s \geqslant 0 \mid q^s(E) = \infty\}.$$

Theorem 2.1. *Let E be a homogeneous Cantor set. Then* $\dim_{\mathrm{p}}(E) \geqslant \dim_{\mathrm{q}}(E)$.

Proof: If $\dim_{\mathrm{q}}(E) > t$, then $q^t(E) = \infty$, that is, $\varlimsup_{m\to\infty} \prod_{k=1}^{m} n_k r_k^t = \infty$.
Let $\delta > 0$ be given. Take k with $|I_\sigma| < \delta$ for all $\sigma \in D_k$. Then, for $\sigma \in D_k$,

$$\begin{aligned} p_\delta^t(I_\sigma \cap E) &\geqslant \sup_{m\geqslant k} \sum_{I_\tau \subset I_\sigma, \tau\in D_m} |I_\tau|^t = |I_\sigma|^t \sup_{m\geqslant k} \sum_{I_\tau \subset I_\sigma, \tau\in D_m} \frac{|I_\tau|^t}{|I_\sigma|^t} \\ &= |I_\sigma|^t \sup_{m\geqslant k} \prod_{l=k}^{m} n_l r_l^t \geqslant |I_\sigma|^t \inf_k \sup_{m\geqslant k} \prod_{l=k}^{m} n_l r_l^t \\ &= |I_\sigma|^t \varlimsup_{m\to\infty} \prod_{k=1}^{m} n_k r_k^t = \infty. \end{aligned}$$

Therefore $p^t(I_\sigma \bigcap E) = \infty$ for all $\sigma \in D_k$.

Consider $E = \bigcup_{n=1}^{\infty} F_n$. Since E is compact, $E = \bigcup_{n=1}^{\infty} \overline{F}_n$. By the Baire Category Theorem, we can take $\overline{F}_{n_0}$ whose interior in E is not empty. Hence, for sufficiently large n, there exist k with $k \geqslant n$ and $\sigma \in D_k$ such that $I_\sigma \bigcap E \subset \overline{F}_{n_0}$. Then

$$p^t(F_{n_0}) = p^t(\overline{F}_{n_0}) \geqslant p^t(I_\sigma \cap E) = \infty.$$

So $p^t(F_{n_0}) = \infty$. Thus we get that $p^t(E) = \infty$ and hence $\dim_{\mathrm{p}}(E) \geqslant t$. ∎

Theorem 2.2. *Let E be a homogeneous Cantor set with $\inf_k r_k > 0$. Then* $\dim_{\mathrm{p}}(E) \leqslant \dim_{\mathrm{q}}(E)$.

Proof: Let $\inf_k r_k = c (> 0)$. Then $r_k \geqslant c$ for all k.
Suppose $\dim_{\mathrm{q}}(E) < t$. Then $q^t(E) = 0$ that is, $\varlimsup_{m\to\infty} \prod_{k=1}^{m} n_k r_k^t = 0$.

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Define $\mu(I_\sigma) = 1/(n_1 n_2 \cdots n_k)$ for $\sigma \in D_k$. We can extend μ to a mass distribution on E. Let $x \in E = \bigcap_{k=1}^{\infty} E_k$. We can choose a sequence of intervals I_{σ_k} satisfying $x \in \bigcap_{k=1}^{\infty} I_{\sigma_k}$, where $\sigma_k \in D_k$. For any $r > 0$, we may find k satisfying $|I_{\sigma_{k+1}}| \leqslant r < |I_{\sigma_k}|$. We have that

$$\begin{aligned}\frac{\mu(B(x,r))}{r^t} \geqslant \frac{\mu(I_{\sigma_{k+1}})}{|I_{\sigma_k}|^t} &= \frac{\mu(I_{\sigma_{k+1}})}{|I_{\sigma_{k+1}}|^t \dfrac{|I_{\sigma_k}|^t}{|I_{\sigma_{k+1}}|^t}} \\ &= \frac{\mu(I_{\sigma_{k+1}})}{|I_{\sigma_{k+1}}|^t \left(\dfrac{1}{r_k}\right)^t} \geqslant \frac{\mu(I_{\sigma_{k+1}})}{|I_{\sigma_{k+1}}|^t \left(\dfrac{1}{c}\right)^t} \text{ (since } r_k \geqslant c) \\ &= \frac{c^t}{n_1 n_2 \cdots n_{k+1} r_1^t \cdots r_{k+1}^t} = \frac{c^t}{\prod_{l=1}^{k+1} n_l r_l^t}.\end{aligned}$$

So we get

$$\underline{\lim}_{r \to \infty} \Big(\mu(B(x,r))\Big)/r^t \geqslant \frac{c^t}{\overline{\lim}_{k\to\infty} \prod_{l=1}^{k+1} n_l r_l^t} = \infty.$$

Thus, by the mass distribution principle ([2]), $p^t(E) = 0$. Hence $\dim_p(E) \leqslant t$. ∎

We can easily get that

$$\dim_q E = \overline{\lim_{k\to\infty}} \frac{\log n_1 \cdots n_k}{-\log r_1 \cdots r_k}$$

by elementary calculation. The condition $\inf_k r_k > 0$ leads to an explicit formula with n_k and r_k of the packing dimension for the homogeneous Cantor sets as follows.

Corollary 2.3. *Let E be a homogeneous Cantor set.*

(1) $\dim_p E \geqslant \overline{\lim_{k\to\infty}} (\log n_1 \cdots n_k)/(-\log r_1 \cdots r_k)$

(2) *If* $\inf_k r_k > 0$ *then* $\dim_p E = \overline{\lim_{k\to\infty}} (\log n_1 \cdots n_k)/(-\log r_1 \cdots r_k)$.

Example 2.4. We take $n_k = 2$ for all k. Then the homogeneous Cantor set E is a symmetric perfect set [**5, 8, 10**] satisfying the assumptions of the Corollary 2.3 (2). Thus we have $\dim_p(E) = \overline{\lim_{k\to\infty}} (k \log 2)/(-\log r_1 \cdots r_k)$. Kardos [5] calculated the Hausdorff dimension as $\dim_H(E) = \underline{\lim}_{k\to\infty} (k \log 2)/(-\log r_1 \cdots r_k)$.

3. Packing measure of homogeneous Cantor sets

We now evaluate a lower bound of packing measure for a homogeneous Cantor set. To do this, we adopt another definition of packing measure.

A *pseudo-packing* of a set E is any family $\mathcal{B}$ of bounded subsets of the real line such that, if $B, B' \in \mathcal{B}$, then $B^c \cap E^c = \phi$ and $B \cap B' \cap E = \phi$ where B^c is the closure of B. A pseudo-packing $\mathcal{B}$ of E by balls centred in E has the following property. If $B_r(x), B_s(y) \in \mathcal{B}$, then $y \notin B_r(x)$. A pseudo-packing gives rise to *a pre-measure* defined as $R^s(E) = \lim_{\delta\to\infty} R^s_\delta(E)$ where $R^s_\delta(E) = \sup\{\Sigma_n |U_n|^s : \{U_n\}$ is a δ - pseudo-packing of E by symmetric intervals centred in E with radii less than $\delta\}$ and *the pseudo-packing measure* is

$$r^s(E) = \inf\{\Sigma_k R^s(E_k) : E \subset \bigcup_k E_k\}.$$

Raymond and Tricot [7] defined the above measure and showed that for any set E, $r^s(E) = p^s(E)$.

Theorem 3.1. *Let E be a homogeneous Cantor set with $d_k \geqslant r_k/2$ and let $s = \dim_p(E)$. Then $p^s(E) \geqslant 2 \varlimsup_{n\to\infty} \prod_{k=1}^{n} n_k r_k^s$.*

Proof: Since E is a closed, bounded set, $P^s(E) = p^s(E) = r^s(E) = R^s(E)$.

For each I_σ, let I_σ^l be the symmetric open interval whose centre is the left endpoint of I_σ and let I_σ^r be the symmetric open interval whose centre is the right endpoint of I_σ and whose radius is $|I_\sigma|/2$. Then $|I_\sigma^l|^s + |I_\sigma^r|^s = 2|I_\sigma|^s$ and $\{I_\sigma^l, I_\sigma^r\}_{\sigma\in D_n}$ is a pseudo-packing of E since $d_n \geqslant r_n/2$. Let J_σ denote the gap between two consecutive sub-intervals of order $k+1$. Then $|J_\sigma| \geqslant \max\{|I_{\sigma,1}|/2, |I_{\sigma,2}|/2\}$ since $d_k \geqslant r_k/2$. Thus we have that

$$\sum_{\sigma\in D_n, |I_\sigma|/2<\delta} \left(|I_\sigma^l|^s + |I_\sigma^r|^s\right) \leqslant R^s_\delta(E).$$

So $2 \varlimsup_{n\to\infty} \prod_{k=1}^{n} n_k r_k^s = 2 \varlimsup_{n\to\infty} \sum_{\sigma\in D_n} |I_\sigma|^s = \varlimsup_{n\to\infty} \sum_{\sigma\in D_n} 2|I_\sigma|^s \leqslant R^s(E)$.

Therefore $2 \varlimsup_{n\to\infty} \prod_{k=1}^{n} n_k r_k^s \leqslant p^s(E)$. ∎

Example 3.2. For the middle-third Cantor set, we have $p^s(E) \geqslant 2$ from Theorem 3.1 since $\varlimsup_{n\to\infty} \prod_{k=1}^{n} n_k r_k^s = 1$. But it is shown in [3] using density theory that $p^s(E) = 4^s > 2$, where $s = \log 2/\log 3$. It is also well known that $H^s(E) = 1$.

Example 3.3. If we apply Theorem 3.1 to example 2.4, we get $p^s(E) \geqslant 2 \varlimsup_{n\to\infty} 2^n \prod_{k=1}^{n} r_k^s$, where s is the packing dimension of E. It follows from [9] that the Hausdorff measure $H^s(E) = \varliminf_{n\to\infty} 2^n \prod_{k=1}^{n} r_k^s$, where s is the Hausdorff dimension of E.

We conclude that duality between these two measures does not hold for a homogeneous Cantor set.

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