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ON STRONGLY STABLE APPROXIMATIONS

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Abstract

Ahues (1987) and Bouldin (1990) have given sufficient conditions for the strong stability of a sequence (T_n) of operators at an isolated eigenvalue of an operator T. This paper provides a unified treatment of their results and also generalizes so as to facilitate their application to a broad class of operators.

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1. Introduction

Let T be a bounded operator on a complex Banach space X and λ be an isolated nonzero eigenvalue of T. Let (T_n) be a sequence of bounded operators on X. In this paper we give sufficient conditions under which the sequence (T_n) is a strongly stable approximation of T at λ. This work is motivated by the recent results of Ahues [1] and Bouldin [3]. Ahues proves the results under the following assumptions:

(A1) T is a compact operator;
(A2) $\|(T - T_n)x\| \to 0$ for every $x \in X$;
(A3) $\|(T - T_n)T_n\| \to 0$.

Bouldin does not require T to be a compact operator, but instead requires λ to be of finite algebraic multiplicity and $\|T_n(T - T_n)\| \to 0$. More precisely, the conditions of Bouldin are the following:

(B1) λ is of finite algebraic multiplicity;

(B2) $\|T_n(T - T_n)\| \to 0$;
(B3) $\|(T - T_n)x\| \to 0$ for every $x \in X$;
(B4) $\|(T - T_n)T_n\| \to 0$.

In his proof, Ahues also requires the condition (B1) of Bouldin which is a consequence of (A1). The purpose of this paper is to unify and generalize the above results so as to include cases where one or more of the conditions (A1), (B1) and (B2) are not satisfied. One of the salient features of this paper is that the strong stability of a stable approximation (T_n) is proved under the assumption that $\|(T - T_n)T_n\| \to 0$ and *one* of the following conditions is satisfied:

(i) $\|(T - T_n)T\| \to 0$;
(ii) $\|(T - T_n)x\| \to 0$ for every $x \in X$, and λ is of finite algebraic multiplicity.

Thus our strong stability result requires neither T to be compact nor $\|T_n(T - T_n)\| \to 0$.

In Section 2 we introduce the definitions and prove some preliminary results. Our main results are proved in Section 3. In Section 4 we compare our results with those of Ahues and Bouldin by way of examples.

2. Preliminaries

For any bounded operator A on a Banach space X we denote the *resolvent set* of A by $\rho(A)$ and the *spectrum* by $\sigma(A)$:

$$\rho(A) := \{z \in \mathbb{C} : A - zI \text{ bijective}\}$$
$$\sigma(A) := \mathbb{C} \backslash \rho(A).$$

We observe that, by the Closed Graph Theorem, a point $z \in \rho(A)$ if and only if $R(z, A) := (A - zI)^{-1}$ exists and is a bounded operator on X. If Γ is a simple closed Jordan curve (s.c.J. curve) in $\rho(A)$, then the *spectral projection* of A associated with the spectral values of A lying inside Γ is given by the integral

$$P(A, \Gamma) := -\frac{1}{2\pi i}\int_\Gamma R(z, A)\, dz. \tag{2.1}$$

By the property of the contour integral it follows that $P(A, \Gamma_1) = P(A, \Gamma_2)$ if Γ_1 and Γ_2 are two s.c.J. curves in $\rho(A)$ such that the part of the spectrum of A enclosed by Γ_1 and Γ_2 are the same. Moreover if rank $P(A, \Gamma) = m$, then Γ encloses a finite number of eigenvalues of A with total algebraic multiplicity m (Kato [5]).

Let T_n and T be bounded operators on X. The sequence (T_n) is said to be a *stable approximation* of T at $z \in \rho(T)$ if there exists $N(z) \in \mathbb{N}$ and $M(z) > 0$ such that for all $n \geq N(z)$,

$$z \in \rho(T_n) \quad \text{and} \quad \|R(z, T_n)\| \leq M(z).$$

The sequence (T_n) is said to be a stable approximation of T on $K \subset \rho(T)$ if (T_n) is stable at each $z \in K$. If $K \subset \rho(T)$ is a compact subset of $\mathbb{C}$, then it can be verified that (T_n) is stable on K if and only if there exist $N(K) \in \mathbb{N}$ and $M(K) > 0$ such that for every $n \geq N(K)$,

$$K \subset \rho(T_n) \quad \text{and} \quad \sup\{\|R(z, T_n)\| : z \in K, \quad n \geq N(K)\} \leq M(K).$$

The sequence (T_n) is said to be a *strongly stable approximation* at an isolated spectral value λ of T if there is a deleted neighbourhood G of λ in $\rho(T)$ such that the following conditions are satisfied:

(i) (T_n) is a stable approximation of T on G;
(ii) for every s.c.J. curve Γ in G enclosing λ, $\operatorname{rank} P(T_n, \Gamma) = \operatorname{rank} P(T, \Gamma)$.

By a deleted neighbourhood G of λ we mean a set of the form

$$G = \{z \in \mathbb{C} : 0 < |z - \lambda| < r\} \quad \text{for some } r > 0.$$

PROPOSITION 2.1. *Let $G \subset \rho(T)$ be a deleted neighbourhood of an isolated spectral value λ of T, and (T_n) be a stable approximation of T on G. If Γ_0 is a s.c.J. curve in G enclosing λ, then*

(i) *(T_n) is a strongly stable approximation of T at λ if and only if* $\operatorname{rank} P(T_n, \Gamma_0) = \operatorname{rank} P(T, \Gamma_0)$,
(ii) *(T_n) is a strongly stable approximation of T at λ if $\|(P - P_n)P\| \to 0$ and $\|(P - P_n)P_n\| \to 0$, where $P = P(T, \Gamma_0)$ and $P_n = P(T_n, \Gamma_0)$.*

PROOF. Result (i) follows from the remarks following the definition of the integral in (2.1); and (ii) is a consequence of (i) and of a well known result ([5, Corollary IV.2.6] and [2, Proposition 2.13 (i)]).

Proposition 2.1 (i) shows that condition (ii) in the definition of strong stability can be replaced by (ii)$'$: for some s.c.J. curve Γ in G enclosing λ,

$$\operatorname{rank} P(T_n, \Gamma) = \operatorname{rank} P(T, \Gamma).$$

We are interested in the sufficient condition for strong stability given in Proposition 2.1 (ii). The following proposition provides a simple proof for the same.

PROPOSITION 2.2. *Let P_1 and P_2 be projections on X such that*

$$\|(P_1 - P_2)P_1\| < 1.$$

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Then

$$\operatorname{rank} P_1 \leq \operatorname{rank} P_2.$$

In particular, if

$$\max\{\|(P_1 - P_2)P_1\|, \ \|(P_1 - P_2)P_2\|\} < 1,$$

then

$$\operatorname{rank} P_1 = \operatorname{rank} P_2.$$

Proof. Consider the operator $Q\colon P_1X \to P_2X$ defined by $Qx = P_2x$, $x \in P_1X$. It is enough to show that Q is injective. Note that for every $x \in P_1X$,

$$Qx = P_2P_1x = P_1x - (P_1 - P_2)P_1x = x - (P_1 - P_2)P_1x.$$

Therefore, whenever $Qx = 0$ and $x \neq 0$, we have

$$\|x\| = \|(P_1 - P_2)P_1x\| \leq \|(P_1 - P_2)P_1\|\|x\| < \|x\|,$$

a contradiction; consequently Q is injective.

Proposition 2.3. *Let A and A_0 be bounded operators on X, and $1 \in \rho(A)$. Then $1 \in \rho(A_0)$ and*

$$\|R(1, A_0)\| \leq \frac{\|R(1, A)\|(1 + \|A - A_0\|)}{1 - \|R(1, A)\|\|(A - A_0)A_0\|}$$

if $\|(A-A_0)A_0\| < 1/\|R(1, A)\|$ and one of the following conditions is satisfied:

(i) $\max\{\|(A - A_0)A\|, \|(A - A_0)A_0\|\} < 1/2$;
(ii) $\|A_0(A - A_0)\| < 1/\|R(1, A)\|$;
(iii) *A_0 is a compact operator;*
(iv) *$A - A_0$ is a compact operator.*

Proof. We denote $A - A_0$ by Δ and consider the following identities:

(a) $(I - \Delta)(I + \Delta) = I - \Delta^2 = (I + \Delta)(I - \Delta)$;
(b) $(I - \Delta)(I - A_0) = (I - A)(I + R(1, A)\Delta A_0)$;
(c) $(I - A_0)(I - \Delta) = (I - A)(I + R(1, A)A_0\Delta)$.

The proof of the proposition follows using the above identities and the following results:

(d) if $\|B\| < 1$, then $1 \in \rho(B)$;
(e) if $0 \in \rho(BC)$, then B is onto and C is one-one;
(f) if B is a compact operator, then $I - B$ is one-to-one if and only if $I - B$ is onto.

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3. The main results

Let T be a bounded operator on a Banach Space X and λ be a non-zero spectral value of T. Let $G \subset \rho(T)$ be a deleted neighbourhood of λ such that $0 \notin G$ and Γ be an s.c.J. curve in G enclosing λ. By the assumption on G, there exists $\delta > 0$ such that $|z| \geq \delta$ for every $z \in \Gamma$.

THEOREM 3.1. *A stable approximation (T_n) of T on G is strongly stable at λ if $\|(T - T_n)T_n\| \to 0$ and one of the following conditions is satisfied:*

(i) $\|(T - T_n)T\| \to 0$;

(ii) $\operatorname{rank} P(T, \Gamma) < \infty$ *and* $\|(T - T_n)x\| \to 0$ *for every* $x \in X$.

PROOF. Let (T_n) be a stable approximation of T on G. Then by the compactness of Γ, it follows that there exists $N = N(\Gamma) \in \mathbb{N}$ and $M = M(\Gamma) > 0$ such that for all $n \geq N$, $\Gamma \subset \rho(T_n)$ and

$$\sup\{\|R(z, T_n)\| : z \in \Gamma, n \geq N\} \leq M. \tag{3.1}$$

Thus the projection $P_n := P(T_n, \Gamma)$ is well defined for $n \geq N$. Now, by Proposition 2.1, it is enough to prove that

$$\|(P - P_n)P\| \to 0 \quad \text{and} \quad \|(P - P_n)P_n\| \to 0,$$

where $P = P(T, \Gamma)$. For $n \geq N$, consider the resolvent identities

$$R(z, T) - R(z, T_n) = R(z, T_n)(T_n - T)R(z, T)$$

and

$$R(z, T) - R(z, T_n) = R(z, T)(T_n - T)R(z, T_n).$$

Then, using the results

$$R(z, T)P = PR(z, T) \quad \text{and} \quad R(z, T_n)P_n = P_nR(z, T_n)$$

and the definitions of P and P_n, we have

$$(P - P_n)P = -\frac{1}{2\pi i}\int_\Gamma R(z, T_n)(T_n - T)PR(z, T)\,dz \tag{3.2}$$

and

$$(P - P_n)P_n = -\frac{1}{2\pi i}\int_\Gamma R(z, T)(T_n - T)P_nR(z, T_n)\,dz \tag{3.3}$$

Now, using the continuity of the map $z \mapsto \|R(z, T)\|$ on the compact set Γ, we have

$$\sup\{\|R(z, T)\| : z \in \Gamma\} < \infty. \tag{3.4}$$

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Using (3.1), (3.4) in (3.2) and (3.3), we have that

$$\|(P - P_n)P\| \le M_1 \|(T - T_n)P\|,$$
$$\|(P - P_n)P_n\| \le M_2 \|(T - T_n)P_n\|$$

for some constants M_1 and M_2. Therefore the result follows once we show that

$$\|(T - T_n)P\| \to 0 \quad \text{and} \quad \|(T - T_n)P_n\| \to 0.$$

To prove this first note the identity

$$I + zR(z, A) = AR(z, A), \qquad z \in \rho(A)$$

for any bounded operator A on X. Then from the definition of P and P_n, we have

$$\begin{aligned}(T - T_n)P &= -\frac{1}{2\pi i}\int_\Gamma (T - T_n)R(z, T)\,dz \\ &= -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(T - T_n)[TR(z, T) - I]\,dz,\end{aligned}$$

and

$$\begin{aligned}(T - T_n)P_n &= -\frac{1}{2\pi i}\int_\Gamma (T - T_n)R(z, T_n)\,dz \\ &= -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(T - T_n)[T_nR(z, T_n) - I]\,dz.\end{aligned}$$

Since Γ does not enclose zero, $-\frac{1}{2\Lambda i}\int_\Gamma \frac{dz}{z} = 0$, so that the above two expressions take the form

$$(T - T_n)P = -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(T - T_n)TR(z, T)\,dz$$

and

$$(T - T_n)Pn = -\frac{1}{2\pi i}\int_\Gamma \frac{1}{z}(T - T_n)T_nR(z, T_n)\,dz.$$

Now, using (3.1), (3.4) and the fact that $|z| \ge \delta$ for every $z \in \Gamma$, for some $\delta > 0$, we have that

$$\|(T - T_n)P\| \le M_3 \|(T - T_n)T\|$$

and

$$\|(T - T_n)P_n\| \le M_4 \|(T - T_n)T_n\|$$

for some constants M_3 and M_4. Thus if

$$\|(T - T_n)T_n\| \to 0$$

and $\|(T - T_n)T\| \to 0$ then the result follows. Also, if $\operatorname{rank} P < \infty$, then P is a compact operator so that $\|(T - T_n)x\| \to 0$ for every $x \in X$ implies

$\|(T - T_n)P\| \to 0$. (Chatelin [2, Theorem 3.2.]). Thus the proof of the theorem is complete.

THEOREM 3.2. *A sequence* (T_n) *of operators is a stable approximation of* T *on* G *if* $\|(T - T_n)T_n\| \to 0$ *,the sequence* $(\|T - T_n\|)$ *is bounded, and one of the following conditions is satisfied:*

(i) $\|(T - T_n)T\| \to 0$;

(ii) $\|T_n(T - T_n)\| \to 0$;

(iii) *there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, T_n *is compact;*

(iv) *there exists* $N \in \mathbb{N}$ *such that for all* $n \geq N$, $T - T_n$ *is compact.*

PROOF. Let $z \in G \subset \rho(T)$. since $\|(T - T_n T_n\| \to 0$, there exists $N_0(z) \in \mathbb{N}$ such that for all $n \geq N_0(z)$,

$$\|(T - T_n)T_n\| < |z|/\|R(z, T)\|.$$

Thus the first condition of Proposition 2.3 is satisfied by taking $A = (1/z)T$ and $A_0 = (1/z)T_n$ for $n \geq N_0(z)$.

The conditions (i), (ii), (iii) and (iv) of the theorem imply the existence of $N_1(z)$, $N_2(z)$, N_3 and N_4 in $\mathbb{N}$ such that

(i) for $n \geq N_1(z)$, $\max\{\|(T - T_n)T\|, \|(T - T_n)T_n\|\} < |z|^2/2$,

(ii) for $n \geq N_2(z)$, $\|T_n(T - T_n)\| < |z|/\|R(z, T)\|$,

(iii) for $n \geq N_3$, T_n is compact,

(iv) for $n \geq N_4$, $T - T_n$ is compact.

These results correspond to the respective conditions of Proposition 2.3 with $A = (1/z)T$ and $A_0 = (1/z)T_n$ for n sufficiently large.

Now as a consequence of the Proposition 2.3, it follows that $1 \in \rho(T_n/z)$, equivalent, $z \in \rho(T_n)$ for sufficiently large n, and

$$\|R(z, T_n)\| \leq \frac{\|R(z, T)\|(|z| + \|T - T_n\|)}{|z| - \|R(z, T)\|\|(T - T_n)T_n\|}$$

where $(\|T-T_n\|)$ is bounded and $\|(T-T_n)T_n\| \to 0$. Thus there exist $N(z) \in \mathbb{N}$ and $M(z) > 0$ such that for $n \geq N(z)$, $z \in \rho(T_n)$ and $\|R(z, T_n)\| \leq M(z)$. This completes the proof of the theorem.

Combining Theorems 3.1 and 3.2 we have

THEOREM 3.3. *A sequence* (T_n) *of operators is a strongly stable approximation of* T *at* λ *if it satisfies*

(C) $\|(T - T_n)T_n\| \to 0$, $(\|T - T_n\|)$ *is bounded*;

one of the conditions

(D1) $\|(T - T_n)T\| \to 0$,

(D2) $\operatorname{rank} P(T, \Gamma) < \infty$, $\|(T - T_n)x\| \to 0$ *for every* $x \in X$,

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and one of the conditions

(E1) $\|(T - T_n)T\| \to 0$,

(E2) $\|T_n(T - T_n)\| \to 0$,

(E3) *there exists* $N \in \mathbb{N}$ *such that for* $n \geq N$, T_n *is compact,*

(E4) *there exists* $N \in \mathbb{N}$ *such that for* $n \geq N$, $T - T_n$ *is compact.*

COROLLARY 3.4 (Ahues). *A sequence* (T_n) *of operators is a strongly stable approximation of* T *at* λ *if* T *is compact,* $\|(T - T_n)T_n\| \to 0$ *and* $\|(T - T_n)x\| \to 0$ *for every* $x \in X$.

COROLLARY 3.5 (Bouldin). *A sequence* (T_n) *is a strongly stable approximation of* T *at* λ *if* $\|(T - T_n)T_n\| \to 0$, $\|T_n(T - T_n)\| \to 0$, $\operatorname{rank} P(T, \Gamma) < \infty$ *and* $\|(T - T_n)x\| \to 0$ *for every* $x \in X$.

4. Examples and remarks

1. If (T_n) is a norm approximation of T, that is $\|T - T_n\| \to 0$ then (C), (D1), and (E2) of Theorem 3.3 are satisfied. Here T need not be a compact operator and an isolated spectral value of T need not be an eigenvalue of finite algebraic multiplicity. Thus, in general, the results of Ahues and Bouldin are not applicable.

For example, if $T = I$, the identity operator of an infinite dimensional Banach space, and $T_n = k_n I$, where k_n are scalars such that $k_n \to 1$, then $\|T - T_n\| \to 0$, but T is not compact and the isolated eigenvalue $\lambda = 1$ is of infinite multiplicity.

2. If (T_n) is a collectively compact approximation of a compact operator T, that is, $\|(T - T_n)x\| \to 0$ for every $x \in X$ and the set

$$\bigcup_{n=n_0}^{\infty} \{(T - T_n x : \|x\| < 1\}$$

is relatively compact for some $n_0 \in \mathbb{N}$, then the conditions (C), (1), (D2), (E3) and (E4) of Theorem 3.3 are satisfied (see Anselone [1]). But (E2) need not hold. (See Bouldin [3, Section 3] for an example). Thus the result of Bouldin is not applicable.

3. Let T be a non-compact operator on a separable Hilbert space X defined by its matrix representation (aij), with $a_{ij} = 0$ for $i \neq j$, that is,

$$T = \sum_{i,j=0} \langle \cdot, ej \rangle a_{ij} e_i, \quad a_{ij} = 0, \ i \neq j$$

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where $\{e_0, e_1, e_2, \dots\}$ is an orthonormal basis, and let T_n be defined as

$$T_n = \sum_{i,j=0}^{n} \langle \cdot, e_j \rangle a_{ij} e_i.$$

Then each T_n is compact and satisfies

$$\|(T - T_n)x\| \langle\to 0 \quad \text{for every } x \in X,$$
$$\|(T - T_n)T_n\| = 0 = \|T_n(T - T_n)\|, \quad n \in \mathbb{N}.$$

This observation of Bouldin [3, Section 3] provides examples which satisfy (C), (D2), (E2) and (E3) of Theorem 3.3, but the result of Ahues does not hold.

4. Let $X = l^1$ and T and T_n be defined as follows:

$$Tx = \sum_{i=1}^{\infty} \left(\sum_{j=1}^{\infty} a_{ij} x(j) \right) e_i,$$

$$T_n x = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} a_{ij} x(j) \right) e_i,$$

where $\{e_1, e_2, \dots\}$ is the standard Schauder basis of l^1, and the scalars a_{ij}, i, $j = 1, 2, \dots$, are given by

$$a_{ij} = \begin{cases} 1 & i = 1 \text{ and } j \geq 1 \\ \dfrac{i}{i+1} & i = j \text{ and } i > 1 \\ 0 & \text{otherwise.} \end{cases}$$

Then the spectrum of T is the set $\{1, 2/3, 3/4, \dots\}$ which has the limit point 1, so that T is not a compact operator. It can be verified that

(i) $\|(T - T_n)x\| \to 0$ for every $x \in X$,
(ii) $\|(T - T_n)T_n\| = 0$, $n \in \mathbb{N}$,
(iii) $\|T_n(T - T_n)\| = 1$, $n \in \mathbb{N}$,
(iv) T_n is compact, $n \in \mathbb{N}$,
(v) $2/3$, $3/4$, ... are simple eigenvalues of T.

Thus, taking $\lambda \in \{2/3, 3/4, \dots\}$, we see that the conditions (C), (D2) and (E3) of Theorem 3.3 are satisfied. But the results of Ahues and Bouldin can not be applied.

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