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TENSOR PRODUCTS OF OPERATORS—STRONG STABILITY AND p-HYPONORMALITY

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Abstract. We say that the operator T on a Hilbert space H into itself is *strongly stable* if $\|T^n x\| \to 0$ as $n \to \infty$, for all $x \in H$. If T is a contraction, then T is said to be *cs-stable* if T has C_0 completely non-unitary part. This note considers the strong stability of operators $A \otimes B$ and the p-hyponormality of operators $A \otimes B$. It is shown that the contraction $A \otimes B$ is cs-stable if and only if so are the contractions cA and $c^{-1}B$ for some scalar c and $A \otimes B$ is p-hyponormal if and only if A and B are. We also characterize p-hyponormal $A \otimes B$ for which the commutator $|A \otimes B|^{2p} - |A^* \otimes B^*|^{2p}$ is compact.

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1. Introduction. Let H be a Hilbert space, and let $B(H)$ denote the algebra of bounded linear operators on H. Given $A, B \in B(H)$, the tensor product $A \otimes B$, on the product space $H \otimes H$, has been considered variously by a number of authors; (see [**3**,**13**,**17**,**20**,**21**] for further references). The operation of taking tensor products $A \otimes B$ preserves many a property of $A, B \in B(H)$, but by no means all of them. Thus, whereas the normaloid property is invariant under tensor products, the spectraloid property is not (see [**20**, pp. 623 and 631]); again, whereas $A \otimes B$ is normal if and only if A and B are [**13**,**19**], there exist paranormal operators A and B such that $A \otimes B$ is not paranormal [**20**, p. 629]. $A \otimes B$ may have a property without (both) A and B having the property. Precisely this happens in the case of strong stability of operators. The operator T is said to be *strongly stable* if $\|T^n x\| \to 0$ as $n \to \infty$, for all $x \in H$ [**15**,**16**]. Strongly stable operators arise as models of discrete time invariant infinite dimensional free bounded linear systems of autonomous homogeneous difference equations $x_{n+1} = Tx_n$, $x_0 = x$. It is clear that $A \otimes B$ is strongly stable whenever A is power bounded (strongly stable) and B is strongly stable (resp., power bounded). If $A \otimes B$ is strongly stable (and so necessarily power bounded) and normaloid (i.e. $\lim_{n\to\infty} \|(A \otimes B)^n\|^{\frac{1}{n}} = \|A\|\|B\|$), then (at least) one of A and B, and $A \otimes B$ are contractions. A general strongly stable operator is cnu (= completely non-unitary) but need not be a contraction or even similar to a contraction [**16**]; a strongly stable contraction is a cnu contraction of the class C_0. See [**18**]. Notice that if $A \otimes B$ has a property P, then so does $\left(cA \otimes c^{-1}B\right)$ for all nonzero scalars c. It is not necessary for A and B to be contractions for $A \otimes B$ to be a contraction: given a contraction $A \otimes B$, the best one can say is that there exists a scalar $c \neq 0$ such that $A_1 = cA$ and $B_1 = c^{-1}B$ are contractions.

Our purpose in this note is a twofold one. We consider the strong stabilty of operators in Section 2, and prove that the operator $A \otimes B$ is strongly stable if and

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only if at least one of the (power bounded) operators A and B is. For the case in which $A \otimes B$ is a contraction, we introduce the concept of cs-stability (to distinguish it from strong stability). We say that the contraction T is *cs-stable* (= the cnu part is strongly stable) if T has C_0 cnu part. We prove that the contraction $A \otimes B$ id cs-stable if and only if the (associated) contractions A_1 and B_1 are cs-stable. In Section 3 we consider the tensor product of p-hyponormal operators. The operator T is said to be p-hyponormal, $0 < p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$. Let $\mathbf{H}(p)$ denote the class of p-hyponormal operators (so that $\mathbf{H}(1)$ denotes the class of 1-hyponormal, or simply hyponormal, operators). Although $\mathbf{H}(p)$, $0 < p < 1$, contains $\mathbf{H}(1)$ as a proper subclass, $\mathbf{H}(p)$ operators have spectral properties very similar to those of $\mathbf{H}(1)$ operators (see [**1**,**2**,**5**,**6**,**7**,**22**], and some of the references cited in these papers, for further information on $\mathbf{H}(p)$ operators). It is shown that $A \otimes B \in \mathbf{H}(p)$ if and only if $A, B \in \mathbf{H}(p)$. We characterize those $A \otimes B \in \mathbf{H}(p)$ for which the commutator $|A \otimes B|^{2p} - |A^* \otimes B^*|^{2p}$ is compact, and prove that if $A \otimes B \in \mathbf{H}(p)$, then either $A \otimes B$ has a non-trival invariant subspace or (at least) one of A and B is the sum of a normal and a compact operator.

In the following, we shall denote the closure of the range and the orthogonal complement of the kernel of an $X \in B(H)$ by $\overline{\operatorname{ran} X}$ and $\ker^{\perp} X$, respectively. The commutator $AB - BA$ of $A, B \in B(H)$ will be denoted by $[A, B]$. We say that a contraction A is cnu (= completely non-unitary) if there exists no non-trivial reducing subspace M of A such that the restriction of A to M, denoted by $A|M$, is unitary. The cnu contraction A is said to be of the class C_0 of contractions if the power sequence $\{A^n\}$ converges strongly to zero; i.e., $\|A^n x\| \to 0$ as $n \to \infty$, for all $x \in H$ [**18**]. In the following the tensor product $H \otimes H$ will denote the completion of the algebraic tensor product of H with H relative to the unique inner product $(x \otimes y_1, x_2 \otimes y_2) = (x_1, x_2)(y_1, y_2)$. The following elementary results on tensor products of operators will be used often (and without further reference) in the sequel: $A_1 \otimes B_1 = A_2 \otimes B_2$ if and only if there exists a scalar $c \neq 0$ such that $A_1 = cA_2$ and $B_1 = c^{-1}B_2$. If A_i and $B_i (i = 1, 2)$ are positive operators, then $A_1 \otimes B_1 = A_2 \otimes B_2$ if and only if there exists a scalar $c > 0$ such that $A_1 = cA_2$ and $B_1 = c^{-1}B_2$. The proofs to these results are to be found in the papers by Hou [**13**] and Stochel [**21**]. (We do not need the full force of the results of Hou or Stochel here.)

It is my great pleasure to thank Professor Carlos Kubrusly for some very enlightening correspondence regarding the strong stability of operators. My thanks are also due to the referee for his suggestions, which have helped improve the presentation of the paper.

2. Stability. The operator T is *strongly stable* if $\|T^n x\| \to \infty$ as $n \to \infty$, for all $x \in H$. A strongly stable operator is power bounded (i.e. there exists a scalar M such that $\sup\|T^n\| \leq M$) and the spectral radius $r(T)$ of T is equal to one. In the case in which the Hilbert space H is separable, an equivalent definition of strong stability is provided by the following result.

Propostion 1. *The power bounded operator T is strongly stable if and only if the only (positive) solution $X \geq 0$ of $T^*XT = X$ is $X = 0$.*

Proof. If T is stongly stable, then

$$(X, x, x) = \lim_{n\to\infty} ((T^*)^n X T^n x, x) \leq \|X\| \lim_{n\to\infty} \|T^n x\|^2 = 0.$$

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Hence $X = 0$. Suppose now that the only solution $X \geq 0$ of $T^*XT = X$ is $X = 0$ but that there exists a non-trivial $x \in H$ such that $\|T^n x\| \not\to 0$ as $n \to \infty$. Then there exists an operator S and a constant $C > 0$ such that $\|T^n x\| \geq C, (Sx, x) > 0$, $\ker S = \{y \in H : \|T^n y\| \to 0 \text{ as } n \to \infty\}$ and $T^*ST = S$; (see [**4**, Lemma 4]). This is a contradiction.

Related to, but distinct from, the strong stability of an operator is the concept of the uniform stability of an operator. $T \in B(H)$ is said to be *uniformly stable* if $\|T^n\| \to 0$ as $n \to \infty$. Uniform stability implies strong stability. It is seen that if $T \in B(H)$ is uniformly stable, then $r(T) < 1$ and T is similar to a strict contraction. Furthermore, $T \in B(H)$ is uniformly stable if and only if there exists an $X \gg 0$ and a scalar α, with $0 < \alpha < 1$, such that $T^*XT \leq \alpha X$; (see [**15**] for more details). Taking our cue from this we make the following definition.

The operator $T = A \otimes B$ on $H \otimes H$ is *uniformly stable* if there exists an operator $Q = Q_1 \otimes Q_2 \gg 0$ and a scalar α, $0 < \alpha < 1$, such that $T^*QT \leq \alpha Q$.

Henceforth A and B will denote non-trivial operators. We prove the following result.

Theorem 1. (a) *Let A and B be power bounded operators on a separable Hilbert space H. Then $A \otimes B$ is stongly stable if and only if at least one of A and B is strongly stable.*

(b) *$A \otimes B$ is uniformly stable if and only if A_1 and B_1 are, where $A_1 = cA$ and $B_1 = c^{-1}B$, for some scalar $c > 0$.*

Proof. (a) To prove our assertion we need only show that if $X_i \geq 0, (i = 1, 2)$, and $(A \otimes B)^*(X_1 \otimes X_2)(A \otimes B) = X_1 \otimes X_2$, then $A^*X_1A = X_1$ and $B^*X_2B = X_2$. The operators A^*X_1A and B^*X_2B being positive, it follows that if $(A \otimes B)^*(X_1 \otimes X_2)$ $(A \otimes B) = X_1 \otimes X_2$, then there exists a scalar $c > 0$ such that $A^*X_1A = cX_1$ and $B^*X_2B = c^{-1}X_2$. Let $\sup_n \|A^n\| \leq M_1$ and $\sup_n \|B^n\| \leq M_2$. Then

$$|c^n|\|X_1\| = \|A^{*n}X_1A^n\| \leq M_1^2\|X_1\|$$

and

$$|c^{-n}|\|X_2\| = \|B^{*n}X_2B^n\| \leq M_2^2\|X_2\|.$$

This implies that $c = 1$, and hence $A^*X_1A = X_1$ and $B^*X_2B = B$.

(b) If A_1 and B_1 are uniformly stable, then

$$r(A \otimes B) = \lim_{n\to\infty} \|(A \otimes B)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \|(A_1 \otimes B_1)^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \{\|A_1^n\|\|B_1^n\|\}^{\frac{1}{n}} < 1,$$

and hence $A \otimes B$ is uniformly stable. Conversely, if $A \otimes B$ is uniformly stable, then

$$A^*Q_1A \otimes B^*Q_2B \leq \alpha(Q_1 \otimes Q_2)$$

for some $0 < \alpha < 1$ and $Q_1 \otimes Q_2 \gg 0$. Since $Q_1 \otimes Q_2$ is invertible if and only if Q_i is $(i = 1, 2)$, there exists a non-zero scalar d such that $X_1 = dQ_1$ and $X_2 = d^{-1}Q_2$ are positive. The operators A^*X_1A and B^*X_2B being positive, there exists a scalar $c > 0$ such that

$$A^*X_1A \leq c^2\sqrt{\alpha}X_1 \quad \text{and} \quad B^*X_2B \leq c^{-2}\sqrt{\alpha}X_2.$$

This implies that A_1 and B_1 are uniformly stable.

Contractions $A \otimes B$. If $A \otimes B$ is a contraction, then $0 \le A^*A \otimes B^*B \le I \otimes I$ and hence there exists a scalar $d > 0$ such that $0 \le A^*A \le dI$ and $0 \le B^*B \le d^{-1}I$. Define operators A_1 and B_1 by $A_1 = c^{-1}A$ and $B_1 = cB$, where $|c|^2 = d$; then $A \otimes B$ is a contraction if and only if $A_1 \otimes B_1$ is a contraction if and only if A_1 and B_1 are contractions.

In deference to (and to distinguish it from) the more established concept of the strong stability of operators, we say that a contraction $T \in B(H)$ is *cs-stable* if T has C_0 cnu part. A characterization of contractions (in $B(H)$) with C_0 cnu part is given by the following result (cf. [**7**, Lemma 1]).

"The contraction T has C_0 cnu part if and only if for every isometry V and operator X such that $T^*X = XV^*$ we also have $TX = XV$".

Taking our cue from this characterisation we say in the following that the contraction $T = A \otimes B$ is cs-stable if for every operator $X = X_1 \otimes X_2$ and isometry V on $H \otimes H$ such that $T^*X = XV^*$ we also have $TX = XV$.

Clearly, $A \otimes B$ is cs-stable if and only if $(A_1 \otimes B_1)$ is cs-stable. Let A_1 and B_1 be cs-stable contractions. Then $T = A_1 \otimes B_1$ is a cs-stable contraction, as the following argument shows. Let $X = X_1 \otimes X_2$, and let V be an isometry on $H \otimes H$ such that $T^*X = XV^*$. Let $|X_i^*|^2 = P_i (i = 1, 2)$. Then

$$A_1^*P_1A_1 \otimes B_1^*P_2B_1 = P_1 \otimes P_2,$$

and so there exists a scalar $d > 0$ such that

$$A_1^*P_1A_1 = dP_i \text{ and } B_1^*P_2B_1 = d^{-1}P_2.$$

A_1 and B_1 being contractions, this implies that

$$d\|P_1\| = \|A_1^*P_1A_1\| \le \|P_1\| \text{ and } d^{-1}\|P_2\| = \|B_1^*P_2B_1\| \le \|P_2\|.$$

Hence $d = 1$ and, since A_1, B_1 are cs-stable, $A_1P_1A_1^* = P_1$ and $B_1P_2B_1^* = P_2$ (see [**4**]). Thus $\overline{\operatorname{ran} P_1} = \overline{\operatorname{ran} X_1}$ reduces A_1, $\overline{\operatorname{ran} P_2} = \overline{\operatorname{ran} X_2}$ reduces B_1, and $A_1|\overline{\operatorname{ran} X_1}$ and $B_1|\overline{\operatorname{ran} X_2}$ are unitary. We have

$$TX = TXV^*V = TT^*XV = XV.$$

The converse also holds, as the following result shows.

THEOREM 2. *The contraction $A \otimes B$ is cs-stable if and only if the contractions A_1 and B_1 are cs-stable.*

Proof. We have to show that $A \otimes B$ is cs-stable $\Rightarrow A_1$ and B_1 are cs-stable. A_1 and B_1 being contractions,

$$\lim_{n\to\infty} A_1^{*^n}A_1^n = X_1^2 \text{ and } \lim_{n\to\infty} B_1^{*^n}B_1^n = X_2^2$$

are well defined positive operators. Hence

$$(X_1 \otimes X_2)^2 = \lim_{n\to\infty}\left(A_1^{*n}A_1^n \otimes B_1^{*n}B_1^n\right) = \lim_{n\to\infty}\left[\left(A_1^* \otimes B_1^*\right)^n (A_1 \otimes B_1)^n\right]$$

is a well defined positive operator. Since

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$$\lim_{n\to\infty} A_1^*(A_1^{*n}A_1^n)A_1 = A_1^*\Big(\lim_{n\to\infty} A_1^{*n}A_1^n\Big)A_1,$$

and

$$\lim_{n\to\infty} B_1^*(B_1^{*^n}B_1^n)B_1 = B_1^*\Big(\lim_{n\to\infty} B_1^{*^n}B_1^n\Big)B_1,$$

$$A_1^*X_1^2A_1 = X_1^2 \text{ and } B_1^*X_2^2B_1 = X_2^2. \tag{1}$$

Also, since $(X_1A_1)(A_1^*X_1) \le X_1^2 = (A_1^*X_1)(X_1A_1), (X_2B_1)(B_1^*X_2) \le X_2^2 = (B_1^*X_2)(X_2B_1)$, X_1A_1 and $X_2B_1 \in \mathbf{H}(1)$. There exist isometries V_1 and V_2 such that X_1A_1 and X_2B_1 have the polar decompositions $X_1A_1 = V_1X_1$ and $X_2B_1 = V_2X_1$. We have

$$(A_1 \otimes B_1)^*(X_1 \otimes X_2) = (X_1 \otimes X_2)(V_1 \otimes V_2)^*, \tag{2}$$

and so (since $A \otimes B$ is cs-stable) also

$$(A_1 \otimes B_1)(X_1 \otimes X_2) = (X_1 \otimes X_2)(V_1 \otimes V_2). \tag{3}$$

We should like now to prove that $[A_1, X_1] = 0 = [B_1, X_2]$. We start by showing that $[|A_1|, X_1] = 0 = [|B_1|, X_2]$.

Equations (2) and (3) together imply that

$$\big(|A_1|^2 \otimes |B_1|^2\big)\big(X_1^2 \otimes X_2^2\big)\big(|A_1|^2 \otimes |B_1|^2\big) = X_1^2 \otimes X_2^2;$$

hence there exists a scalar $d > 0$ such that $|A_1|^2X_1^2|A_1|^2 = dX_1^2$ and $|B_1|^2X_2|B_1|^2 = d^{-1}X_2^2$. Since A_1 and B_1 are contractions, $d = 1$, and hence we have

$$|A_1|^2X_1^2|A_1|^2 = X_1^2 \text{ and } |B_1|^2X_1^2|B_1|^2 = X_2^2. \tag{4}$$

Let A_1 and B_1 have the polar decompositions $A_1 = U|A_1|$ and $B_1 = W|B_1|$. Let $x \in H$, and let $\{x_n\}$ be the sequence defined by $x_n = X_1^2|A_1|^{2n}x$. Then

$$\begin{aligned}\|x_n\| &= \big\|X_1^2|A_1|^{2n}x\big\| = \big\||A_1|^2X_1^2|A_1|^{2n+2}x\big\| \quad \text{(by (4))}\\ &= \big\||A_1|^2x_{n+1}\big\| \le \big\|x_{n+1}\big\|,\end{aligned}$$

and the sequence $\{\|x_{n+1}\|\}$ is a monotonic increasing sequence bounded above. Also,

$$\begin{aligned}\|x_{n+1}\|^2 &= \big(x_{n+2}, |A_1|^2x_{n+1}\big) \text{ (by (4))}\\ &= \big(x_{n+2,}x_n\big) \le \left(\frac{\|x_n\| + \|x_{n+2}\|}{2}\right)^2;\end{aligned}$$

i.e., the sequence $\{\|x_n\|\}$ is convex. Hence $\{\|x_n\|\}$ is a constant sequence. In particular, $\|X_1^2x\| = \big\||X_1^2||A_1|^2x\big\|$, for all $x \in H$ and

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$$\left\|\left(|A_1|^2X_1^2 - X_1^2|A_1|^2\right)x\right\|^2 = \left\||A_1|^2X_1^2x\right\|^2 + \left\|X_1^2|A_1|^2x\right\| - 2\mathrm{Re}\left(X_1^2x, |A_1|^2X_1^2|A_1|^2x\right) \leq 0,$$

i.e. $\left[|A_1|^2, X_1^2\right] = 0$, or equivalently $[|A_1|, X_1] = 0$. Taken together with (4) this implies that $|A_1|^{4n}X_1^2 = X_1^2$ for all $n = 0, 1, 2, \cdots$. Hence

$$|A_1|X_1^2 = X_1^2 = X_1^2|A_1|. \tag{5}$$

A similar argument shows that $[|B_1|, X_2] = 0$ and

$$|B_1|X_2^2 = X_2^2 = X_2^2|B_1|. \tag{6}$$

We show next that $[U, X_1] = 0 = [W, X_2]$.

Equations (2) and (3) taken together imply also that $\overline{\mathrm{ran}(X_1 \otimes X_2)}$ reduces $A_1 \otimes B_1$ and $(A_1 \otimes B_1)\big|\overline{\mathrm{ran}(X_1 \otimes X_2)}$ is unitary. Consequently,

$$X_1 \otimes X_2 = (A_1 \otimes B_1)(X_1 \otimes X_2)\left(V_1^* \otimes V_2^*\right)$$

and hence

$$X_1^2 \otimes X_2^2 = \left(A_1X_1^2A_1^*\right) \otimes \left(B_1X_2^2B_1^*\right).$$

Arguing as above (see (4)) it now follows that

$$A_1X_1^2A_1^* = X_1^2 \text{ and } B_1X_2^2B_1^* = X_2^2. \tag{7}$$

This implies that

$$U^*X_1^2 = U^*A_1X_1^2A_1^* = |A_1|X_1^2|A_1|U^* = X_1^2U^* \text{ (see (5))}$$

and

$$W^*X_2^2 = W^*B_1X_2^2B_1^* = |B_1|X_2^2|B_1|W^* = X_2^2W^* \text{ (see (6))}.$$

Thus $[U, X_1] = 0 = [W, X_2]$, and hence $[A_1, X_1] = 0 = [B_1, X_2]$.

The commutativity of A_1 and X_1 when taken along with (1) implies that

$$X_1^2 = \lim_{n\to\infty} A_1^{*n}X_1^2A_1^n = X_1^2 \lim_{n\to\infty} A_1^{*n}A_1^n = X_1^4.$$

Hence X_1 is a projection, and we have from (1) and (7) that $A_1|\overline{\mathrm{ran}X_1}(= A_1|\mathrm{ran}X_1)$ is unitary and $A_1\big|\mathrm{ran}(I - X_1) \in C_0$. A similar argument shows that X_2 is a projection, $B_1|\mathrm{ran}X_2$ is unitary and $B_1\big|\mathrm{ran}(I - X_2) \in C_0$. This completes the proof.

Remark 1. Our definition of cs-stability of the contraction $A \otimes B$ is a particular case of "generalised Putnam-Fuglede (commutativity) theorems" (see [**6**,**7**,**8**,**9**,**10**,**12**,**19**]). Let $\delta_{AB} : B(H) \to B(H)$ denote the generalised derivation $\delta_{AB}(X) = AX - XB$, let $d_{AB} : B(H) \to B(H)$ denote the elementary operator $d_{AB}(X) = AXB - X$, and let $D_{AB} = \delta_{AB}$ or d_{AB}. Let P_1 and P_2 be two classes of operators. The pair (P_1, P_2) is said to have the *(generalised) Putnam-Fuglede (commutativity) property*, denoted

$(P_1, P_2) \in PF(D)$, if $\ker D_{AB} \subseteq \ker D_{A^*B^*}$, for every operator $A \in P_1$ and operator $B^* \in P_2$. The Putnam-Fuglede property holds for a number of pairs of classes P_1 and P_2, chief amongst them classes P_1 and P_2 consisting of normal or subnormal or hyponormal operators (see [**9**,**10**,**19**]). (See also [**8**] for more information on classes P_1 and P_2 for which $(P_1, P_2) \in PF(\delta) \Longleftrightarrow (P_1, P_2) \in PF(d)$, and further references.)

REMARK 2. The tensor product $A \otimes B$ can be identified with multiplication on the Hilbert space $C_2(H)$ of Hilbert-Schmidt class operators on H. More precisely, $A \otimes B$ can be identified with the mapping $\tau_{AB^*}|C_2(H)$, where $\tau_{AB^*}(X) = AXB^*$ [**3**]. Theorem 2 implies that $\tau_{AB^*}|C_2(H)$ is a contraction with C_0 cnu part if and only if A_1 and B_1 are contractions with C_0 cnu part.

4. H(p) operators. We say that the operator T is *p-hyponormal*, $0 < p \leq 1$, if $|T^*|^{2p} \leq |T|^{2p}$. Let $\mathbf{H}(p)$ denote the class of p-hyponormal operators (so that $\mathbf{H}(1)$ denotes the class of hyponormal operators). The class $\mathbf{H}(p)$ is monotonic decreasing on p; i.e., if $T \in \mathbf{H}(p)$, then $T \in \mathrm{H}(q)$ for all $0 < q \leq p$, and we may assume without loss of generality that $0 < p < \frac{1}{2}$. (Indeed one may assume, without loss of generality that $p = 2^{-n}$, for some integer $n > 1$.) $\mathbf{H}(\frac{1}{2})$ operators were introduced by Xia (see [**22**, p238] for the appropriate reference), and $\mathbf{H}(p)$ operators for a general $0 < p < \frac{1}{2}$ have since been considered by a number of authors (see [**1**,**2**,**5**,**6**,**22**] for further references). Although the class of $\mathbf{H}(p)$ operators $0 < p < \frac{1}{2}$, is strictly larger than the class of hyponormal operators, $\mathbf{H}(p)$ operators share a large number of properties with hyponormal operators. Throughout the following we assume that A, B are non-trivial $\mathbf{H}(p)$ operators ($0 < p < \frac{1}{2}$) which are linearly independent (i.e., there exists no scalar γ such that $A = \gamma B$). We start by considering the p-hyponormality of the tensor product $A \otimes B$.

THEOREM 3. *$A \otimes B \in \mathbf{H}(p) \Longleftrightarrow A$ and $B \in \mathbf{H}(p)$.*

Proof. Suppose that $A \otimes B \in \mathbf{H}(p)$. Let $|A|$ and $|B|$ have the spectral decompositions

$$|A| = \int \lambda dE(\lambda) \text{ and } |B| = \int \mu dF(\mu),$$

and let $f : (0, \infty) \to (0, \infty)$ be such that $f(xy) = f(x)f(y)$. Then

$$f(|A| \otimes |B|) = \int\int f(\lambda\mu) dE(\lambda) \otimes dF(\mu) = \left(\int f(\lambda) dE(\lambda)\right) \otimes \left(\int f(\mu) dF(\mu)\right)$$
$$= f(|A|) \otimes f(|B|).$$

Choosing $f(x) = x^p$ we have

$$\begin{aligned} |A^*|^{2p} \otimes |B^*|^{2p} &= f\left(|A^*|^2 \otimes |B^*|^2\right) = f\left(|A^* \otimes B^*|^2\right) \\ &= |A^* \otimes B^*|^{2p} \\ &\leq |A \otimes B|^{2p} = f\left(|A \otimes B|^2\right) = f\left(|A|^2 \otimes |B|^2\right) = |A|^{2p} \otimes |B|^{2p}. \end{aligned}$$

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Hence there exists a scalar $c > 0$ such that

$$|A^*|^{2p} \leq c|A|^{2p} \text{ and } |B^*|^{2p} \leq c^{-1}|B|^{2p}.$$

Since

$$\||A|^p\|^2 = \left\||A^*|^p\right\|^2 = \sup_{\|x\|=1}\left(|A^*|^{2p}x, x\right) \leq \sup_{\|x\|=1}\left(c|A|^{2p}x, x\right) = c\||A|^p\|^2$$

and

$$\||B|^p\|^2 = \left\||B^*|^p\right\|^2 = \sup_{\|x\|=1}\left(|B^*|^{2p}x, x\right) \leq \sup_{\|x\|=1}\left(c^{-1}|B|^{2p}x, x\right) = c^{-1}\||B|^p\|^2,$$

we must have $c = 1$, and then $A, B \in \mathbf{H}(p)$.

Conversely, if $A, B \in \mathbf{H}(p)$, then

$$\begin{aligned}
&\left(|A|^{2p} - |A^*|^{2p}\right) \otimes \left(|B|^{2p} - |B^*|^{2p}\right) \geq 0 \\
&\quad \Rightarrow \left(|A|^{2p} \otimes |B|^{2p}\right) - \left(|A^*|^{2p} \otimes |B^*|^{2p}\right) \\
&\quad \geq |A|^{2p} \otimes |B^*|^{2p} + |A^*|^{2p} \otimes |B|^{2p} - 2|A^*|^{2p} \otimes |B^*|^{2p} \\
&\quad = \left(|A|^{2p} - |A^*|^{2p}\right) \otimes |B^*|^{2p} + |A^*|^{2p} \otimes \left(|B|^{2p} - |B^*|^{2p}\right) \geq 0.
\end{aligned}$$

Hence $A \otimes B \in \mathbf{H}(p)$.

Corollary 1. *$\tau_{AB^*}\big|C_2(H) \in \mathbf{H}(p)$ if and only if $A, B \in \mathbf{H}(p)$.*

Proof. As noted earlier, $\tau_{AB^*}\big|C_2(H)$ can be identified with $A \otimes B$.

Remark 3. The same sort of characterisation (as in Corollary 1) cannot be valid for more general elementary operators. Thus, for example, the elementary operator $X \rightarrow AXB^* + A^*XB$ restricted to $C_2(H)$ is self-adjoint for all $A, B \in B(H)$.

Remark 4. Suppose that A, B are doubly commuting (i.e., $AB = BA$ and $AB^* = B^*A$) hyponormal operators. Then

$$\begin{aligned}
(A + B)^*(A + B) &= A^*A + A^*B + B^*A + B^*B \\
&\geq AA^* + AB^* + BA^* + BB^* = (A + B)(A + B)^*,
\end{aligned}$$

so that $A + B$ is hyponormal. This implies that $A \otimes I + I \otimes B \in \mathbf{H}(1)$ for all operators $A, B \in \mathbf{H}(1)$. Given $A, B \in \mathbf{H}(p)$, does $A \otimes I + I \otimes B \in \mathbf{H}(p)$?

Let $A \otimes B \in \mathbf{H}(p)$, and let D denote the commutator

$$(0 \leq)D = |A \otimes B|^{2p} - \left|(A \otimes B)^*\right|^{2p} = |A|^{2p} \otimes |B|^{2p} - |A^*|^{2p} \otimes |B^*|^{2p}.$$

The proof of our next result, which considers the compactness of the commutator D, uses the following simpler version of [**13**, Theorem 3.1].

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LEMMA 1. *If A_1, A_2 are linearly independent operators and $A_1 \otimes B_1 + A_2 \otimes B_2$ is compact, for some operators B_1 and B_2, then B_1 and B_2 are compact.*

Recall that an operator T is said to be *essentially normal* if the commutator $T^*T - TT^*$ is compact.

THEOREM 4. *D is compact if and only if either*
(i) *A and B are normal compact or*
(ii) *$A(B)$ is normal compact and B (respectively, A) is essentially normal.*

Proof. We have three possibilities:
either (a) $|A|^{2p}$ and $|A^*|^{2p}$, also $|B^*|^{2p}$ and $|B^*|^{2p}$, are linearly independent,
or (b) only $|A|^{2p}$ and $|A^*|^{2p}$ (only $|B|^{2p}$ and $|B^*|^{2p}$) are linearly independent,
or (c) neither $|A|^{2p}$ and $|A^*|^{2p}$ nor $|B|^{2p}$ and $|B^*|^{2p}$ are linearly independent.
We start by showing that possibility (a) cannot occur.

If $|A|^{2p}$ and $|A^*|^{2p}$ are linearly independent and D is compact, then Lemma 1 implies that $|B|^{2p}$ is compact. Hence $|B|$, and so also B, is compact. Since a compact p-hyponormal operator is normal **[5]** and $B \in \mathbf{H}(p)$ by Theorem 3, B is normal compact. But then $|B|^{2p} = |B^*|^{2p}$. Hence (a) cannot occur, and the only viable possibilities are either (b) or (c). Notice that if B is normal, then $D = \left(|A|^{2p} - |A^*|^{2p}\right) \otimes |B|^{2p}$ is compact if and only if $|A|^{2p} - |A^*|^{2p}$ and $|B|^{2p}$ are compact. Since this is possible if and only if B is normal compact and either A is normal compact or A is essentially normal, it follows that possibility (b) occurs if and only if either (i) or (ii) holds. Suppose now that possibility (c) occurs, and that there exists a scalar r (necessarily, $r \leq 1$) such that $|A|^{2p} = r|A^*|^{2p}$. Then

$$D = |A|^{2p} \otimes \left(|B|^{2p} - r|B^*|^{2p}\right)$$

is compact if and only if A is normal compact and $|B|^{2p} - r|B^*|^{2p}$ is essentially normal. Since the normality of A implies that $r = 1$, we conclude (as in the case in which (b) occurs) that (c) occurs if and only if either (i) or (ii) holds. This completes the proof.

Recall from **[2]** that if the operator T is such that the negative part of $|T|^{2p} - |T^*|^{2p}$ is trace class, where $0 < p < \frac{1}{2}$, then

$$\operatorname{trace}\left(|T|^{2p} - |T^*|^{2p}\right) \leq \frac{1}{\pi} m(T+X) \int_{\sigma(T+X)} r^{2p-1} dr d\theta,$$

for any operator X satisfying $\operatorname{trace}(|X|^p) < \infty$. (Here $m(T+X)$ denotes the multiplicity of $T+X$). Thus, if A is a finitely multicyclic $\mathcal{H}(p)$ operator, then $\left(|A|^{2p} - |A^*|^{2p}\right)$ is trace class. Since trace class operators form an ideal, we have that

$$|A|^{2^2 p} - |A^*|^{2^2 p} = |A|^{2p}\left(|A|^{2p} - |A^*|^{2p}\right) + \left(|A|^{2p} - |A^*|^{2p}\right)|A^*|^{2p}$$

is trace class. Letting $p = 2^{-n}$, and finitely repeating this argument, it follows that $|A|^{2^{n+1}p} - |A^*|^{2^{n+1}p} = |A|^2 - |A^*|^2$ is trace class. Thus a finitely multicyclic $\mathcal{H}(p)$ operator has trace class commutator, and hence is essentially normal. Combining this with the theorem above, it follows that if $A \otimes B \in \mathbf{H}(p)$ is finitely multicyclic, then (ker $A = $ ker $B = \{0\}$ and) either (i) A, B are finitely multicyclic normal operators or (ii) A (resp., B) $\in \mathbf{H}(p)$ is finitely multicyclic and B (resp. A) is normal compact.

COROLLARY 2. *Given $A \otimes B \in \mathbf{H}(p)$, either $A \otimes B$ has a non-trivial invariant subspace or (at least) one of A and B is the sum of a normal and a compact operator.*

Proof. If $A \otimes B \in \mathbf{H}(p)$ does not have a non-trivial invariant subspace, then $A \otimes B$ has a rationally cyclic vector, so that the commutator D is trace class, and hence compact (see above). Applying Theorem 4 we conclude that A (resp., B) is normal and B (resp., A) is essentially normal. (Clearly (i) of the statement of Theorem 4 cannot happen for the reason that if A and B are normal, then A and B have non-trivial invariant subspaces, say H_1 and H_2; the completion of $H_1 \otimes H_2$ is then a non-trivial invariant subspace for $A \otimes B$.) For definiteness, let us assume that A is normal and B is essentially normal. Then $B \in \mathcal{H}(p)$ does not have a non-trivial invariant subspace, both B and B^* have empty point spectrum (so that both B and B^* have the single valued extension property), and B is biquasitriangular [**14**, Theorem 2.3.21]. Hence $B = N + K$ for some normal N and compact K [**11**, Corollary 4.2]. This completes the proof.

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