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A NOTE ON THE RELATIONSHIPS BETWEEN CONVEXITY AND INVEXITY

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Abstract

Using the fact that a differentiable quasi-convex function is also pseudo-convex at every point x of its domain where $\nabla f(x) \neq 0$, recent results relating different forms of convexity and invexity are strengthened.

In [1] Ben-Israel and Mond provide the following simple and nice characterisation of invex functions:

THEOREM 1. *Let $f: R^n \to R$ be differentiable. Then f is invex if and only if every stationary point is a global minimum.*

For other proofs of this statement, see [2, 4]. On page 4 of the same article the authors present a diagram showing the various relationships between convex, pseudo-convex, invex, quasi-convex and quasi-invex functions. This diagram can be improved by making use of the following result, due to Crouzeix and Ferland [3].

THEOREM 2. *Let f be a differentiable and quasi-convex function on an open convex set $X \subseteq R^n$. Then f is pseudo-convex on X if and only if f has a minimum at $x \in X$ whenever $\nabla f(x) = 0$.*

We provide a new and perhaps simpler proof of this result. The necessary part of the theorem follows from the definition of pseudo-convex functions

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(see [5]). As for sufficiency, let $x^0 \in X$, $\nabla f(x^0) = 0 \Rightarrow x^0$ is a (global) minimum point of $f(x)$ on X, i.e. $(x - x^0)' \ \nabla f(x^0) = 0 \Rightarrow f(x) \geqq f(x^0)$, $\forall x \in X$.

It is obvious that $f(x)$ is then locally pseudo-convex at x^0, with respect to X (see [5]). Let us now prove that: $f(x)$ quasi-convex on X; $x^0 \in X$; $\nabla f(x^0) \neq 0\lambda$ implies $f(x)$ pseudo-convex at x^0, i.e. $(x-x^0)' \nabla f(x^0) \geq 0 \Rightarrow f(x) \geq f(x^0)$, $\forall x \in X$.

Let us consider a point $x^1 \in X$ such that

$$(x^1 - x^0)' \nabla f(x^0) \geq 0 \tag{1}$$

but for which it is

$$f(x^1) < f(x^0). \tag{2}$$

Thus x^1 belongs to the non void set

$$X_0 = \{x | x \in X, \ f(x) \leq f(x^0)\},$$

whose elements, thanks to the quasi-convexity of $f(x)$, verify the relation

$$x \in X_0 \Rightarrow (x - x^0)' \nabla f(x^0) \leq 0. \tag{3}$$

Let us now consider the sets, both non void,

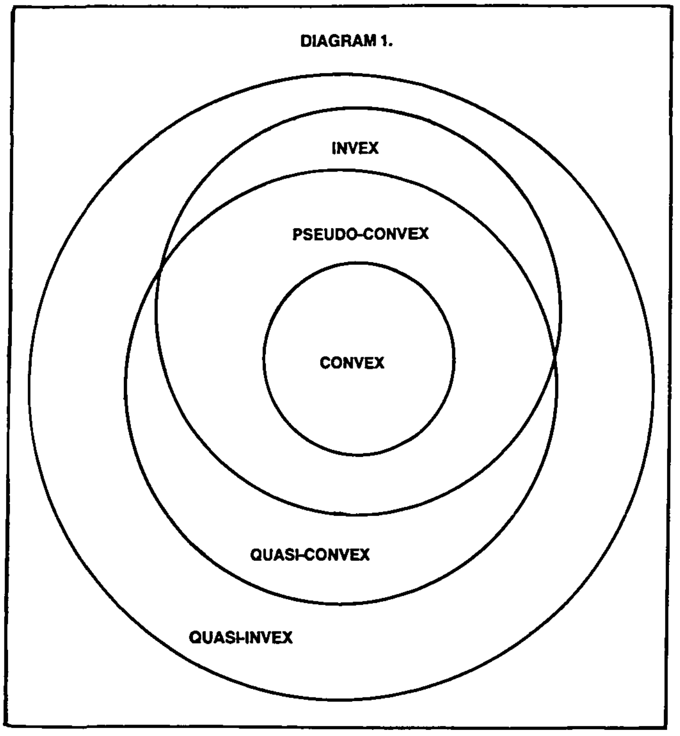
$$W = \{x | x \in X, (x - x^0)' \nabla f(x^0) \geq 0\}, \qquad X_{00} = X_0 \cap W.$$

The following implication obviously holds:

$$x \in X_{00} \Rightarrow x \in H_0 = \{x | x \in X, (x - x^0)' \nabla f(x^0) = 0\}.$$

It is therefore evident that X_{00} is included in the hyperplane (since $\nabla f(x^0) \neq 0$) $H = \{x | x \in R^n, (x - x^0)' \nabla f(x^0) = 0\}$, a hyperplane supporting X_0, owing to (3). Relations (1) and (2) point out that x^1 belongs to W and X_0 and hence to X_{00}, H_0, H. Moreover (2) says that x^1 lies in the interior of X_0: therefore x^1 at the same time belongs to the interior of a set and to a hyperplane supporting the same set, which is absurd. So relation (2) is false and (1) implies $f(x^1) \geq f(x^0)$.

The quasi-convex function $f(x)$ is thus pseudo-convex at every point x of X where $\nabla f(x) \neq 0$. Consequently we note that sufficient conditions to test the quasi-convexity of a function, in a convex set where $\nabla f(x) \neq 0$, $\forall x \in X$, really locate the class of pseudo-convex functions. This is, for example, the case of the determinantal conditions for twice continuously differentiable functions established by Arrow and Enthoven and generalised by other authors. (see [3]).



Taking Theorem 2 into account, the diagram on page 4 in [1] must be modified as above.

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