THE FIXING OF THE GAUSSIAN GRAVITATIONAL CONSTANT AND THE CORRESPONDING GEOCENTRIC GRAVITATIONAL CONSTANT [1]

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Résumé. — L'auteur analyse les avantages de l'utilisation de la constante de la gravitation héliocentrique. Il propose qu'une constante de la gravitation géocentrique soit définie d'une manière analogue. Cette constante présenterait les mêmes avantages.

Abstract. — The author analyses the advantages of the use of the gaussian heliocentric gravitational constant. He proposes that a geocentric gravitational constant should be defined in an analogous way, that would present similar advantages.

Zusammenfassung. — Verf. untersucht die Vorteile der Verwendung der Gaussschen heliozentrischen Gravitationskonstanten. Er schlägt vor, eine geozentrische Gravitationskonstante zu definieren, was ähnliche Vorteile bieten würde.

Резюме. — Автор анализирует выгоды пользования гелиоцентрической гравитационной постоянной. Он предлагает установить аналогичное определение геоцентрической гравитационной постоянной, предоставляющей те-же преимущества.

A. **Introduction.** — The possibility of the introduction of a " geo-unit " (g. u.), or " geocentric astronomical unit ", to serve geocentric orbits in the same way as the " astronomical unit " (a. u.) serves heliocentric orbits has aroused both strong support and strong opposition.

The support has come from persons familiar with the device by which the proper definition of the astronomical unit has made it possible to fix the value of the Gaussian gravitational constant, k or k_s, and so to reduce greatly the revisions required by the adoption of improved basic

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physical data. They see in the proposal the possibility of fixing likewise a value of the corresponding geocentric constant, k_e. (Note : k_e^2 is sometimes written k_E^2, GE, GM, or $GM_\oplus$, where G is the gravitational constant, and $E = M = M_\oplus$ is the mass of the Earth.)

The opposition has come in part from those who have been strongly grounded in the C. G. S. system. A. H. Cook, in his letter to me of 1962 June 6, expresses his viewpoint as follows : " the determination of the best consistent values of the constants is a question of observation and we must be careful not to establish conventional values until the observational estimates are reasonably stable; ... with the sole assumption of a value of the velocity of light, all these quantities can, and should, be expressed in the fundamental units of physics-the kilogramme, the second derived from the caesium atomic frequency, and the metre derived from the wavelength of the 6 056 Å line of Kr_{86} ". In private conversation Cook also specifically questioned the wisdom, and even the validity, of the fixing of k_s and of the use of the astronomical unit.

With the advent of planetary observations in " range " and " range-rate " (topocentric distance and radial velocity), expressed fundamentally in " light-seconds ", it is most certainly time for us to reinvestigate the question as to the proper units for planetary ephemerides, for space navigation, and for determinations of closely associated physical constants. Is it still possible for us to use a fixed value of the Gaussian k or k_s ? The answering of this question should be a preliminary to any consideration of the g. u. and of the fixing of k_e.

As for the pre-radar-observation problem, three points should be made clear at the start :

(1) The fixing of k_s is not the same as the adoption of a " conventional value " of a physical constant $\left(\text{e. g. the international adoption of Hayford's spheroid of 1909, with an equatorial radius of 6 378 388 m and a flattening of } \frac{1}{297}\right)$. It is instead the replacement of a physical constant by a mathematical one. The uncertainty of the physical constant is transferred into an uncertainty in the mean distance of the Earth from the Sun, which distance is no longer exactly one a. u., without uncertainty. The astronomical unit is defined instead as the mean distance of a hypothetical, massless, and unperturbed planet whose period is $P_0 = \frac{2\pi}{k_s}$. The mean distance of the Earth from the Sun is thus made to depart only slightly from one a. u., negligibly for many purposes, but it becomes a quantity determined by observation just as are all other planetary mean distances. (This redefinition may be looked upon as the last skirmish in the Copernican revolution against a geocentric universe !)

(2) The definition of the astronomical unit in such a way as to fix k_s is purely a matter of convenience, and has nothing to do with questions of accuracy. The convenience is to be found in the centering of this physical uncertainty in the orbit of the Earth, and in its removal from all other heliocentric ephemerides, except as their improvement depends upon observations made from the Earth. Such an improvement could be based upon an astronomical unit defined, as originally, as the mean distance of the Earth from the Sun, or upon the kilometer, or upon the light-second. The resulting accuracy would be the same, but the complexities would be greater than with the present practice.

(3) The use of the a. u. and the fixed k_s is most certainly not required. One may use the kilometer or the light-second if he satisfies himself that he has good reasons for doing so.

A considerable number of people think of the Gaussian k_s as being defined by $k_s^2 = Gm_{\odot}$ where G is the gravitational constant and $m_{\odot}$ is the mass of the Sun, so that the dimensions of k_s^2 are L^3T^{-2} instead of $L^3M^{-1}T^{-2}$ as with G; the units of L and T are of course adjusted to the a. u. and the day in place of the centimeter and the second. It is equally simple, however, and actually preferable if k_s^2 is to be used in force equations as well as acceleration equations, to think of k_s as defined by $k_s^2 = G$, so that it has the same dimensions and differs only in units. Thus we may think of the C. G. S. system as having an alternative in a. u.-$m_{\odot}$-day system. The astronomer has come to accept the necessity of these two systems, and possibly others, and the resulting necessity of transfers from one system to another. If anything, he finds his astronomical system of units more fundamental and better determined than the C. G. S. system; but he does not make odious comparisons between systems. He recognizes instead that both systems are known more accurately than the ratios of the astronomical unit to the kilometer (or the light-second), and of the mass of the Sun to the gram.

B. **Is it still possible to use the a. u. and a fixed k_s ?** — The advent of electronic observations of planetary " ranges " and " range-rates " essentially in light-seconds (l. s.) raises seriously the question as to whether this unit or the kilometer (or other metric unit) may not be better than the astronomical unit (a. u.) for future heliocentric orbit work. The question may be considered in the light of three sub-questions :

(1) Is the matter still merely one of convenience, or is accuracy also involved ?

(2) What are the circumstances if range or range-rate observations are of the same order of accuracy as optical observations of direction (e. g., right ascension, α, and declination, δ) ?

(3) Will these circumstances be altered if range or range-rate observations become markedly more accurate than those of α and δ (e. g., with errors of order 10^{-9} instead of 10^{-6}) ?

The basic principles underlying the answers to these questions can be studied by comparing the calculations based upon the a. u. with those based upon the l. s. for a vastly simplified model : Let the Earth and Venus be massless planets traveling in unperturbed, circular and coplanar orbits. Let the coordinates observed be the geocentric range, ρ^*, and celestial longitude, λ, in light-seconds and radians, respectively. (There would be no celestial latitude; and range-rate, $\dot{\rho}^*$, might be studied by an extension of the following formulae, but would not contribute additional substance to the discussion.) Let the observed-minus-computed (O—C) residuals in ρ^* and λ be used to improve the ratio, R, of the a. u. to the l. s., and the elements of the two planetary orbits, but to improve the latter without changing the circular character of the orbits. Let the constants of gravitation be

$$(1) \qquad \begin{cases} k = k_s(\text{a. u.})^{\frac{3}{2}}(m_\odot)^{-\frac{1}{2}}(\text{day})^{-1} & (k_s = 0.01720209895), \\ k^* = \text{R}^{\frac{3}{2}}\,k_s(\text{l. s.})^{\frac{3}{2}}(m_\odot)^{-\frac{1}{2}}(\text{day})^{-1}, \end{cases}$$

where R is approximately 500, of course; where $m_\odot$ might be omitted, of course, by those who prefer to consider that $k_s^2 = \text{G}m_\odot$; and where the day, rather than the second, is taken as the unit for k^* and associated calculations purely for convenience, and any possible differences between astronomical and terrestrial standards of time are neglected. Then, remembering that k_s is fixed by the definition of the a. u. cited in the foregoing,

$$(2) \qquad \Delta k = 0, \qquad \frac{\Delta k^*}{k^*} = \frac{3}{2}\,\frac{\Delta \text{R}}{\text{R}}.$$

The remaining constants of the problem, subject to correction along with R, then, are the elements of the two circular orbits :

$$(3) \qquad \begin{cases} \text{L}_{10} = \text{the mean longitude of the Earth at the epoch, } t_0, \\ \text{L}_{20} = \text{the mean longitude of Venus at the epoch, } t_0 \end{cases}$$

and

$$(4) \qquad \begin{cases} a_1 = \text{the mean distance of the Earth in a. u.}, \\ a_2 = \text{the mean distance of Venus in a. u.} \end{cases}$$

or

$$(5) \qquad \begin{cases} a_1^* = \text{R}a_1 = \text{the mean distance of the Earth in l. s.}, \\ a_2^* = \text{R}a_2 = \text{the mean distance of Venus in l. s.} \end{cases}$$

Then the mean angular motions are derived from

$$(6) \qquad n_i = \frac{k}{a_i^{\frac{3}{2}}} = \frac{k^*}{(a_i^*)^{\frac{3}{2}}}$$

so that

$$(7) \qquad \frac{\Delta n_i}{n_i} = -\frac{3}{2}\frac{\Delta a_i}{a_i} = \frac{3}{2}\left(\frac{\Delta R}{R} - \frac{\Delta a_i^*}{a_i^*}\right).$$

Equation (7) is a crucial one in the comparison of light-second procedures with astronomical-unit procedures. The uncertainty of R may be the principal (or even the only) constituent of the uncertainty of a_i^*; yet any attempt at combining ΔR and Δa_i^* leads inexorably to the a. u. ! Next, the mean longitudes at the time of observation, t, and their uncertainties,

$$(8) \qquad L_i = L_{i0} + n_i(t - t_0), \qquad \Delta L_i = \Delta L_{i0} + \Delta n_i(t - t_0).$$

Then, for a. u.,

$$(9) \qquad x_i = a_i \cos L_i, \qquad y_i = a_i \sin L_i;$$

$$(10) \qquad \begin{cases} \Delta x_i = x_i \dfrac{\Delta a_i}{a_i} - y_i \Delta L_i = x_i \dfrac{\Delta a_i}{a_i} + \dot{x}_i(t - t_0)\dfrac{\Delta n_i}{n_i} - y_i \Delta L_{i0} \\ \qquad = \left[x_i - \dfrac{3}{2}\dot{x}_i(t - t_0)\right]\dfrac{\Delta a_i}{a_i} - y_i \Delta L_{i0}, \\ \Delta y_i = \left[y_i - \dfrac{3}{2}\dot{y}_i(t - t_0)\right]\dfrac{\Delta a_i}{a_i} + x_i \Delta L_{i0}. \end{cases}$$

where, of course,

$$(11) \qquad \dot{x}_i = \frac{dx_i}{dt} = -n_i y_i, \qquad \dot{y}_i = \frac{dy_i}{dt} = +n_i x_i.$$

Shifting to geocentric co-ordinates,

$$(12) \qquad \begin{cases} \xi = \rho \cos \lambda = x_2 - x_1, \qquad \Delta\xi = \Delta x_2 - \Delta x_1, \\ \eta = \rho \sin \lambda = y_2 - y_1, \qquad \Delta\eta = \Delta y_2 - \Delta y_1; \end{cases}$$

$$(13) \qquad \tan \lambda = \frac{\eta}{\xi}, \qquad \rho^2 \Delta\lambda = \xi\Delta\eta - \eta\Delta\xi;$$

$$(14) \qquad \rho = \sqrt{\xi^2 + \eta^2}, \qquad \rho\,\Delta\rho = \xi\,\Delta\xi + \eta\,\Delta\eta;$$

$$(15) \qquad \rho^* = R\rho, \qquad \Delta\rho^* = R\,\Delta\rho + \rho\,\Delta R.$$

Thus, if we collect terms, we may write

$$(16) \qquad \begin{cases} \Delta\rho^* = \dfrac{\partial\rho^*}{\partial L_{10}}\Delta L_{10} + \dfrac{\partial\rho^*}{\partial L_{20}}\Delta L_{20} + \dfrac{\partial\rho^*}{\partial a_1}\Delta a_1 + \dfrac{\partial\rho^*}{\partial a_2}\Delta a_2 + \dfrac{\partial\rho^*}{\partial R}\Delta R, \\ \Delta\lambda = \dfrac{\partial\lambda}{\partial L_{10}}\Delta L_{10} + \dfrac{\partial\lambda}{\partial L_{20}}\Delta L_{20} + \dfrac{\partial\lambda}{\partial a_1}\Delta a_1 + \dfrac{\partial\lambda}{\partial a_2}\Delta a_2 + \dfrac{\partial\lambda}{\partial R}\Delta R \end{cases}$$

and note the following two special partial differential coefficients,

$$(17) \qquad \frac{\partial \rho^*}{\partial \mathrm{R}} = \rho = \frac{\rho^*}{\mathrm{R}}, \qquad \frac{\partial \lambda}{\partial \mathrm{R}} = 0.$$

In calculations based upon l. s. the equations are largely analogous to equations (9) to (14), the last of which yield ρ^* and $\Delta\rho^*$ directly, without need for equations (15). An important difference is found in equations (10); for example

$$(18) \qquad \Delta x_i^* = \left[x_i^* - \frac{3}{2} \dot{x}_i^* (t - t_0) \right] \frac{\Delta a_i^*}{a_i^*} - y_i^* \, \Delta \mathrm{L}_{i0} + \frac{3}{2} \dot{x}_i^* (t - t_0) \frac{\Delta \mathrm{R}}{\mathrm{R}}$$

so that we may trace through the following partial differential coefficients, leading to those of equations (16) modified by the replacement of a_1 and a_2 by a_1^* and a_2^* :

$$(19) \qquad \frac{\partial \xi^*}{\partial \mathrm{R}} = \frac{3 \dot{\xi}^*}{2 \mathrm{R}} (t - t_0), \qquad \frac{\partial \eta^*}{\partial \mathrm{R}} = \frac{3 \dot{\eta}^*}{2 \mathrm{R}} (t - t_0).$$

$$(20) \qquad \frac{\partial \rho^*}{\partial \mathrm{R}} = \frac{3 \dot{\rho}^*}{2 \mathrm{R}} (t - t_0), \qquad \frac{\partial \lambda}{\partial \mathrm{R}} = \frac{3 \dot{\lambda}}{2 \mathrm{R}} (t - t_0)$$

where

$$(21) \qquad \begin{cases} \dot{\xi}^* = \dot{x}_2^* - \dot{x}_1^*, & \rho^* \dot{\rho}^* = \xi^* \dot{\xi}^* + \eta^* \dot{\eta}^*, \\ \dot{\eta}^* = \dot{y}_2^* - \dot{y}_1^*, & (\rho^*)^2 \dot{\lambda} = \xi^* \dot{\eta}^* - \eta^* \dot{\xi}^*. \end{cases}$$

Contrasting the a. u. equations (17) with the l. s. equations (20), we see the reason for the pre-radar astronomical preference for the a. u., as presently defined, over any other unit (e. g., the kilometer as well as the light-second). The reason is expressed by $\frac{\partial \lambda}{\partial \mathrm{R}} = 0$, i. e., directional observations in general are free of R or of any similar ratio, when the a. u. is the unit of distance.

The first of equations (15) and (17) are expressive of the fact that, even when radar range observations are introduced into the problem, the use of the a. u. in the integrations postpones the introduction of R until the very last step in the comparison with the observations, and accordingly simplifies its partial differential coefficient in the differential correction or error analysis. Equations (20), it should be remembered, are derived for the present very much simplified model; they represent only skeleton versions of the integrations and partial differential coefficients of an actual orbital integration. The answers to sub-questions (2) and (3), p. 97, accordingly still favor the astronomical unit.

The factor $(t - t_0)$ in the partial differential coefficients (20) may seem to have both favorable and unfavorable aspects. The apparently unfavorable aspect is in the time-increasing effect of an uncertainty in R; the apparently favorable aspect is in the time-increasing accuracy with which R may be determined from observation. Both of these aspects are illusory. Since Δa_i^* is largely if not entirely determined by ΔR, the two $(t - t_0)$ terms of equation (18), and of similar equations, are working against one another and the net accuracy of the determination of ΔR is the same by either method. Thus the answer to sub-question (1), p. 97, is that the matter is still one of convenience, not accuracy.

It is proposed in my accompanying paper that *astrodynamical ephemerides* of the Earth and Venus be determined along with the value of the solar parallax (i. e., in fact, of R) by such a means as a least squares solution leaving the astronomical values of the mean distances unchanged. In the foregoing simplified model this proposal would set Δa_1 and Δa_2 equal to zero in equation (16); the orbital elements to be corrected along with R would be represented solely by L_{10} and L_{20}. The simplification of the determination based upon the astronomical unit is self-evident. The determination based upon the light-second would require the determination of five quantities instead of 3, with the not unexpected result in the end that

$$\frac{\Delta a_1^*}{a_1^*} = \frac{\Delta a_2^*}{a_2^*} = \frac{\Delta R}{R}. \tag{22}$$

But to simplify the determination by taking advantage of this expected result would be to replace the l. s. by the a. u. !

Other special problems, such as the determination of an orbit when the Earth's ephemeris can be considered accurate, strengthen our conclusion that the astronomical unit and a fixed k_s are still to be preferred to the light-second or the kilometer and associated (and " unfixed ") values of the gravitational constant.

I should state again that the device of " fixing " the gravitational constant can be used, for example, with the kilometer, but only at the expense of introducing an " astronomical kilometer " tied to the astronomical unit and eventually differing from the " terrestrial kilometer ". I want to make it clear, however, that I have never advocated such a move; in fact, I join with A. H. Cook in a horrified shudder at the mere thought of it !

C. **The proposal for a " g. u. " and a fixed k_e.** — In considering the proposal that we accept a " geo-unit " (g. u.) or " geocentric astronomical unit ", determined by a fixed value of the geocentric gravitational constant (k_e), as a possible unit for the integration of geocentric ephemerides, we should first note the following parallels to the situation involving the a. u. :

(1) The g. u. would be the mean distance of a hypothetical, massless, unperturbed satellite whose period is $P_0 = \frac{2\pi}{k_e}$, of a spherically homo-

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geneous Earth whose mass is the same as that of the actual Earth. With k_e thus treated as a mathematical constant, the uncertainty associated with it as a physical constant would be transferred to the Earth's equatorial radius (a_e) as expressed in g. u. The adoption of the g. u. as alternative to the a_e or the kilometer as unit of distance, accordingly, does not involve the adoption of a " conventional value ". The proposal bears only a superficial resemblance to a conventional adoption because we would attempt to start with a g. u. equal to our best present value of a_e-but no possible use of the g. u. makes it necessary that it remain equal to a_e, i. e., that it remain a " conventional value " of a_e.

(2) The proposal is one of convenience and not one of accuracy. Again the convenience is the removal of first-order uncertainties from the dynamically centered ephemerides, this time geocentric. The first-order uncertainty comes only in the last stage of a topocentric ephemeris, i. e., in the shift from the center of the Earth to the " point of observation " (which may be either observing the vehicle or observed from the vehicle).

If the " point of observation " is another vehicle (or satellite), by the way, neither a_e nor kilometer would have to enter the discussion at all. The uncertainty in the " point of observation " would be derived from the uncertainties in the orbit or ephemeris of the other vehicle. The greatest convenience of the g. u. as unit of distance would be in the comparison of orbital integrations based upon it. If revisions of any of the compared calculations were necessary, they would be only in the final determinations of the observation residuals, and would be arrived at merely by reducing the observations to a common basis.

(3) No one would be required to use the " geo-unit " if in his judgement the kilometer or the light-second served his purposes better. Probably the g. u. would tend to supplant a_e as a unit of distance, on the other hand (*cf.* section D); their advantages are not identical, but for the most part the g. u. has the advantages of a_e in addition to the fixing of k_e, with resultant non-revision of the core ephemeris integrations.

D. **The use of k_e as a " fundamental constant "**. — In favoring k_e^2 over g_e, the equatorial acceleration of gravity, or $a_{\mathbb{C}}$, the geocentric mean distance of the Moon, as a " fundamental constant ", the I. A. U. Symposium No. 21, on the *System of Astronomical Constants*, Paris, 1963 May 27-31, greatly simplified the basic framework of interrelating equations. For example, the solar parallax, $\pi_\odot$, if expressed in radians, and if defined as the ratio of the Earth's equatorial radius, a_e, to the astronomical unit (a. u.), is determined by

$$\pi_\odot^3 = \frac{k_s^2}{k_e^2}\,\frac{m_\oplus}{m_\odot}\cdot\frac{1}{(86\,400)^2} \tag{23}$$

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assuming that k_s^2 is expressed in (a. u.)^3/day^2, and that k_e^2 is expressed in $(a_e)^3$/sec^2.

We only complicate this relationship if we replace k_s^2, quite unnecessarily, by the equation by which it was determined originally,

$$(24) \qquad k_s^2 = \left(\frac{2\pi}{P_{\oplus☾}}\right)^2 \left(\frac{a_{\oplus☾}}{A}\right)^3 \frac{m_\odot}{m_\odot + m_\oplus + m_☾}$$

where $m_\odot$, $m_\oplus$, $m_☾$ are the masses of the Sun, Earth, Moon; where $P_{\oplus☾}$ and $a_{\oplus☾}$ are the heliocentric period and mean distance of the Earth-Moon system; where A is the length of the astronomical unit expressed in terms of the same unit as $a_{\oplus☾}$; and where $\frac{a_{\oplus☾}}{A}$ includes the effects of perturbations. Inasmuch as k_s^2 is fixed in numerical value in equation (23), it is quite pointless to replace it by the other quantities of equation (24), each with its own uncertainty, but such that the several uncertainties compensate and cancel one another.

Similarly we might replace k_e^2 by one of the following equations,

$$(25) \qquad k_e^2 = \left(\frac{2\pi}{P_☾}\right)^2 \left(\frac{a_☾}{a_e}\right)^3 \frac{m_\oplus}{m_\oplus + m_☾}$$

where, additionally, $P_☾$ is the geocentric period of the Moon; where a_e is the Earth's equatorial radius expressed in terms of the same unit as $a_☾$; and where $\frac{a_☾}{a_e}$ is modified to include the effects of perturbations; or

$$(26) \qquad k_e^2 = \frac{g_e a_e^2}{F_1}$$

where, additionally, a_e is the Earth's equatorial radius; and where F_1 includes the effects of perturbations. But, beginning with the present time, k_e^2 will probably be better determined from geodetic satellites than from either of the foregoing formulae; it is better, accordingly to treat it as fundamental, utilizing all sources of information for its accurate determination.

If, next, we extend the foregoing discussion to consider the possible complete replacement of the Earth's equatorial radius by the " geo-unit " (g. u.), we note immediately that by redefining $\pi_\odot$ as the ratio of the g. u. to the a. u. we do not alter equation (23) in form, but permit the use in it of a fixed value of k_e^2 as well as of k_s^2. The relative uncertainty of $\pi_\odot$, then, becomes simply one third of the relative uncertainty of $\frac{m_\oplus}{m_\odot}$. The simplifications o both formula and definition of $\pi_\odot$ should not be dismissed lightly.

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