SOME REMARKS ON SEMIGROUP PRESENTATIONS

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Dedicated to H. S. M. Coxeter

1. Let a semigroup A be given by generators $a_1, a_2, \ldots, a_d$ and defining relations $u_1 = v_1, u_2 = v_2, \ldots, u_e = v_e$ between these generators, the u_i, v_i being words in the generators. We then have a *presentation of* A, and write

$$A = \mathrm{sgp}(a_1, \ldots, a_d; u_1 = v_1, \ldots, u_e = v_e).$$

The same generators with the same relations can also be interpreted as the presentation of a group, for which we write

$$A^* = \mathrm{gp}(a_1, \ldots, a_d; u_1 = v_1, \ldots, u_e = v_e).$$

There is a unique homomorphism ϕ: $A \to A^*$ which maps each generator $a_i \in A$ on the same generator $a_i \in A^*$. This is a monomorphism if, and only if, A is embeddable in a group; and, equally obviously, it is an epimorphism if, and only if, the image $A\phi$ in A^* is a group. This is the case in particular if $A\phi$ is finite, as a finite subsemigroup of a group is itself a group. Thus we see that *$A\phi$ and A^* are both finite* (in which case they coincide) *or both infinite* (in which case they may or may not coincide). If A^*, and thus also $A\phi$, is finite, A may be finite or infinite; but if A^*, and thus also $A\phi$, is infinite, then A must be infinite too. It follows in particular that *A is infinite if e, the number of defining relations, is strictly less than d, the number of generators.*

2. We now assume that d is chosen minimal, so that A cannot be generated by fewer than d elements. For what values of d is then $e = d$ compatible with A being finite?

The corresponding question for groups has not, I believe, been fully answered. Examples of finite groups are known for which $e = d$ when d (always minimal) is 1, 2, or 3; and Mennicke **(2)** has conjectured that for $d \geqslant 4$ a finite group must have $e > d$. This has been proved only for some special classes of groups, such as p-groups.

In the case of semigroups the answer is much simpler, indeed almost trivial: *The semigroup*

$$A = \mathrm{sgp}(a_1, a_2, \ldots, a_d; a_1 a_2 = a_2, a_2 a_3 = a_3, \ldots, a_d a_1 = a_1)$$

is finite, namely of order d, and it cannot be generated by fewer than d elements. To see this, one shows that $a_i a_j = a_j$ for all i and j. Assume that $a_i a_{i+k} = a_{i+k}$

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has already been established, where k is positive and suffixes are taken modulo d. The case $k = 1$ is given. Then

$$a_i\, a_{i+k+1} = a_i\, a_{i+k}\, a_{i+k+1} = a_{i+k}\, a_{i+k+1} = a_{i+k+1}.$$

It follows that our semigroup satisfies the law

$$xy = y;$$

its carrier (that is, set of elements) is just $\{a_1, a_2, \ldots, a_d\}$, and every subset of this is the carrier of a subsemigroup. Hence no fewer than all its d elements suffice to generate it. The corresponding group A^* is easily seen to be trivial.

One can vary this example so as to make the corresponding group non-trivial. Consider the semigroup

$$B = \operatorname{sgp}(a_1, a_2, \ldots, a_{d-1}, b;\ a_1 a_2 = a_2, \ldots, a_{d-2}\, a_{d-1} = a_{d-1},$$
$$a_{d-1}\, b^n = b^n,\ b^n a_1 = a_1).$$

This is obtained from the semigroup A we have just considered by putting $a_d = b^n$, with a new generator b. Here n is to be a positive integer. We show that *B is finite.* First we remark that $a_1, a_2, \ldots, a_{d-1}, b^n$ generate a finite subsemigroup of B, and that in this subsemigroup

$$xy = y$$

is a law. Thus every element of B can be written as a string of powers of b alternating with single occurrences of a_i. Now

$$a_i\, b^m a_j = a_i\, b^m b^n a_j = a_i\, b^n b^m a_j = b^n b^m a_j = b^m b^n a_j = b^m a_j;$$

hence every element of B can be reduced to the form

$$b^p a_i\, b^q,$$

where b^p or b^q may be absent, or to the form b^p. Finally

$$b^{2n} = b^n a_1\, b^n = a_1\, b^n = b^n;$$

thus the powers of b that occur can be reduced to b^{2n-1} or lower powers, and it follows that B is finite. It is not difficult to determine the order of B. The group B^* is cyclic of order n.

3. The semigroup B we have just examined illustrates also a different point, namely the adjunction of an nth root of an element, in this case a_d, to a semigroup, in this case A. We shall show presently that such an adjunction is always possible. For the moment we only remark that here we have adjoined an nth root *freely*, in a sense that can, but will not, be made precise. If an nth root for $n > 1$ is adjoined freely to a non-cyclic group, the resulting group is always infinite. It is remarkable that in semigroups this is not so, as our example shows.

THEOREM 3.1. *Solutions of*

$$x^n = a_0$$

can always be adjoined to the arbitrary semigroup A, whatever the element $a_0 \in A$ and the integer $n > 0$ may be.

Differently put, A can be embedded in a semigroup B containing an element b such that $b^n = a_0$. See **(3)** for a discussion of adjunction of elements to groups—applicable also, *mutatis mutandis*, to semigroups—and in particular Theorem 5.1 for the corresponding fact. The proof is a simple application of the wreath product of semigroups introduced in **(6)**. Denote by A' the semigroup obtained from A by adjoining a unit element, which we denote by 1; if A already possesses a unit element, we may adjoin another one, or put $A' = A$. Let $C = \mathrm{sgp}(c;\ c^{n+1} = c^n)$ act on the set $Y = \{1, 2, \ldots, n + 1\}$ by putting, for $y \in Y$,

$$y^{c^i} = \min(y + i, n + 1).$$

Now we form the wreath product

$$B = A' \mathrm{Wr}\ C,$$

with C acting on Y as described. The elements of B are pairs (f, c^i), where f is a function on Y to A, and $1 \leqslant i \leqslant n$. The functions $g_a\colon Y \to A'$ defined by

$$g_a(1) = a, \qquad g_a(y) = 1 \quad \text{for } 2 \leqslant y \leqslant n + 1$$

form a semigroup, under the usual pointwise multiplication, isomorphic to A'; and so do the elements

$$(g_a, c^n) \in B.$$

One readily verifies that if

$$b = (g_a, c),$$

then

$$b^i = (g_a, c^i)$$

and in particular

$$b^n = (g_a, c^n).$$

If we embed A in B by identifying $a \in A$ with $(g_a, c^n) \in B$, then B is thus seen to contain nth roots of every element of A, and the theorem follows. Instead of the particular monogenic semigroup C we have used, one can use the cyclic group of order n, acting on its own carrier by right multiplication, and using the pairs with constant functions as first component and the unit element of C as second component for the embedding of A—this is the method applied in the case of groups.

We note as a corollary of the proof that *if A is a finite semigroup, then the adjunction of nth roots of elements of A can be carried out in a finite semigroup.* The free adjunction of an nth root may, as we have seen, also result in a finite semigroup, but will in general give an infinite semigroup.

As in the case of groups (**3**, Theorem 6.2) one can also prove that *every semigroup can be embedded in a divisible semigroup*, that is one in which every element has nth roots for every $n > 0$.

4. We present some more isolated examples of finite semigroups with $d = 2$ generators and $e = 2$ defining relations. It is not difficult to make more examples, but a systematic study may be difficult, as it is in the case of groups.

Consider

$$A_1 = \mathrm{sgp}(a, b; ba = a^2b, b^2 = a^3). \tag{4.1}$$

The corresponding group

$$A_1^* = \mathrm{gp}(a, b; ba = a^2b, b^2 = a^3)$$

is easily seen to be the non-abelian group of order 6 (the symmetric group of degree 3). To determine A_1 itself, we adapt the well-known Todd–Coxeter method (**7**) to deal with semigroup presentations. This will not only give us information on the order of the semigroup defined by a given presentation, but also provide a representation of the semigroup by mappings of a set S into itself. There is no need to describe the method in detail, as the reader will have no difficulty in interpreting the example set out below in the spirit of Coxeter and Moser (**1**, §2.2, especially 1st edition, p. 16; 2nd edition, p. 15). We exhibit the completed tables in Table I for the semigroup presented by (4.1).

TABLE I

1	2	3	4	5	6	7	8	9	10	11	12	13	14
	b	a	a	a	b	b	b	a	a	a		a	b
1	3	6	2	4	6	3	7	2	4	7	1	2	3
2	5	8	4	7	8	5	9	4	7	9	2	4	5
3	7	9	6	10	9	7	8	6	10	8	3	6	7
4	6	10	7	9	10	6	11	7	9	11	4	7	6
5	9	11	8	12	11	9	10	8	12	10	5	8	9
6	11	13	10	8	13	11	12	10	8	12	6	10	11
7	8	12	9	11	12	8	13	9	11	13	7	9	8
8	13	9	12	10	9	13	8	12	10	8	8	12	13
9	10	8	11	13	8	10	9	11	13	9	9	11	10
10	9	11	8	12	11	9	10	8	12	10	10	8	9
11	12	10	13	9	10	12	11	13	9	11	11	13	12
12	11	13	10	8	13	11	12	10	8	12	12	10	11
13	8	12	9	11	12	8	13	9	11	13	13	9	8

Column 1 contains the numbers 1, 2, . . . that form the set S on which the semigroup will act. Columns 2 and 3 are headed by the generators, in the order in which they occur, making up the left-hand side of the first relation; columns 4, 5, 6 similarly deal with the right-hand side of the first relation, and columns

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7, 8 and 9, 10, 11 with the two sides of the second relation. Column 12 is a repetition, inserted only for the sake of convenience, of column 1; and columns 13 and 14, headed by the generators, are also put in for convenience only, and are not necessary (they list the effect of the generators on the elements of S). Thus the row labelled "s" (where $s \in S$) in columns 1 and 12 has in column 13 the effect of mapping s by a, and in column 14 the effect of mapping s by b. Columns 2, 4, 7, 9 contain in row "s" the effect of mapping s by the generator heading the column: thus the columns 2, 7, 14 are the same, and also the columns 4, 9, 13 are the same. Columns 3, 5, 6, 8, 10, 11 contain the effect of mapping the elements in the preceding column by the generator heading the column. The relations are taken into account by ensuring that columns 3 and 6 are equal, and also columns 8 and 11 are equal. The fact that the table "closes" after a finite number of rows—here 13—have been constructed means that the semigroup presented by (4.1) is finite, and of order $13 - 1 = 12$: every element of the semigroup A_1 corresponds to a unique element of the set $S = \{1, 2, \ldots, 13\}$; but $1 \in S$ does not correspond to any element of A_1, but rather to a unit element adjoined to A_1, which ensures that different elements of A_1 induce distinct mappings of S into S.

The last columns of the table, columns 12, 13, 14, allow one to find an expression in terms of the generators for every element of A_1; we identify the elements of A_1 by the element of S into which they map $1 \in S$. Thus $2 = 1a = a$, $3 = 1b = b$, $4 = 2a = 1a^2 = a^2$, $5 = ab$, $6 = ba$, $7 = b^2$, $8 = aba$, $9 = ab^2$, $10 = ba^2$, $11 = bab$, $12 = aba^2$, $13 = abab$; and one can also read off the relations, such as $6 = ba = a^2b$, $7 = b^2 = a^3$, $8 = aba = b^3$, $9 = ab^2 = b^2a$, and so on. Finally one can without difficulty write down the multiplication table of A_1 (Table II), with columns 13 and 14 as its first two columns. It will

TABLE II

	2	3	4	5	6	7	8	9	10	11	12	13
2	4	5	7	6	8	9	10	11	12	13	8	9
3	6	7	10	11	9	8	13	12	11	10	9	8
4	7	6	9	8	10	11	12	13	8	9	10	11
5	8	9	12	13	11	10	9	8	13	12	11	10
6	10	11	8	9	13	12	11	10	9	8	13	12
7	9	8	11	10	12	13	8	9	10	11	12	13
8	12	13	10	11	9	8	13	12	11	10	9	8
9	11	10	13	12	8	9	10	11	12	13	8	9
10	8	9	12	13	11	10	9	8	13	12	11	10
11	13	12	9	8	10	11	12	13	8	9	10	11
12	10	11	8	9	13	12	11	10	9	8	13	12
13	9	8	11	10	12	13	8	9	10	11	12	13

be noticed that the bottom right-hand 6×6 square, with the rows and columns labelled $8, \ldots, 13$, is simply the multiplication table of the group A_1^*, with 13 as unit element, 9 and 11 as elements of order 3, and 8, 10, 12 as elements of order 2.

5. We add another example, namely

$$A_2 = \mathrm{sgp}(a, b; a = bab, b = aba). \tag{5.1}$$

The corresponding group

$$A_2^* = \mathrm{gp}(a, b; a = bab, b = aba)$$

is the quaternion group, in a presentation equivalent to one mentioned (without proof) in (**5**, p. 192).

TABLE III

	a	b	a	b	b	a	b	a		a	b
1	2	3	6	2	3	2	5	3	1	2	3
2	4	5	3	4	5	4	8	5	2	4	5
3	6	4	7	6	4	6	2	4	3	6	4
4	7	8	5	7	8	7	6	8	4	7	8
5	3	7	9	3	7	3	4	7	5	3	7
6	8	2	4	8	2	8	9	2	6	8	2
7	9	6	8	9	6	9	3	6	7	9	6
8	5	9	2	5	9	5	7	9	8	5	9
9	2	3	6	2	3	2	5	3	9	2	3

Table III, arranged like Table I (except that we have not numbered the columns), shows that the semigroup A_2 presented by (5.1) has order 8. Thus it must coincide with the quaternion group A_2^*, and we have in (5.1) a semigroup presentation of the quaternion group in terms of two generators and two defining relations.

This can also be shown by a less computational method; we illustrate this method with a similar example, which is not amenable to treatment by the adapted Todd–Coxeter method. Consider the semigroup

$$A_3 = \mathrm{sgp}(a, b, c; a = bab, b = cbc, c = aca). \tag{5.2}$$

This is a natural generalization, to the case of three generators, of (5.1). Now

$$a = bab = bacbc = bacbaca.$$

Put

$$bacbac = e;$$

then

$$a = ea.$$

Next

$$c = aca = eaca = ec,$$

$$b = cbc = ecbc = eb.$$

Thus e is a left neutral element. By symmetry, the element $cabcab$ is a right neutral element. But if a semigroup (or even a groupoid, as associativity does not enter this) has a left neutral and also a right neutral element, then they coincide and are a unit element. Thus A_3 has a unit element e. Moreover, using the obvious symmetry of the presentation (5.2) again, we see that

$$e = (bac)^2 = (cab)^2 = (acb)^2 = (bca)^2 = (cba)^2 = (abc)^2.$$

Thus a has a left inverse, namely $cbacb$, and also a right inverse, namely $bcabc$: these then coincide, and are a two-sided inverse of a. Correspondingly b and c have inverses, and A_3 is a group. It follows that A_3 coincides with the group

$$A_3^* = \mathrm{gp}(a, b, c; a = bab, b = cbc, c = aca).$$

This could not have been proved by showing A_3 and A_3^* to have the same order, because their order is infinite (**4**, Example (5.6)).

The fact that a semigroup presentation may yield a group is not in itself remarkable; for every group presentation can be made into a semigroup presentation by introducing the inverses of the group generators as further semigroup generators, with suitable further relations. The presentations (5.1) and (5.2) are remarkable in not requiring any further generators or relations.

Like the original Todd–Coxeter method, our adapted Todd–Coxeter method will readily lend itself to programming for automatic computation. Also, again like the original Todd–Coxeter method, it will often lead to the generation of rows that subsequently have to be identified with earlier rows.

6. The original Todd–Coxeter method is more general than the case of it that we have so far adapted to semigroups. It enumerates cosets of a subgroup of a given group, and provides a permutation representation of the given group by right multiplication of these cosets. Can this more general method also be adapted to the case of semigroups?

The initial difficulty that one meets is that cosets of subsemigroups in semigroups cannot be defined so as to have all the desirable properties which cosets of subgroups have in groups.

We here adopt as our definition not the analogue of a single coset of a subgroup in a group, but the set of all (right) cosets of the subgroup. We accordingly consider a *right semicongruence* $\mathfrak{r}$ on our semigroup A, that is an equivalence relation that admits right multiplication by elements of A: thus if x, y, z are elements of A and if $(x, y) \in \mathfrak{r}$, then also $(xz, yz) \in \mathfrak{r}$. The equivalence classes or "$\mathfrak{r}$-blocks" then play the role of cosets. If K is such an $\mathfrak{r}$-block, then the elements $h \in A$ such that $Kh \subseteq K$ form a subsemigroup H of A, the "fixing semigroup" of K; this may be void. The different non-void fixing semigroups that arise from the blocks of a right semicongruence can then be considered as "conjugate."

If we start from a subsemigroup H of the semigroup A, we can then form the least right semicongruence that contains H in a block: then H will be contained in the fixing semigroup of that block; but it may be properly contained in it. This is the nearest we can get to the "cosets of H."

This being said, we can then, however, extend our adaptation of the Todd–Coxeter method to the full generality of the original method. This is perhaps best illustrated by an example. We take the semigroup A_1 given by (4.1) and in it the subsemigroup generated by b; its elements are b, b^2, b^3, b^4, and it satisfies the relation

$$b^5 = b^3.$$

We again exhibit just the finished table (Table IV), using 1′, 2′, ... to distinguish the entries from those of Table I. To ensure that the subsemigroup generated by b is contained in a block, it suffices to ensure that b and b^2 are given the same number, namely 3′.

TABLE IV

1	2	3	4	5	6	7	8	9	10	11	12	13	14
	b	a	a	a	b	b	b	a	a	a		a	b
1′	3′	6′	2′	4′	6′	3′	3′	2′	4′	3′	1′	2′	3′
2′	5′	3′	4′	3′	3′	5′	6′	4′	3′	6′	2′	4′	5′
3′	3′	6′	6′	7′	6′	3′	3′	6′	7′	3′	3′	6′	3′
4′	6′	7′	3′	6′	7′	6′	7′	3′	6′	7′	4′	3′	6′
5′	6′	7′	3′	6′	7′	6′	7′	3′	6′	7′	5′	3′	6′
6′	7′	3′	7′	3′	3′	7′	6′	7′	3′	6′	6′	7′	7′
7′	6′	7′	3′	6′	7′	6′	7′	3′	6′	7′	7′	3′	6′

The blocks of the corresponding right semicongruence and their fixing semigroups are given in Table V, in which the elements of A_1 are referred to by their numbers as in Tables I and II, and the blocks by their numbers as in Table IV.

TABLE V

Number of block	Block	Fixing semigroup carrier
2′	{2}	∅
3′	{3, 7, 8, 13}	{3, 7, 8, 13}
4′	{4}	∅
5′	{5}	∅
6′	{6, 9, 12}	{5, 7, 10, 13}
7′	{10, 11}	{6, 7, 12, 13}

It should be recalled that 1 is not an element of A_1, but can be adjoined as unit element; correspondingly 1′ is not a block, but can be similarly adjoined.

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