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ON A DIOPHANTINE EQUATION

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In this note, we find all solutions of the diophantine equation $x^2+3^m=y^n$, where (x,y,m,n) are non-negative integers with $x\neq 0$ and $n\geqslant 3$.

In this note, we investigate the equation

$$x^2+3^m=y^n \tag{1}$$

when $x>0$ and $n\geqslant 3$.

For $n=2$ the problem is not interesting because in this case the given equation (1) has infinitely many solutions and all of them are of the form

$$\begin{cases} x=\dfrac{3^a-3^b}{2}, \\ y=\dfrac{3^a+3^b}{2}, \\ m=a+b, \end{cases} \qquad \text{for some integers } a>b\geqslant 0.$$

The fact that equation (1) has no solution when $m=0$ was shown by Lebesgue (see [**7**]) and the fact that (1) has no solution for $m=1$ was proved by Cohn (see [**4**]). Recently, Arif and Muriefah (see [**1**]) found all solutions of equation (1) when m is odd. They are all of the form $x=10\cdot 3^{3t}$, $y=7\cdot 3^{2t}$, $m=5+6t$ and $n=3$. The same authors investigated equation (1) for m even in [**2**].

Our result is the following:

THEOREM. *All solutions of equation* (1) *with m even are of the form $x=46\cdot 3^{3t}$, $m=4+6t$, $y=13\cdot 3^{2t}$ and $n=3$.*

We begin by showing that it suffices to treat equation (1) when $3\nmid x$. Indeed, assume that $x=3^a x_1$ for some $a\geqslant 1$ and $3\nmid x_1$. Write $y=3^b y_1$ where $b\geqslant 0$ and $3\nmid y_1$. Equation (1) becomes

$$3^{2a}x_1^2+3^m=3^{nb}y_1^n. \tag{2}$$

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We distinguish 3 cases:

CASE 1. $2a > m$.

Equation (2) becomes

$$\left(3^{a-m/2}x_1\right)^2 + 1 = 3^{nb-m}y_1^n. \tag{3}$$

From equation (3) it follows that $nb = m$. If we denote by $X = 3^{a-m/2}x_1$ and by $Y = y_1$, we get

$$X^2 + 1 = Y^n \tag{4}$$

which has no solution by Lebesgue's result.

CASE 2. $2a = m$.

Equation (2) becomes

$$x_1^2 + 1 = 3^{nb-m}y_1^n. \tag{5}$$

Since -1 is not a quadratic residue modulo 3, it follows that $nb = m$. Hence, equation (5) becomes

$$x_1^2 + 1 = y_1^n$$

which is again Lebesgue's equation.

CASE 3. $2a < m$.

Equation (2) becomes

$$x_1^2 + 3^{m-2a} = 3^{nb-2a}y_1^n. \tag{6}$$

From equation (6), it follows that $nb = 2a$. Equation (6) is now

$$x_1^2 + 3^{m_1} = y_1^n \tag{7}$$

with $m_1 = m - 2a$ even. Equation (7) is precisely equation (1) for m_1 even and $3 \nmid x_1$.

From now on we assume that $(x,\ y,\ m,\ n)$ is a solution of (1) with $3 \nmid x$. Notice that x is even and that y is odd — indeed, if x is odd then $x^2 + 3^m \equiv 2 \pmod 8$, hence it cannot be the power of an even number (nor of a odd number).

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We treat two cases:

THE CASE $4 \mid n$

In this case, we may assume that $n = 4$. Equation (1) can be rewritten as

$$3^m = (y^2 - x)(y^2 + x). \tag{8}$$

Since $y^2 - x$ and $y^2 + x$ are coprime, it follows that

$$y^2 - x = 1,$$
$$y^2 + x = 3^m.$$

Hence, $2y^2 = 3^m + 1$ or

$$\left(3^{m/2}\right)^2 - 2y^2 = -1. \tag{9}$$

The equation

$$X^2 - 2Y^2 = -1$$

is a Pell equation and its positive solutions are given by $X_1 = 1$, $Y_1 = 1$, $X_2 = 7$, $Y_2 = 5$ and

$$X_n = 6X_{n-1} - X_{n-2},\ Y_n = 6Y_{n-1} - Y_{n-2}. \tag{10}$$

It follows that $3 \nmid X_n$, which contradicts the fact that $X = 3^{m/2}$.

Thus, equation (1) has no solution such that $4 \mid n$.

THE CASE $4 \nmid n$

Since $n \geqslant 3$ and $4 \nmid n$, it follows that there exists an odd prime p such that $p \mid n$. We may assume that $n = p$. Equation (1) becomes

$$x^2 + 3^m = y^p. \tag{11}$$

Since $x^2 \equiv y^2 \equiv 1 \pmod 3$, it follows that $y \equiv 1 \pmod 3$. Rewrite equation (11) as

$$\left(x + i3^{m/2}\right)\left(x - i3^{m/2}\right) = y^p.$$

Since $\mathbf{Z}[i]$ has class number 1 and $\gcd\left(x + i3^{m/2},\ x - i3^{m/2}\right) = 1$, it follows that the exists two integers a and b such that $y = a^2 + b^2$ and

$$\begin{cases} x + i3^{m/2} = (a + ib)^p, \\ x - i3^{m/2} = (a - ib)^p. \end{cases} \tag{12}$$

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Notice that $ab \neq 0$. Solving system (12), we get

$$x = \frac{(a+ib)^p + (a-ib)^p}{2},$$

(13) $$3^{m/2} = \frac{(a+ib)^p - (a-ib)^p}{2i}.$$

Since p is odd, it follows from the first equation (13) that $a \mid x$. In particular, $3 \nmid a$. Moreover, from the second equation (13), it follows that $b \mid 3^{m/2}$.

We treat first the case $p = 3$. In this case, the second equation (13) becomes

(14) $$3^{m/2} = b(3a^2 - b^2).$$

Reducing equation (14) modulo 3, it follows that $3 \mid b$. In particular, $9 \mid b(3a^2 - b^2)$ which gives $m/2 \geqslant 2$. If $m/2 = 2$, we get

$$9 = b(3a^2 - b^2)$$

and $b = \pm 3$. This leads to $a = 2$, $b = 3$, which gives the solution $(x,\ y,\ m,\ n) = (46,\ 13,\ 4,\ 3)$ which was previously found by Cohn (see [**5**]).

We now show that equation (14) has no solution for $m > 4$. Indeed, let $b = \pm 3^u$ for some u, $0 < u < m/2$. Equation (14) becomes

$$3a^2 - 3^{2u} = \pm 3^{m/2-u}$$

or

(15) $$a^2 = 3^{2u-1} \pm 3^{m/2-u-1}.$$

Since $3 \nmid a$, it follows that $u = m/2 - 1$ and

(16) $$a^2 = 3^{m-3} \pm 1.$$

The equation with -1 leads to

$$a^2 + 1 = 3^{m-3}$$

for some $m \geqslant 6$ which is impossible by Lebesgue's result. The equation with $+1$ leads to

(17) $$a^2 = 3^{m-3} + 1$$

with $m \geqslant 6$. From a result of Chao Ko (see [**6**]), we know that the only nontrivial solution of the equation

$$X^2 = Y^n + 1$$

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for some $n \geqslant 3$ is given by $X = 3$, $Y = 2$. Hence, equation (17) has no solution.

From now on we assume that $p > 3$. We first show that $b = \pm 3^{m/2}$. Notice first that $b \neq \pm 1$. Indeed, if $b = \pm 1$, then $y = a^2 + b^2 = a^2 + 1$. Since $y \equiv 1 \pmod 3$, it follows that $a \equiv 0 \pmod 3$ which is a contradiction. Hence, $b = \pm 3^u$ for some u, $0 < u \leqslant m/2$. Assume that $b = \pm 3^u$ for some $u < m/2$. After simplifying the second equation (13) by b and reducing it modulo 3 we get $pa^{p-1} \equiv 0 \pmod 3$ which is impossible for $p > 3$ prime and $3 \nmid a$. Hence, $b = \pm 3^{m/2}$. From [**5**, Lemma 4 and Lemma 5], it follows that $b = -3^{m/2}$, $C = 3^m \equiv 1 \pmod{16}$ and $p \equiv -1 \pmod{12}$. In particular, $4 \mid m$. From the same paper of Cohn, we also know that a is even and that if q is any odd prime dividing a, then

$$3^{m(q-1)} \equiv 1 \left(\bmod\ q^2\right) \tag{18}$$

and that if $q^\alpha \| a$, then $q^{2\alpha} \| \left(3^{m(q-1)} - 1\right)$.

We now return to the second equation (13). Let $\varepsilon = a + ib$ and $\overline{\varepsilon} = a - ib$. Since $b = -3^{m/2}$, it follows that

$$\frac{\varepsilon^p - \overline{\varepsilon}^p}{\varepsilon - \overline{\varepsilon}} = -1. \tag{19}$$

Notice that the sequence

$$u_k = \frac{\varepsilon^k - \overline{\varepsilon}^k}{\varepsilon - \overline{\varepsilon}} \quad \text{for all } k \geqslant 0 \tag{20}$$

is a Lucas sequence. By the results of [**3**], we know that in this case u_k has a primitive divisor for all prime values of $k > 13$. Moreover, for $k \in \{5, 7, 11, 13\}$ there are precisely 10 Lucas sequences for which u_k does not have a primitive divisor and all these 10 sequences can be found in [**3**, Table 1]. One can easily see that none of these 10 sequences has the property that the roots of the characteristic equation are in $\mathbb{Z}[i]$. Hence, $|u_p| > 1$, which contradicts (19). It follows that there are no solutions for $p > 3$.

One can now employ the arguments from Case 3 at the beginning of the paper to conclude that the general solution of equation (1) for m even is given by $x = 46 \cdot 3^{3t}$, $m = 4 + 6t$, $y = 13 \cdot 3^{2t}$ and $n = 3$.

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