

A FORMULA FOR THE NUMBER OF SPANNING TREES IN CIRCULANT GRAPHS WITH NONFIXED GENERATORS AND DISCRETE TORI

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Abstract

We consider the number of spanning trees in circulant graphs of βn vertices with generators depending linearly on n . The matrix tree theorem gives a closed formula of βn factors, while we derive a formula of $\beta - 1$ factors. We also derive a formula for the number of spanning trees in discrete tori. Finally, we compare the spanning tree entropy of circulant graphs with fixed and nonfixed generators.

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1. Introduction

A spanning tree of a connected graph G is a connected subgraph of G without cycles with the same vertex set as G . The number of spanning trees in a graph G , $\tau(G)$, is an important graph invariant. It can be computed from the well-known matrix tree theorem due to Kirchhoff (see, for example, [1]). Let G be a graph on n vertices labelled by v_1, \dots, v_n . The adjacency matrix $A = (A_{ij})$ of G is the $n \times n$ matrix in which $A_{ij} = 1$ if v_i and v_j are adjacent and $A_{ij} = 0$ otherwise. The degree matrix $D = (D_{ij})$ is the $n \times n$ diagonal matrix in which the diagonal element D_{ii} is the degree of the corresponding vertex v_i . We only consider $2d$ -regular graphs, so that $D = 2dI_n$, where I_n is the $n \times n$ identity matrix. The Laplacian matrix Δ_G of a $2d$ -regular graph G on n vertices is defined by

$$\Delta_G = 2dI_n - A.$$

The matrix tree theorem states that

$$\tau(G) = \frac{1}{n} \det^* \Delta_G \tag{1.1}$$

where $\det^* \Delta_G$ denotes the product of the nonzero eigenvalues of the Laplacian matrix of G . In this paper, we prove closed formulas for $\tau(G)$ for two types of graphs in terms

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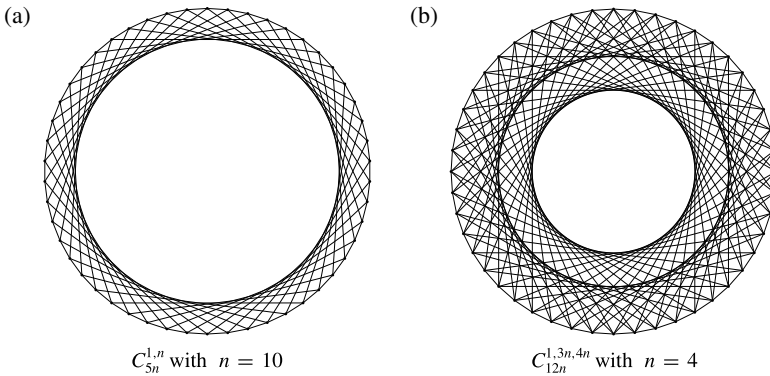


FIGURE 1. Examples of circulant graphs.

of eigenvalues of the Laplacian on a subgraph of G . The formulas are particularly interesting when the number of vertices is larger than the other parameters of the graph.

Let $1 \leq \gamma_1 \leq \dots \leq \gamma_d \leq \lfloor n/2 \rfloor$ be positive integers. A circulant graph $C_n^{\gamma_1, \dots, \gamma_d}$ is the $2d$ -regular graph with n vertices labelled $0, 1, \dots, n - 1$ such that each vertex $v \in \mathbb{Z}/n\mathbb{Z}$ is connected to $v \pm \gamma_i \pmod n$ for all $i \in \{1, \dots, d\}$. The first type of graph to be studied is the circulant graph with the first generator equal to 1 and the $d - 1$ others linearly depending on the number of vertices, that is, $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$, where $1 \leq \gamma_1 \leq \dots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor$ and β are integers. Two examples are illustrated in Figure 1. It is known that the number of spanning trees in circulant graphs with n vertices satisfies a linear recurrence relation with constant coefficients in n (see [4]). For $\beta \in \{2, 3, 4, 6, 12\}$, closed formulas have been obtained by Zhang *et al.* [8], using techniques inspired by Boesch and Prodinger [2] and properties of Chebyshev polynomials. As noted in [8], this method does not work for other values of β . In Section 2, we derive a theorem (Theorem 2.1) in a simple way which gives a closed formula for all integer values of β . This gives an answer to an open question in [3] and [8] and proves the conjecture stated in [6]. The second type of graph studied is the d -dimensional discrete torus defined by the quotient $\mathbb{Z}^d / \Lambda \mathbb{Z}^d$ with nearest neighbours connected, where Λ is a diagonal integer matrix. In the final section, we deduce the tree entropy of a sequence of circulant graphs with nonfixed generators and compare it to the one with fixed generators.

2. Spanning trees in circulant graphs with nonfixed generators

Let $V(G)$ be the set of vertices of a graph G and $f : V(G) \rightarrow \mathbb{R}$ a function. To derive the eigenvalues of the Laplacian it is more convenient to use the variant definition as an operator acting on the space of functions, that is,

$$\Delta_G f(x) = \sum_{y \sim x} (f(x) - f(y))$$

where the sum is over all vertices adjacent to x . Since the circulant graph $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$ is the Cayley graph of the group $\mathbb{Z}/\beta n\mathbb{Z}$, the eigenvectors of the Laplacian are given

by the characters

$$\chi_k(x) = e^{2\pi i k x / \beta n}, \quad k = 0, 1, \dots, \beta n - 1,$$

where $x \in \mathbb{Z} / \beta n \mathbb{Z}$. As in [1, Proposition 3.5], the eigenvalues are given by

$$\lambda_k = 2d - 2 \cos(2\pi k / \beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta), \quad k = 0, 1, \dots, \beta n - 1. \quad (2.1)$$

THEOREM 2.1. *Let $1 \leq \gamma_1 \leq \dots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor$ be positive integers and $\mu_k = 2(d-1) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta)$, $k = 1, \dots, \beta - 1$, be the nonzero eigenvalues of the Laplacian on the circulant graph $C_\beta^{\gamma_1, \dots, \gamma_{d-1}}$. For all $n \in \mathbb{N}_{\geq 1}$, the number of spanning trees in the circulant graph $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$ is given by*

$$\begin{aligned} \tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}) &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(\left(\frac{\mu_k}{2} + 1 + \sqrt{\mu_k^2 / 4 + \mu_k} \right)^n + \left(\frac{\mu_k}{2} + 1 - \sqrt{\mu_k^2 / 4 + \mu_k} \right)^n - 2 \cos\left(\frac{2\pi k}{\beta}\right) \right). \end{aligned}$$

REMARK 2.2. It would be interesting to see if this pattern appears in other types of graphs, that is, with the number of spanning trees expressed in terms of the eigenvalues of the Laplacian on a subgraph of the original graph.

PROOF. Applying the matrix tree theorem (1.1) to the graph $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$, with eigenvalues given by (2.1), gives

$$\tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}) = \frac{1}{\beta n} \prod_{k=1}^{\beta n-1} \left(2d - 2 \cos(2\pi k / \beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta) \right).$$

Since there are n spanning trees in the cycle C_n^1 ,

$$n = \tau(C_n^1) = \frac{1}{n} \prod_{k=1}^{n-1} (2 - 2 \cos(2\pi k / n)). \quad (2.2)$$

The product over $k = 1, \dots, \beta n - 1$ can be split as a product over multiples of β , that is, $k = \beta k'$ with $k' = 1, \dots, n - 1$, and over nonmultiples of β , that is, $k = k' + l\beta$ with $k' = 1, \dots, \beta - 1$ and $l = 0, 1, \dots, n - 1$. The product over the multiples of β reduces to (2.2), so it follows that

$$\begin{aligned} \tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}) &= \frac{n}{\beta} \prod_{\substack{k=1 \\ \beta \nmid k}}^{\beta n-1} \left(2d - 2 \cos(2\pi k / \beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta) \right) \\ &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \prod_{l=0}^{n-1} \left(2d - 2 \cos(2\pi(k + l\beta) / \beta n) - 2 \sum_{m=1}^{d-1} \cos(2\pi(k + l\beta) \gamma_m / \beta) \right) \end{aligned}$$

$$= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \prod_{l=0}^{n-1} \left(2 \cosh \left(\operatorname{argcosh} \left(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta) \right) \right) - 2 \cos(2\pi k / \beta n + 2\pi l / n) \right). \tag{2.3}$$

We now evaluate the product over l by the following calculation:

$$\begin{aligned} \prod_{l=0}^{n-1} (2 \cosh \theta - 2 \cos((\omega + 2\pi l) / n)) &= e^{-n\theta} \prod_{l=0}^{n-1} (e^{2\theta} - 2 \cos((\omega + 2\pi l) / n) e^\theta + 1) \\ &= e^{-n\theta} \prod_{l=0}^{n-1} (e^\theta - e^{i(\omega+2\pi l)/n})(e^\theta - e^{-i(\omega+2\pi l)/n}). \end{aligned} \tag{2.4}$$

The complex numbers $e^{i(\omega+2\pi l)/n}$ and $e^{-i(\omega+2\pi l)/n}$, for $l = 0, 1, \dots, n - 1$, are the $2n$ roots of the following polynomial in e^θ :

$$e^{2n\theta} - 2e^{n\theta} \cos \omega + 1 = 0.$$

Therefore the product (2.4) is equal to

$$e^{-n\theta} (e^{2n\theta} - 2e^{n\theta} \cos \omega + 1) = 2 \cosh(n\theta) - 2 \cos \omega.$$

Using this relation in (2.3) with $\theta = \operatorname{argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta))$ and $\omega = 2\pi k / \beta$, we have

$$\tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}) = \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(2 \cosh \left(n \operatorname{argcosh} \left(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta) \right) \right) - 2 \cos(2\pi k / \beta) \right). \tag{2.5}$$

The theorem then follows by expressing the formula in terms of the eigenvalues on $C_{\beta}^{\gamma_1, \dots, \gamma_{d-1}}$ and using the relation $\operatorname{argcosh} x = \log(x + \sqrt{x^2 - 1})$ for $x \geq 1$. Indeed, writing $\mu_k = 2(d - 1) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta)$,

$$\begin{aligned} \tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}) &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} (2 \cosh(n \operatorname{argcosh}(1 + \mu_k / 2)) - 2 \cos(2\pi k / \beta)) \\ &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(2 \cosh \left(n \log \left(1 + \mu_k / 2 + \sqrt{\mu_k^2 / 4 + \mu_k} \right) \right) - 2 \cos(2\pi k / \beta) \right) \\ &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} \left(\left(\mu_k / 2 + 1 + \sqrt{\mu_k^2 / 4 + \mu_k} \right)^n \right. \\ &\quad \left. + \left(\mu_k / 2 + 1 - \sqrt{\mu_k^2 / 4 + \mu_k} \right)^n - 2 \cos(2\pi k / \beta) \right). \quad \square \end{aligned}$$

REMARK 2.3. The techniques used here to derive Theorem 2.1 might not be generalisable to circulant graphs with two or more fixed generators. As an example, to

compute the number of spanning trees in the graph $C_{\beta n}^{1,2,\gamma n}$ we would need to find a closed formula for the product

$$\prod_{l=0}^{n-1} (2 \cosh \theta - 2 \cos((\omega + 2\pi l)/n) - 2 \cos(2(\omega + 2\pi l)/n))$$

where $\theta = \operatorname{argcosh}(3 - \cos(2\pi k\gamma/\beta))$ and $\omega = 2\pi k/\beta$. We were not able to do that.

EXAMPLE 2.4. The formula of Theorem 2.1 reproves Theorems 4, 5, 6, 8 and corrects a typographical error in Theorem 7 in [8]. For example, [8, Theorem 5] states that

$$\tau(C_{3n}^{1,n}) = \frac{n}{3} [(\sqrt{7/4} + \sqrt{3/4})^{2n} + (\sqrt{7/4} - \sqrt{3/4})^{2n} + 1]^2$$

which is a particular case of the formula with $d = 2$, $\gamma_1 = 1$, $\beta = 3$, on noting that $\mu_k = 2 - 2 \cos(2\pi k/3)$, $k = 1, 2$, are the nonzero eigenvalues on the cycle C_3^1 . As another example, [8, Theorem 8] states that

$$\tau(C_{6n}^{1,2n,3n}) = \frac{n}{6} [(\sqrt{11/4} + \sqrt{7/4})^{2n} + (\sqrt{11/4} - \sqrt{7/4})^{2n} - 1]^2 [(\sqrt{2} + 1)^n + (\sqrt{2} - 1)^n]^2 \times [(\sqrt{7/4} + \sqrt{3/4})^{2n} + (\sqrt{7/4} - \sqrt{3/4})^{2n} + 1]^2$$

which is a particular case of the formula with $d = 3$, $\gamma_1 = 2$, $\gamma_2 = 3$, $\beta = 6$ and $\mu_k = 4 - 2 \cos(2\pi k/3) - 2 \cos(\pi k)$, $k = 1, \dots, 5$, being the nonzero eigenvalues on the circulant graph $C_6^{2,3}$.

REMARK 2.5. We emphasise that the circulant graph $C_{\beta n}^{1,\gamma_1 n, \dots, \gamma_{d-1} n}$ consists of n copies of $C_{\beta}^{\gamma_1, \dots, \gamma_{d-1}}$ which are embedded in the cycle $C_{\beta n}^1$. This explains the eigenvalues on $C_{\beta}^{\gamma_1, \dots, \gamma_{d-1}}$ appearing in the formula.

3. Spanning trees in discrete tori

In this section we establish a formula for the number of spanning trees in the discrete torus $\mathbb{Z}^d / \Lambda \mathbb{Z}^d$ with nearest neighbours connected, where $\Lambda = \operatorname{diag}(\alpha_1, \dots, \alpha_{d-1}, n)$ is a diagonal matrix with positive integer coefficients. Let $k = (k_1, \dots, k_d)$, $x = (x_1, \dots, x_d) \in \mathbb{Z}^d / \Lambda \mathbb{Z}^d$ and $k_{\Lambda} = \Lambda^{-1}k$. The eigenvectors of the Laplacian are given by

$$g_{k_{\Lambda}}(x) = e^{2\pi i \langle k_{\Lambda}, x \rangle}$$

where $\langle \cdot, \cdot \rangle$ denotes the usual inner product. Denote by e_i , $i = 1, \dots, d$, the canonical basis of \mathbb{Z}^d . Since each vertex $x \in \mathbb{Z}^d / \Lambda \mathbb{Z}^d$ is connected to its nearest neighbours, that is, x is adjacent to $x - e_i$ and $x + e_i$, for all $i = 1, \dots, d$, we obtain the eigenvalues on $\mathbb{Z}^d / \Lambda \mathbb{Z}^d$ by applying the Laplacian on the eigenvectors $g_{k_{\Lambda}}(x)$:

$$\lambda_k = 2d - 2 \sum_{i=1}^{d-1} \cos(2\pi k_i / \alpha_i) - 2 \cos(2\pi k_d / n) \quad \text{where } k \in \mathbb{Z}^d / \Lambda \mathbb{Z}^d.$$

The formula given in the following theorem is interesting when n is larger than $\det(\Lambda)$. It improves the asymptotic result given in [6, Example 4.4.3].

THEOREM 3.1. Let $A = \text{diag}(\alpha_1, \dots, \alpha_{d-1})$ and, for $\ell = 1, \dots, \det(A) - 1$, let

$$\{\mu_\ell\}_\ell = \left\{ 2(d-1) - 2 \sum_{i=1}^{d-1} \cos\left(\frac{2\pi k_i}{\alpha_i}\right) : k_i = 0, 1, \dots, \alpha_i - 1, (k_1, \dots, k_{d-1}) \neq 0 \right\}$$

be the nonzero eigenvalues of the Laplacian on $\mathbb{Z}^{d-1}/A\mathbb{Z}^{d-1}$. For all $n \in \mathbb{N}_{\geq 1}$, the number of spanning trees in the discrete torus $\mathbb{Z}^d/\Lambda\mathbb{Z}^d$ is given by

$$\tau(\mathbb{Z}^d/\Lambda\mathbb{Z}^d) = \frac{n}{\det(A)} \prod_{\ell=1}^{\det(A)-1} \left(\left(\frac{\mu_\ell}{2} + 1 + \sqrt{\mu_\ell^2/4 + \mu_\ell} \right)^n + \left(\frac{\mu_\ell}{2} + 1 - \sqrt{\mu_\ell^2/4 + \mu_\ell} \right)^n - 2 \right).$$

PROOF. From the matrix tree theorem,

$$\begin{aligned} \tau(\mathbb{Z}^d/\Lambda\mathbb{Z}^d) &= \frac{1}{\det(A)n} \prod_{i=1}^{d-1} \prod_{\substack{k_i=0 \\ (k_1, \dots, k_d) \neq 0}}^{\alpha_i-1} \prod_{k_d=0}^{n-1} \left(2d - 2 \sum_{i=1}^{d-1} \cos(2\pi k_i/\alpha_i) - 2 \cos(2\pi k_d/n) \right) \\ &= \frac{n}{\det(A)} \prod_{i=1}^{d-1} \prod_{\substack{k_i=0 \\ (k_1, \dots, k_{d-1}) \neq 0}}^{\alpha_i-1} \prod_{k_d=0}^{n-1} \left(2 \cosh\left(\text{argcosh}\left(d - \sum_{i=1}^{d-1} \cos(2\pi k_i/\alpha_i)\right)\right) - 2 \cos(2\pi k_d/n) \right) \\ &= \frac{n}{\det(A)} \prod_{i=1}^{d-1} \prod_{\substack{k_i=0 \\ (k_1, \dots, k_{d-1}) \neq 0}}^{\alpha_i-1} \left(2 \cosh\left(n \text{argcosh}\left(d - \sum_{i=1}^{d-1} \cos(2\pi k_i/\alpha_i)\right)\right) - 2 \right) \end{aligned}$$

where the second equality comes from (2.2) and the third equality comes from the same device as in the proof of Theorem 2.1, namely

$$\prod_{k=0}^{n-1} (2 \cosh \theta - 2 \cos(2\pi k/n)) = 2 \cosh(n\theta) - 2.$$

The theorem then follows by expressing the formula in terms of the eigenvalues on $\mathbb{Z}^{d-1}/A\mathbb{Z}^{d-1}$ and from the relation $\text{argcosh } x = \log(x + \sqrt{x^2 - 1})$, for $x \geq 1$. \square

4. Spanning tree entropy of circulant graphs

For a sequence of regular graphs G_n with vertex set $V(G_n)$, one can consider the number of spanning trees as a function of n . Assuming that the limit

$$z = \lim_{n \rightarrow \infty} \frac{\log \tau(G_n)}{|V(G_n)|}$$

exists, it is sometimes called the associated tree entropy [7]. From Theorem 2.1, the tree entropy of the circulant graph $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$ with nonfixed generators as $n \rightarrow \infty$, denoted by $z_{\text{NF}}(\beta; \gamma_1, \dots, \gamma_{d-1})$, is given in the following corollary.

COROLLARY 4.1. *Let $1 \leq \gamma_1 \leq \dots \leq \gamma_{d-1} \leq \lfloor \beta/2 \rfloor$ and β be positive integers. The tree entropy of the circulant graph $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$ as $n \rightarrow \infty$ is given by*

$$\begin{aligned} z_{\text{NF}}(\beta; \gamma_1, \dots, \gamma_{d-1}) &= \frac{1}{\beta} \sum_{k=1}^{\beta-1} \operatorname{argcosh} \left(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta) \right) \\ &= \int_0^\infty \left(e^{-t} - \frac{1}{\beta} \sum_{k=0}^{\beta-1} e^{-\mu_k t} e^{-2t} I_0(2t) \right) \frac{dt}{t} \end{aligned}$$

where $\mu_k = 2(d-1) - 2 \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta)$, $k = 0, 1, \dots, \beta-1$, are the eigenvalues of the Laplacian on the circulant graph $C_\beta^{\gamma_1, \dots, \gamma_{d-1}}$ and I_0 is the modified I -Bessel function of order zero.

PROOF. Let $f_k := \operatorname{argcosh}(1 + \mu_k/2) > 0$, $k = 1, \dots, \beta-1$. From (2.5), the number of spanning trees in $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$ is given by

$$\begin{aligned} \tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}) &= \frac{n}{\beta} \prod_{k=1}^{\beta-1} (e^{nf_k} + e^{-nf_k} - 2 \cos(2\pi k / \beta)) \\ &= \frac{n}{\beta} e^{n \sum_{k=1}^{\beta-1} f_k} \prod_{k=1}^{\beta-1} (1 + e^{-2nf_k} - 2 \cos(2\pi k / \beta) e^{-nf_k}). \end{aligned}$$

But

$$\lim_{n \rightarrow \infty} \log(1 + e^{-2nf_k} - 2 \cos(2\pi k / \beta) e^{-nf_k}) = 0 \quad \text{for } k = 1, \dots, \beta-1,$$

and so

$$\prod_{k=1}^{\beta-1} (1 + e^{-2nf_k} - 2 \cos(2\pi k / \beta) e^{-nf_k}) = e^{o(1)} \quad \text{as } n \rightarrow \infty.$$

Therefore, the asymptotic number of spanning trees in $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$ is given by

$$\tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}) = \frac{n}{\beta} e^{n \sum_{k=1}^{\beta-1} \operatorname{argcosh}(d - \sum_{m=1}^{d-1} \cos(2\pi k \gamma_m / \beta)) + o(1)} \quad \text{as } n \rightarrow \infty.$$

This proves the first equality. The second equality comes from [6, Proposition 2.4] which expresses the $\operatorname{argcosh}$ in terms of an integral of the modified I -Bessel function: for all $x \geq 2$,

$$\int_0^\infty (e^{-t} - e^{-xt} I_0(2t)) \frac{dt}{t} = \operatorname{argcosh}(x/2). \quad \square$$

As mentioned in Section 2, the circulant graph $C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n}$ consists of n copies of $C_\beta^{\gamma_1, \dots, \gamma_{d-1}}$ which are embedded in the cycle $C_{\beta n}^1$. This structure is reflected by the appearance of the term $\theta_{C_\beta^{\gamma_1, \dots, \gamma_{d-1}}}(t) e^{-2t} I_0(2t)$ in the asymptotic formula, where $\theta_{C_\beta^{\gamma_1, \dots, \gamma_{d-1}}}(t) = \sum_{k=0}^{\beta-1} e^{-\mu_k t}$ is the theta function on $C_\beta^{\gamma_1, \dots, \gamma_{d-1}}$ and $e^{-2t} I_0(2t)$ is the typical term appearing in the asymptotics of the number of spanning trees in the cycle. Indeed,

the tree entropy on the cycle is (see [6, Section 3.2])

$$z_{\text{cycle}} = \int_0^\infty (e^{-t} - e^{-2t} I_0(2t)) \frac{dt}{t} = 0.$$

Consider the sequence of circulant graphs $C_{\beta n}^{1,n,\gamma_1 n, \dots, \gamma_{d-1} n}$ when $n \rightarrow \infty$ with $z_{\text{NF}}(\beta; 1, \gamma_1, \dots, \gamma_{d-1})$ denoting the corresponding tree entropy. In the following proposition we show that it is greater than the entropy of circulant graphs with fixed generators.

PROPOSITION 4.2. *For all positive integers $\gamma_1, \dots, \gamma_d$, there exists an integer $B \geq 2$ such that for all $\beta \geq B$,*

$$z_{\text{NF}}(\beta; 1, \gamma_1, \dots, \gamma_{d-1}) > z_{\text{F}}(1, \gamma_1, \dots, \gamma_d)$$

where $z_{\text{F}}(1, \gamma_1, \dots, \gamma_d)$ is the tree entropy of the circulant graph $C_n^{1,\gamma_1, \dots, \gamma_d}$ with fixed generators.

PROOF. By letting $\beta \rightarrow \infty$ in the corollary, the sum over the Laplacian eigenvalues converges to a Riemann integral, so that

$$\lim_{\beta \rightarrow \infty} z_{\text{NF}}(\beta; 1, \gamma_1, \dots, \gamma_{d-1}) = \int_0^\infty (e^{-t} - e^{-2(d+1)t} I_0(2t) I_0^{1,\gamma_1, \dots, \gamma_{d-1}}(2t, \dots, 2t)) \frac{dt}{t}$$

where $I_0^{1,\gamma_1, \dots, \gamma_{d-1}}$ is the d -dimensional modified I -Bessel function of order zero defined by (see [6, Section 2.4])

$$I_0^{1,\gamma_1, \dots, \gamma_{d-1}}(2t, \dots, 2t) = \frac{1}{2\pi} \int_{-\pi}^\pi e^{2t(\cos w + \sum_{m=1}^{d-1} \cos(\gamma_m w))} dw.$$

It can be expressed in terms of a series of modified I -Bessel functions:

$$I_0^{1,\gamma_1, \dots, \gamma_{d-1}}(2t, \dots, 2t) = \sum_{(k_1, \dots, k_{d-1}) \in \mathbb{Z}^{d-1}} I_{\sum_{i=1}^{d-1} \gamma_i k_i}(2t) \prod_{i=1}^{d-1} I_{k_i}(2t).$$

On the other hand, from [6, Theorem 1.1], the tree entropy of the circulant graph $C_n^{1,\gamma_1, \dots, \gamma_d}$ with fixed generators as $n \rightarrow \infty$ is given by

$$z_{\text{F}}(1, \gamma_1, \dots, \gamma_d) = \int_0^\infty (e^{-t} - e^{-2(d+1)t} I_0^{1,\gamma_1, \dots, \gamma_d}(2t, \dots, 2t)) \frac{dt}{t}$$

where, for all $t > 0$,

$$\begin{aligned} I_0^{1,\gamma_1, \dots, \gamma_d}(2t, \dots, 2t) &= \sum_{(k_1, \dots, k_d) \in \mathbb{Z}^d} I_{\sum_{i=1}^d \gamma_i k_i}(2t) \prod_{i=1}^d I_{k_i}(2t) \\ &> I_0(2t) \sum_{(k_1, \dots, k_{d-1}) \in \mathbb{Z}^{d-1}} I_{\sum_{i=1}^{d-1} \gamma_i k_i}(2t) \prod_{i=1}^{d-1} I_{k_i}(2t) \\ &= I_0(2t) I_0^{1,\gamma_1, \dots, \gamma_{d-1}}(2t, \dots, 2t). \end{aligned}$$

Therefore $\lim_{\beta \rightarrow \infty} z_{\text{NF}}(\beta; 1, \gamma_1, \dots, \gamma_{d-1}) > z_{\text{F}}(1, \gamma_1, \dots, \gamma_d)$. □

Related to this comparison between circulant graphs with fixed and nonfixed generators, one might wonder, for example in the simplest case of $C_{\beta n}^{1,n}$, how taking limits first in β and then in n would compare to taking limits first in n and then in β . From [5, Lemma 5] and by letting $\beta \rightarrow \infty$ in [5, Theorem 4], one easily sees that for all positive integers $\gamma_1, \dots, \gamma_{d-1}$,

$$\lim_{\gamma_d \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\log \tau(C_n^{\gamma_1, \dots, \gamma_d})}{n} = \lim_{\beta \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\log \tau(C_{\beta n}^{1, \gamma_1 n, \dots, \gamma_{d-1} n})}{\beta n},$$

which by definition is

$$\lim_{\gamma_d \rightarrow \infty} z_F(\gamma_1, \dots, \gamma_d) = \lim_{\beta \rightarrow \infty} z_{NF}(\beta; \gamma_1, \dots, \gamma_{d-1}).$$

In the particular case of $d = 2$ it shows that the limits over n and β commute, that is,

$$\lim_{\beta \rightarrow \infty} \lim_{n \rightarrow \infty} \frac{\log \tau(C_{\beta n}^{1,n})}{\beta n} = \lim_{n \rightarrow \infty} \lim_{\beta \rightarrow \infty} \frac{\log \tau(C_{\beta n}^{1,n})}{\beta n},$$

which does not seem obvious *a priori*.

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