5

Quantum Field Theory and the Path Integral

5.1 Preliminaries

We will consider the case of a classical scalar field theory and its quantization. Later in this book we will consider both vector and spinor fields. A classical scalar field $\phi(x^\mu)$ is a real-valued function of the coordinates of space and time. The meaning that it is a scalar field is that the value that the function takes is invariant under Lorentz transformations. All inertial observers measure the same value for the field at a given spacetime point.

$$\phi(x^\mu) = \phi'(x'^\mu) \tag{5.1}$$

where

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \tag{5.2}$$

with the standard notation $x^0 = t$ and x^i, $i = 1, 2, \cdots d$ are the spatial coordinates. The transformation matrix $\Lambda^\mu{}_\nu$ satisfies

$$\Lambda^\mu{}_\nu \eta^{\nu\sigma} \Lambda^\tau{}_\sigma = \eta^{\mu\tau} \tag{5.3}$$

with $\text{diag}\,[\eta^{\nu\sigma}] = (1, -1, -1, \cdots)$ the usual Minkowski space metric, which is the defining condition for a Lorentz transformation. In general, an equation of motion for a classical scalar field is a non-linear partial differential equation. We will restrict ourselves to the case of second-order equations, then Lorentz invariance dictates the form

$$\partial_\nu \partial^\nu \phi(x^\mu) + V'(\phi(x^\mu)) = 0. \tag{5.4}$$

Written out, this equation is

$$\left(\frac{d^2}{dt^2} - \nabla^2\right)\phi(x^\mu) + V'(\phi(x^\mu)) = 0. \tag{5.5}$$

Such an equation comes from the variation of an action, $S[\phi(x^\mu)]$, which is a functional of the field $\phi(x^\mu)$, *i.e.*

$$\delta S[\phi(x^\mu)] = \left.\frac{S[\phi(x^\mu)+\epsilon\delta\phi(x^\mu)] - S[\phi(x^\mu)]}{\epsilon}\right|_{\epsilon=0} = 0$$
$$\forall\, \delta\phi(x^\mu) \Rightarrow \partial_\nu\partial^\nu\phi(x^\mu) + V'(\phi(x^\mu)) = 0. \tag{5.6}$$

Then we find the action giving rise to equations of motion, Equation (5.4), is given by

$$S[\phi(x^\mu)] = \int d^dx \left(\frac{1}{2}\partial_\nu\phi(x^\mu)\partial^\nu\phi(x^\mu) - V(\phi(x^\mu))\right) \equiv \int d^dx\mathcal{L} \tag{5.7}$$

where $\mathcal{L}$ is called the Lagrangian density. The kinetic energy is

$$T = \int d^{d-1}x \left(\frac{1}{2}\partial_t\phi(x^\mu)\partial_t\phi(x^\mu)\right) \tag{5.8}$$

while the potential energy is

$$V = \int d^{d-1}x \left(\frac{1}{2}\vec{\nabla}\phi(x^\mu)\cdot\vec{\nabla}\phi(x^\mu) + V(\phi(x^\mu))\right) \tag{5.9}$$

which define the Lagrangian as $L = T - V$ and the action is simply

$$S[\phi(x^\mu)] = \int dt\,(T - V). \tag{5.10}$$

This defines a dynamical system which is an exact analogy to the particle mechanical systems we have been considering in the previous chapters. There are just a few simple conceptual changes. The dynamical variable is a function of space, which evolves through time. For a mechanical system the variables were the positions of particles in space and these positions were evolving through time. Now the spatial coordinates x^i are not the positions of any particle. They are just parameters or labels, and they do not evolve in time. An important point to observe is that a dynamical variable which is a function of space, rather than a point in space, comprises an infinite number of degrees of freedom, in contradistinction to the case of particle mechanics where we typically consider only a finite number of particles. This is easy to make explicit by expanding the scalar field in terms of a fixed orthonormal basis of functions $\phi_n(x^i)$, $n = 0,1,2,\cdots$,

$$\phi(x^i,t) = \sum_{n=0}^{\infty} c_n(t)\phi_n(x^i). \tag{5.11}$$

We can thus exchange the dynamical field $\phi(x^i,t)$ for an infinite number of dynamical variables $\{c_n(t)\}_{n=0,1,\cdots,\infty}$.

This difference is the cause of almost all the problems that arise in the quantization of fields. We will proceed with the philosophy that these problems correspond to the extreme ultraviolet or infrared degrees of freedom, this philosophy perhaps to be justified only *a posteriori*. We plead ignorance as to what dynamics actually exist at extremely high energies and simply reject theories where the answers to questions involving processes at only low energies depend on the dynamics at very high energies! Furthermore, we invoke the principles of locality and causality, which stated simply means that configurations at the other end of the universe cannot affect the local dynamics here. In this way we consider only theories which are unaffected by cutting off the infrared degrees of freedom. Thus, effectively, we are interested in theories with an enormous but actually finite number of degrees of freedom, since we can cut the theory off in both the infrared and the ultraviolet. However, this number of degrees of freedom is assumed to be so huge that it is well-approximated by ∞, so long as that limit is sensible.

5.2 Canonical Quantization

5.2.1 Canonical Quantization of Particle Mechanics

The canonical quantization of fields proceeds formally as for particle mechanics. First we briefly review how it works for particle mechanics. We find the classical canonical variables p_i and q_i, $p_i = \frac{\partial L}{\partial \dot{q}_i}$ and the Hamiltonian $H = \sum_i p_i \dot{q}_i - L$. The equations of motion are:

$$\begin{aligned} \dot{q}_i &= \{q_i, h(q_j, p_k)\} \\ \dot{p}_i &= \{p_i, h(q_j, p_k)\} \end{aligned} \tag{5.12}$$

where $\{\cdot,\cdot\}$ is the Poisson bracket,

$$\{A,B\} = \sum_i \frac{\partial A}{\partial q_i}\frac{\partial B}{\partial p_i} - \frac{\partial A}{\partial p_i}\frac{\partial B}{\partial q_i}. \tag{5.13}$$

Quantization proceeds with the replacement

$$\{A,B\} \to -\frac{i}{\hbar}\left[\hat{A},\hat{B}\right] \tag{5.14}$$

yielding, for example, the canonical commutation relations:

$$[\hat{q}_i, \hat{p}_j] = i\hbar\delta_{i,j}. \tag{5.15}$$

All dynamical variables become operators, $O \to \hat{O}$, which act on vectors in a Hilbert space.

5.2.2 Canonical Quantization of Fields

Applying the above to the case of classical fields, we define the conjugate momenta in an analogous way,

$$\Pi(x^i,t) = \frac{\delta L}{\delta\dot{\phi}(x^i,t)}. \tag{5.16}$$

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Then

$$H = \int d^{d-1}x \left(\Pi(x^i,t)\dot{\phi}(x^i,t) - \mathcal{L}\right)$$
$$= \int d^{d-1}x \left(\frac{1}{2}\Pi^2(x^i,t) + \frac{1}{2}\vec{\nabla}\phi(x^i,t)\cdot\vec{\nabla}\phi(x^i,t) + V(\phi(x^i,t))\right). \quad (5.17)$$

The Poisson bracket is now given by (for local functions of $\phi(x^i), \Pi(x^i)$, we can dispense with the functional derivatives and just write partial derivatives, as they give the same answer)

$$\{A,B\} = \int d^{d-1}x \frac{\partial A}{\partial\phi(x^i,t)}\frac{\partial B}{\partial\Pi(x^i,t)} - \frac{\partial A}{\partial\Pi(x^i,t)}\frac{\partial B}{\partial\phi(x^i,t)} \quad (5.18)$$

which includes the fundamental Poisson brackets

$$\{\phi(x^i,t),\Pi(x^j,t)\} = \delta^{d-1}(x^i - x^j). \quad (5.19)$$

We impose the same quantization prescription as in the particle mechanics case, given by Equation (5.14). This yields the celebrated equal time canonical commutation relations

$$\left[\hat{\phi}(x^i,t),\hat{\Pi}(y^i,t)\right] = i\hbar\delta^{d-1}(x^i - y^i). \quad (5.20)$$

The (Heisenberg) equations of motion follow from the commutators:

$$i\hbar\frac{d}{dt}\hat{\phi}(x^i,t) = \left[\hat{\phi}(x^i,t),\hat{H}\right] \quad (5.21)$$
$$i\hbar\frac{d}{dt}\hat{\Pi}(x^i,t) = \left[\hat{\Pi}(x^i,t),\hat{H}\right] \quad (5.22)$$

There is a lot of mathematical subtlety in the definition of the product of the quantum field operators of a one-spacetime point which is required in the definition of the Lagrangian and Hamiltonian. Indeed, the quantum field operators that satisfy Equation (5.20) cannot be simple operators but in fact are operator-valued distributions. The operator products required to define the Lagrangian and the Hamiltonian are not straightforwardly well-defined. Canonical quantization can be made to work reasonably well for the case of linear field theories, for example see [107].

So far we have been considering the quantization in the Heisenberg picture. The variables are dynamical while the states are constant. We can equally well consider the quantization in the Schrödinger picture, with the transformation

$$\hat{\phi}(x^i,t) \to \hat{\phi}^S(x^i) = U(t)\hat{\phi}(x^i,t)U^\dagger(t)$$
$$\hat{\Pi}(x^i,t) \to \hat{\Pi}^S(x^i) = U(t)\hat{\Pi}(x^i,t)U^\dagger(t). \quad (5.23)$$

Then we find,

$$\frac{\partial\hat{\phi}^S(x^i)}{\partial t} = \frac{\partial\hat{\Pi}^S(x^i)}{\partial t} = 0, \quad (5.24)$$

i.e. the fundamental quantum fields in the Schrödinger picture are time-independent, if $U(t)$ satisfies

$$i\hbar\frac{d}{dt}U(t) = \hat{H}U(t). \tag{5.25}$$

The formal solution of this differential equation is $U(t) = e^{-it\hat{H}/\hbar}$. Evidently $\hat{H}$ commutes with $U(t)$. The corresponding transformation of the Hamiltonian yields

$$\hat{H} \to \hat{H}^S = U(t)\hat{H}U^\dagger(t) = \hat{H}. \tag{5.26}$$

This states that the Hamiltonian for time-independent problems does not depend on the representation. If we have an eigenstate of $\hat{H}$,

$$\hat{H}\left|\Psi\right\rangle = \mathcal{E}\left|\Psi\right\rangle \tag{5.27}$$

then

$$\hat{H}^S U(t)\left|\Psi\right\rangle = U(t)\hat{H}U^\dagger(t)U(t)\left|\Psi\right\rangle = \mathcal{E}U(t)\left|\Psi\right\rangle. \tag{5.28}$$

Thus

$$i\hbar\frac{d}{dt}\left(U(t)\left|\Psi\right\rangle\right) = U(t)\hat{H}\left|\Psi\right\rangle = \hat{H}^S\left(U(t)\left|\Psi\right\rangle\right) = \mathcal{E}\left(U(t)\left|\Psi\right\rangle\right) \tag{5.29}$$

which is just the Schrödinger equation.

5.3 Quantization via the Path Integral

Now the path integral for a quantum particle mechanics amplitude in Minkowski time, as given by Equation (2.42), yields

$$\langle y|\, e^{-\frac{iT\hat{h}(X.P)}{\hbar}}\,|x\rangle = \mathcal{N}\int_x^y \mathcal{D}z(t)e^{i\frac{S[z(t)]}{\hbar}}. \tag{5.30}$$

This formula was proven assuming nothing of the nature of the space in which x and y took their values. Typically they were coordinates in $\mathbf{R}^n$, but they could have been in any configuration space of unconstrained variables (with constraints additional terms can appear [76]). Actually we have

$$\langle q_f|\, e^{-\frac{iT\hat{H}(\hat{q}.\hat{p})}{\hbar}}\,|q_i\rangle = \mathcal{N}\int_{q_i}^{q_f} \mathcal{D}q(t)e^{i\frac{S[q(t)]}{\hbar}}, \tag{5.31}$$

where $q(t)$ could be any generalized coordinate, for example, an angular variable of a rotator or the radius of a bubble which changes its size.

Then, for quantum field theory, we simply let q take values in the space of configurations of a classical field. This gives

$$\langle \phi_f|\, e^{-\frac{iT\hat{H}\left(\hat{\phi},\hat{\Pi}\right)}{\hbar}}\,|\phi_i\rangle = \mathcal{N}\int_{\phi_i}^{\phi_f} \mathcal{D}\phi(x^\mu)e^{-i\frac{S[\phi(x^\mu)]}{\hbar}}. \tag{5.32}$$

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The states $|\phi_i\rangle$ and $|\phi_f\rangle$ correspond to a quantum field localized on the configurations $\phi_i(x^\mu)$ and $\phi_f(x^\mu)$, respectively. The states $|\phi\rangle$ are directly analogous to the states $|\vec{x}\rangle$ that we considered earlier in particle quantum mechanics. These were eigenstates of the (Schrödinger) position operator $\hat{\vec{X}}$

$$\hat{\vec{X}}\,|\vec{x}\rangle = \vec{x}\,|\vec{x}\rangle\,. \tag{5.33}$$

In that respect, the states $|\phi\rangle$ are taken to be eigenstates of the field operator

$$\hat{\phi}^S(x^i)\,|\phi\rangle = \phi(x^i)\,|\phi\rangle\,. \tag{5.34}$$

The states $|\phi\rangle$ are also improper vectors, as the states $|\vec{x}\rangle$ were, and true states are obtained by smearing with some profile function

$$|F\rangle = \int \mathcal{D}\phi F(\phi)\,|\phi\rangle \tag{5.35}$$

where $F(\phi)$ is a functionally square integrable functional. The inner product is defined by

$$\langle F|\,G\rangle = \int \mathcal{D}\phi F^*(\phi)G(\phi). \tag{5.36}$$

We call the Feynman path integral in this case the functional integral. It is a rather formal object in Minkowski space, but it can be used to generate the usual perturbative expansion of matrix elements, in a rather efficient manner. (Its analogue in Euclidean space, which we will use, can be rigorously defined in some cases.)

5.3.1 The Gaussian Functional Integral

We can essentially perform only one functional integral and that, too, not necessarily in closed form. This is the Gaussian functional integral. However, if we can do the Gaussian functional integral it is sufficient to generate the perturbative expansion. Consider the functional $W[J]$ of some external source field $J(x^\mu)$ defined by

$$\begin{aligned} W[J] &= \mathcal{N}\int \mathcal{D}\phi e^{\frac{i}{\hbar}\int d^dx\left(\frac{1}{2}\partial_\mu\phi(x^i,t)\partial^\mu\phi(x^i,t)-\frac{1}{2}m^2\phi^2(x^i,t)-V\left(\phi(x^i,t)\right)+J(x^i,t)\phi(x^i,t)\right)} \\ &\equiv \sum_{N=0}^{\infty}\frac{i^N}{\hbar^N N!}\int d^dx_1\cdots d^dx_N J(x_1)\cdots J(x_N)G^N(x_1,\cdots,x_N), \end{aligned} \tag{5.37}$$

where the integrations are done over all of spacetime and we impose the boundary conditions on the field the $\phi(x^\mu)\to 0$ as $|x^\mu|\to\infty$. Then the so-called N point Green functions of the theory are obtained via functional differentiation

$$G^N(x_1,\cdots,x_N) = \left(\frac{\hbar}{i}\right)^N\left(\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_N)}\right)W[J]\bigg|_{J=0}. \tag{5.38}$$

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Correspondingly, $W[J]$ is called the generating functional since it can be used to generate all the Green functions of the theory. We will show that the $G^N(x_1,\cdots,x_N)$ corresponds in principle to the matrix elements

$$\langle 0|T\left(\hat{\phi}^H(x_1)\cdots\hat{\phi}^H(x_N)\right)|0\rangle \tag{5.39}$$

where the state $|0\rangle$ is the eigenstate of the Schrödinger field operator with eigenvalue $\phi(x^i)=0$, *i.e.* $\hat{\phi}^S(x^i)|0\rangle=0$.

For a Hamiltonian that depends on time, $\hat{H}^S(t)$, which is the case here with an arbitrary external source $J(x^\mu)$,

$$\hat{H}^S(t)=\hat{H}^0+\hat{H}^{int.} \tag{5.40}$$

with

$$\hat{H}^0=\int d^{d-1}x\left(\frac{1}{2}\hat{\Pi}(x^i)\hat{\Pi}(x^i)+\frac{1}{2}\vec{\nabla}\hat{\phi}^S(x^i)\cdot\vec{\nabla}\hat{\phi}^S(x^i)+V\left(\hat{\phi}^S(x^i)\right)\right) \tag{5.41}$$

and

$$\hat{H}^{int.}(t)=\int d^{d-1}x\left(J(x^i,t)\hat{\phi}^S(x^i)\right), \tag{5.42}$$

one can easily prove that the path integral gives rise to

$$\mathcal{N}\int\mathcal{D}z(t)e^{\frac{i}{\hbar}S[z(t)]}=\lim_{T\to\infty}\langle y|\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt\hat{H}^S(t)}\right)|x\rangle \tag{5.43}$$

where $\mathbf{T}(A(t_1)B(t_2))=\theta(t_1-t_2)A(t_1)B(t_2)+\theta(t_2-t_1)B(t_2)A(t_1)$, the usual time-ordered product. The time-ordered product here yields the limiting value of the (infinite) ordered product of infinitesimal unitary time translations over each of N infinitesimal time elements, $\epsilon=T/N$ between $-T/2$ and $T/2$, ordered so that the latest time occurs to the left

$$\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\frac{T}{2}}^{\frac{T}{2}}dt\hat{H}^S(t)}\right)$$
$$=\lim_{N\to\infty}e^{-\frac{i}{\hbar}\epsilon\hat{H}^S(\frac{T}{2})}e^{-\frac{i}{\hbar}\epsilon\hat{H}^S(\frac{T}{2}-\epsilon)}\cdots e^{-\frac{i}{\hbar}\epsilon\hat{H}^S(-\frac{T}{2}+2\epsilon)}e^{-\frac{i}{\hbar}\epsilon\hat{H}^S(-\frac{T}{2}+\epsilon)}. \tag{5.44}$$

The Hamiltonian being time-dependent because of the, in principle, time-dependent external source $J(x^i,t)$. The derivation of the path integral goes through as before by inserting a complete set of states between the infinitesimal unitary transformations. (There is a completely analogous expression for the case of the Euclidean path integral, where the time-ordering is replaced by Euclidean time-ordering, which is sometimes called path-ordering.) Thus we find with

$$W[J]=\lim_{T\to\infty}\langle 0|\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt\hat{H}^S(t)}\right)|0\rangle \tag{5.45}$$

then

$$
\begin{aligned}
&\frac{-i\hbar\delta}{\delta J(x_1)} \cdots \frac{-i\hbar\delta}{\delta J(x_N)} W[J]\Big|_{J=0} = \\
&\quad = \frac{-i\hbar\delta}{\delta J(x_1)} \cdots \frac{-i\hbar\delta}{\delta J(x_N)} \langle 0|\,\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{t_1}^{\infty} dt \hat{H}^S(t)}\right) \\
&\quad\quad \times \mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{t_2}^{t_1} dt \hat{H}^S(t)}\right) \cdots \mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{t_N} dt \hat{H}^S(t)}\right) |0\rangle\Big|_{J=0} \\
&\quad\quad \text{for} \quad t_1 > t_2 > \cdots > t_N \\
&\quad = \langle 0|\,\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{t_1}^{\infty} dt \hat{H}^S(t)}\right) \hat{\phi}^S(x_1^i)\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{t_2}^{t_1} dt \hat{H}^S(t)}\right) \hat{\phi}^S(x_2^i)\cdots\hat{\phi}^S(x_N^i) \\
&\quad\quad \times \mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{t_N} dt \hat{H}^S(t)}\right) |0\rangle\Big|_{J=0} \\
&\quad = \langle 0|\,\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \hat{H}^S(t)}\right) \mathbf{T}\left(e^{\frac{i}{\hbar}\int_{-\infty}^{t_1} dt \hat{H}^S(t)}\right) \hat{\phi}^S(x_1^i)\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{t_1} dt \hat{H}^S(t)}\right) \\
&\quad\quad \times \mathbf{T}\left(e^{\frac{i}{\hbar}\int_{-\infty}^{t_2} dt \hat{H}^S(t)}\right) \hat{\phi}^S(x_2^i)\cdots\hat{\phi}^S(x_N^i)\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{t_N} dt \hat{H}^S(t)}\right) |0\rangle\Big|_{J=0} \\
&\quad = \langle 0|\,\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{\infty} dt \hat{H}^S(t)}\right) \hat{\phi}^H(x_1^\mu)\hat{\phi}^H(x_2^\mu)\cdots\hat{\phi}^H(x_N^\mu)\,|0\rangle\Big|_{J=0} \\
&\quad \to \langle E=0|\,\mathbf{T}\left(\hat{\phi}^H(x_1^\mu)\hat{\phi}^H(x_2^\mu)\cdots\hat{\phi}^H(x_N^\mu)\right)|E=0\rangle\Big|_{J=0},
\end{aligned}
\tag{5.46}
$$

where we have explicitly written the Heisenberg fields as $\hat{\phi}^H(x^\mu) = \mathbf{T}\left(e^{\frac{i}{\hbar}\int_{-\infty}^{t} dt' \hat{H}^S(t')}\right) \hat{\phi}^S(x^i)\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-\infty}^{t} dt' \hat{H}^S(t')}\right)$ while the Schrödinger operators are defined with respect to $t=-\infty$. Here $|0\rangle$ still corresponds to the state with $\phi(x)=0$ while the state $|E=0\rangle$ corresponds to the true zero-energy vacuum state. However, the last identification in Equation (5.46) requires explanation as it is not exactly the same as Equation (5.39). As we will see, once we define the functional integral more carefully, instead of computing the matrix element in Equation (5.39), the functional integral projects uniformly onto that which corresponds to the matrix element in the state of zero energy, the vacuum state. At the present juncture the definition of the functional integration is extremely formal, and neither the operator-valued matrix element in Equation (5.39) nor its functional integral representation exist.

If we nevertheless continue formally, we find

$$
\begin{aligned}
W[J] &= \mathcal{N}\int \mathcal{D}\phi e^{\frac{-i}{\hbar}\int d^dx V\left(-i\hbar\frac{\delta}{\delta J(x)}\right)} \times \\
&\quad \times e^{\frac{i}{\hbar}\int d^dx\left(\frac{1}{2}\partial_\mu\phi(x^i,t)\partial^\mu\phi(x^i,t)-\frac{1}{2}m^2\phi^2(x^i,t)+J(x^i,t)\phi(x^i,t)\right)} \\
&= e^{\frac{-i}{\hbar}\int d^dx V\left(-i\hbar\frac{\delta}{\delta J(x)}\right)} W^0[J].
\end{aligned}
\tag{5.47}
$$

$W^0[J]$ is a Gaussian functional integral, which we can explicitly perform. We use the formula, which as written is only formal but becomes valid if defined via an

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appropriate analytic continuation

$$\int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{i\frac{1}{2}\left(ax^2+2bx\right)} = \int_{-\infty}^{\infty} \frac{dx}{\sqrt{2\pi}} e^{i\frac{a}{2}\left(x-\frac{b}{a}\right)^2} e^{-i\frac{1}{2}b\left(\frac{1}{a}\right)b} = \frac{1}{\sqrt{-ia}} e^{-i\frac{1}{2}b\left(\frac{1}{a}\right)b} \tag{5.48}$$

which generalizes to

$$\int \frac{d^n x}{(2\pi)^{\frac{n}{2}}} e^{i\frac{1}{2}\left((\vec{x},A\cdot\vec{x})+2(\vec{b},\vec{x})\right)} = (\det(-iA))^{-\frac{1}{2}} e^{-i\frac{1}{2}\left((\vec{b},A^{-1}\cdot\vec{b}\right)} \tag{5.49}$$

for finite dimensional matrices. Boldly generalizing to the infinite dimensional case, for $W^0[J]$ we find, with $A \to -\left(\partial_\mu\partial^\mu + m^2\right)$ and $b \to J$ (and absorbing an infinite product of i's into the normalization constant),

$$W^0[J] = \frac{\mathcal{N}}{\sqrt{\det\left(\partial_\mu\partial^\mu + m^2\right)}} e^{-\frac{i}{2}\int d^dx d^dy \left(J(x)\,\langle x|\frac{1}{-\left(\partial_\mu\partial^\mu+m^2\right)}|y\rangle\, J(y)\right)} \tag{5.50}$$

5.3.2 The Propagator

It only remains to calculate

$$\langle x| \frac{1}{-\left(\partial_\mu\partial^\mu + m^2\right)} |y\rangle = \int \frac{d^dk}{(2\pi)^d} e^{-ik_\mu(x-y)^\mu} \frac{1}{k_\mu k^\mu - m^2}. \tag{5.51}$$

We seem to be on the right path to defining the functional integral; however, we come up against another problem: this Green function is ambiguous. This problem is only solved via analytic continuation. In the Fourier representation, for example, there are poles in the k_0 integration at $k_0 = \pm\sqrt{|\vec{k}|^2 + m^2}$. We cannot integrate through the poles, we must provide a prescription for integrating around them. Such a prescription translates directly into fixing the asymptotic boundary condition on the solutions of the problem, for ϕ

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi = J. \tag{5.52}$$

Clearly any solution for ϕ is ambiguous up to a solution of the homogeneous equation

$$\left(\partial_\mu\partial^\mu + m^2\right)\phi_0 = 0. \tag{5.53}$$

Correspondingly, the Green function to Equation (5.52) is also ambiguous by the addition of an arbitrary solution of the homogeneous equation. The asymptotic boundary conditions on ϕ fix the Green function. These boundary conditions are equivalent to giving the pole prescription.

5.3.3 Analytic Continuation to Euclidean Time

The existence of homogeneous solutions corresponds to zero modes in the operator $A=-\left(\partial_\mu\partial^\mu+m^2\right)$; hence, the original integral was ill-defined. The problem can be traced back to the matrix element

$$W[J]=\lim_{T\to\infty}\langle 0|\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt\hat{H}^S(t)}\right)|0\rangle\,. \tag{5.54}$$

The operator in the matrix element can be written, for an arbitrary future time t,

$$\begin{aligned}\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^S(t')}\right)&=e^{-\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^0}e^{\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^0}\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^S(t')}\right)\\&\equiv e^{-\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^0}U(t,-T/2).\end{aligned} \tag{5.55}$$

Then $U(t,-T/2)$ satisfies the differential equation

$$\begin{aligned}i\hbar\frac{\partial U(t,-T/2)}{\partial t}&=e^{-\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^0}\hat{H}^{int.}(t)e^{\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^0}U(t,-T/2)\\&\equiv\hat{H}^I(t)U(t,-T/2)\end{aligned} \tag{5.56}$$

where

$$\begin{aligned}\hat{H}^I(t)&=\int d^{d-1}xJ(x^i,t)e^{-\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^0}\hat{\phi}^S(x^i)e^{\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^0}\\&\equiv\int d^{d-1}xJ(x^i,t)\hat{\phi}^I(x^i,t)\end{aligned} \tag{5.57}$$

defines the interaction representation Hamiltonian and the interaction representation field $\hat{\phi}^I(x^i,t)$. The solution of the differential Equation (5.56) is unique with boundary condition $U(-T/2,-T/2)=1$ and given by

$$U(t,-T/2)=\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{t}dt'\hat{H}^I(t')}\right). \tag{5.58}$$

Thus

$$W[J]=\lim_{T\to\infty}\langle 0|e^{-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt'\hat{H}^0}\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt'\hat{H}^I(t')}\right)|0\rangle. \tag{5.59}$$

The state $|0\rangle$ corresponds to an eigenstate of the Schrödinger field operator with the eigenvalue zero, and is not an energy eigenstate of the Hamiltonian, hence

$$|0\rangle=\sum_E C_E|E\rangle \tag{5.60}$$

where

$$\hat{H}^0|E\rangle=E|E\rangle\,. \tag{5.61}$$

Then the matrix element in Equation (5.59) is given by

$$W[J]=\lim_{T\to\infty}\sum_{E,E'}e^{-\frac{i}{\hbar}TE'}C_{E'}^*C_E\langle E'|\mathbf{T}\left(e^{-\frac{i}{\hbar}\int_{-T/2}^{T/2}dt'\hat{H}^I(t')}\right)|E\rangle \tag{5.62}$$

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This expression is generally not well-defined. The infinite phases give an ever-oscillatory contribution which does not exist in the limit $T\to\infty$. We are in fact interested in the matrix element and its various moments which give rise to the Green functions, as $J\to 0$. Even in this limit, we get that $W[J=0]$ is ill-defined; if any of the $C_E\neq 0$ for any $E\neq 0$, then

$$W[J=0]\to\sum_E e^{-\frac{i}{\hbar}(\infty)E}|C_E|^2. \tag{5.63}$$

Thus, somehow we must project onto the ground state, defined to have $E=0$. This would happen if we can add a negative imaginary part to E. Equivalently, if we rotate

$$t\to\tau=-it \quad d^dx\to -id^dx \tag{5.64}$$

the action goes to

$$S\to iS^E=i\int d^dx\left(\frac{1}{2}\left(\partial_\mu\phi\partial_\mu\phi+m^2\right)+V(\phi)-J\phi\right), \tag{5.65}$$

and the matrix element is

$$\langle 0|T\left(e^{-\frac{1}{\hbar}\int_{-\infty}^{\infty}dt\hat{H}(t)}\right)|0\rangle\sim\langle E=0|T\left(e^{-\frac{1}{\hbar}\int_{-\infty}^{\infty}dt\hat{H}(t)}\right)|E=0\rangle\,. \tag{5.66}$$

$|E=0\rangle$ is the zero-energy vacuum state of the theory with $J=0$. Then the functional integral gives

$$\mathcal{N}'\int\mathcal{D}\phi e^{\frac{-S^E}{\hbar}}=\langle E=0|T\left(e^{-\frac{1}{\hbar}\int_{-\infty}^{\infty}dt\hat{H}(t)}\right)|E=0\rangle \tag{5.67}$$

and the Minkowski space functional integral is defined by the analytic continuation of this object to real times.

The rotation $t\to -i\tau$ yields the Euclidean operator $\left(-\partial_\mu\partial_\mu+m^2\right)\phi$ which has no zero modes,

$$\left(-\partial_\mu\partial_\mu+m^2\right)\phi=0\Rightarrow\phi=0. \tag{5.68}$$

Thus

$$\mathcal{N}'\int\mathcal{D}\phi\, e^{\frac{-S^E}{\hbar}}$$
$$=\frac{\mathcal{N}'}{\sqrt{\det\left(-\partial_\mu\partial_\mu+m^2\right)}}e^{-\int d^dxV\left(\hbar\frac{\delta}{\delta J(x)}\right)}e^{-\int d^dxd^dy\left(J(x)\langle x|\frac{1}{\left(-\partial_\mu\partial_\mu+m^2\right)}|y\rangle J(y)\right)}, \tag{5.69}$$

where

$$\langle x|\frac{1}{(-\partial_\mu\partial_\mu+m^2)}|y\rangle=\int\frac{d^dk}{(2\pi)^d}e^{ik_\mu(x-y)_\mu}\frac{1}{(k_\mu k_\mu+m^2)} \tag{5.70}$$

which is now well-defined.

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The analytic continuation back to Minkowski space $(x_0 - y_0) \to i(x_0 - y_0)$ gives the Minkowski Green function with the "correct" Feynman prescription at the poles

$$\langle x| \frac{1}{-(\partial_\mu \partial^\mu + m^2)} |y\rangle = \int \frac{d^d k}{(2\pi)^d} e^{-ik_\mu (x-y)^\mu} \frac{1}{(k_\mu k^\mu - m^2 + i\epsilon)}. \quad (5.71)$$

Thus once the Minkowski space functional integral is defined via the analytic continuation back from Euclidean space, it clearly gives the vacuum expectation value

$$\begin{aligned} W[J] &= \langle E=0| \mathbf{T} \left(e^{-\frac{i}{\hbar} \int_{-\infty}^{\infty} dt \hat{H}^S(t)} \right) |E=0\rangle \\ &= e^{\frac{-i}{\hbar} \int d^d x V\left(-i\hbar \frac{\delta}{\delta J(x)}\right)} e^{-\frac{i}{2} \int d^d k \frac{\tilde{J}(k)\tilde{J}(-k)}{\left(k_\mu k^\mu - m^2 + i\epsilon\right)}}. \end{aligned} \quad (5.72)$$

For example, the Feynman propagator is obtained from

$$\Delta_F(x_1, x_2) = \langle E=0| \mathbf{T} (\phi(x_1)\phi(x_2)) |E=0\rangle = \int \frac{d^d k}{(2\pi)^d} \frac{e^{-ik_\mu (x-y)^\mu}}{k_\mu k^\mu - m^2 + i\epsilon}. \quad (5.73)$$