A NORMAL FORM FOR RIEMANN MATRICES

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A matrix ω having p rows and $2p$ columns of complex number elements is called a *Riemann matrix of genus* p if there exists a rational $2p$-rowed skew matrix C such that

$$(1) \qquad \omega C \omega' = 0, \qquad \gamma = i\omega C\bar{\omega}'$$

is positive definite Hermitian. The matrix C is then called a *principal matrix* of ω. If ω and ω_0 are two Riemann matrices of the same genus, we say that ω and ω_0 are *isomorphic* if there exists a non-singular p-rowed complex matrix α, and a non-singular $2p$-rowed rational matrix A, such that

$$(2) \qquad \omega_0 = \alpha\omega A.$$

Then the matrix

$$(3) \qquad C_0 = A^{-1}C(A^{-1})'$$

has the property

$$\omega_0 C_0 \omega_0' = (\alpha\omega A)[A^{-1}C(A^{-1})']A'\omega'\alpha' = \alpha(\omega C\omega')\alpha' = 0,$$

and

$$\gamma_0 = i\omega_0 C_0 \bar{\omega}' = i(\alpha\omega A)[A^{-1}C(A^{-1})']A'\bar{\omega}'\bar{\alpha}' = i\alpha(\omega C\bar{\omega}')\bar{\alpha}' = \alpha\gamma\bar{\alpha}'$$

is positive definite Hermitian when γ is positive definite Hermitian. It follows that, if (2) holds, then ω is *a Riemann matrix with C as principal matrix if and only if ω_0 is a Riemann matrix with C_0 as principal matrix*. The relation of isomorphism for Riemann matrices is readily seen to be an equivalence relation.

Two Riemann matrices ω and ω_0 are said to be *equivalent* if (2) holds with A a *unimodular* integral matrix. The relation of equivalence is also easily seen to be an equivalence relation. It is also clear that, if C is a principal matrix of ω, and t is any *positive* rational number, the matrix tC is a principal matrix of ω. Then every Riemann matrix ω has a principal matrix C whose elements are all integers. Indeed C may be selected so that the greatest common divisor of its (integral) elements is 1.

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THEOREM. *Let ω be a Riemann matrix of genus p and C be a principal matrix of ω, so that the elements of C can be taken to be all integers. Then ω is equivalent to a Riemann matrix*

$$\omega_0 = (E^{-1}, \tau) \tag{4}$$

for a symmetric p-rowed complex matrix τ, where $E = \operatorname{diag}\{e_1, \ldots, e_p\}$, the e_i are non-zero integers such that e_{i+1} divides e_i for $i = 1, \ldots, p-1$, and $\tau = \gamma + \delta i$ for real matrices γ and δ where δ is positive definite. Indeed the matrix

$$C_0 = \begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix} \tag{5}$$

is the usual canonical form for the skew matrix C under unimodular congruence, and C_0 is a principal matrix of ω_0.

While the result above is well known, the literature does not seem to contain a direct matrix-theoretic proof. We shall now provide such a proof.

We use the fact that there exists a unimodular matrix A such that

$$A^{-1}C(A')^{-1} = C_0$$

has the form (5). The matrix A^{-1} is a unimodular integral matrix when A is, and the integers e_i are the so-called *elementary divisors* of C. Then $\omega_1 = \omega A$ has C_0 as principal matrix. We may then write

$$\omega_1 = \omega A = (\tau_1, \sigma_1) \tag{6}$$

for p-rowed square matrices τ_1 and σ_1 with complex elements. It is clear that the complex $2p$-rowed square matrix

$$\Omega_1 = \begin{pmatrix} \omega_1 \\ \bar{\omega}_1 \end{pmatrix} \tag{7}$$

has the property that

$$i \overset{2p}{\begin{pmatrix} \omega_1 \; (p \\ \bar{\omega}_1 \; (p \end{pmatrix}} \begin{pmatrix} C_0 \end{pmatrix} \begin{pmatrix} \bar{\omega}'_1 & \omega'_1 \end{pmatrix} = i \begin{pmatrix} \omega_1 C_0 \\ \bar{\omega}_1 C_0 \end{pmatrix} \begin{pmatrix} \bar{\omega}'_1 & \omega'_1 \end{pmatrix} = i \begin{pmatrix} \omega_1 C_0 \bar{\omega}'_1 & \omega_1 C_0 \omega'_1 \\ \bar{\omega}_1 C_0 \bar{\omega}'_1 & \bar{\omega}_1 C_0 \omega'_1 \end{pmatrix} \tag{8}$$

$$i \quad \Omega_1 \quad C_0 \quad \bar{\Omega}_1' \qquad\qquad = \begin{pmatrix} \gamma_1 & 0 \\ 0 & \gamma_1' \end{pmatrix}$$

where $\gamma_1 = i\omega_1 C_0 \bar{\omega}_1'$ is positive definite Hermitian. Hence γ_1 and γ' are non-singular and so Ω_1 and C_0 are non-singular. Evidently the fact that γ is non-singular implies that each of its matrix factors must have rank at least p. Since ω_1 is a p by $2p$ matrix it must have rank p.

Let r be the rank of τ_1, where then $r \leqslant p$. There exists a permutation matrix P such that the first r columns of $\tau_1 P$ are linearly independent. Then P is a p-rowed unimodular matrix and the matrix

$$\omega_2 = \alpha\omega_1 \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} = (\alpha\tau_1 P, \alpha\sigma_1 P) = (\tau_2, \sigma_2) \tag{9}$$

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is *equivalent* to ω_1 and hence to ω for every non-singular p-rowed complex matrix α. Moreover, α can be selected so that

$$\tau_2 = \begin{pmatrix} I_r & \lambda \\ 0 & 0 \end{pmatrix} \tag{10}$$

for an identity matrix I_r of r rows and an r by $p - r$ complex matrix λ. What we actually have done is to use the fact that the rows of $\tau_1 P$ are all linear combinations of certain r of its rows, we can first select an $\alpha^{(')}$ which is a permutation matrix making the r basal rows the first r rows, and we can then multiply by an $\alpha^{('')}$ which has the effect of replacing the remaining $p - r$ rows by zero rows. Since the first r columns of $\alpha^{('')}\alpha^{(')}\tau_1 P$ remain linearly independent, we can multiply by a non-singular matrix $\alpha^{(''')}$ such that $\alpha^{(''')}\alpha^{('')}\alpha^{(')}\tau_1 P$ is the matrix τ_2 of (10).

The matrix

$$\begin{aligned} C_2 &= \begin{pmatrix} P' & 0 \\ 0 & P' \end{pmatrix}\begin{pmatrix} 0 & E \\ -E & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} \\ &= \begin{pmatrix} 0 & P'E \\ -P'E & 0 \end{pmatrix}\begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} = \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}, \end{aligned} \tag{11}$$

where $E_2 = P'EP$, is obtained from E by merely permuting its diagonal elements and so is a diagonal matrix.

Let us now write

$$\sigma_2 = \begin{pmatrix} \sigma_{21} & \sigma_{22} \\ \sigma_{23} & \sigma_{24} \end{pmatrix}, \qquad E_2 = \begin{pmatrix} F_2 & 0 \\ 0 & G_2 \end{pmatrix}, \tag{12}$$

where σ_{21} is an r-rowed square complex matrix (the sizes of the complex matrices σ_{22}, σ_{23}, σ_{24} being determined by this), F_2 has r rows, and so G_2 has $p - r$ rows of integer elements. We form

$$\begin{aligned} \omega_2 C_2 \omega_2' &= (\tau_2, \sigma_2)\begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix}\begin{pmatrix} \tau_2' \\ \sigma_2' \end{pmatrix} = (-\sigma_2 E_2, \tau_2 E_2)\begin{pmatrix} \tau_2' \\ \sigma_2' \end{pmatrix} \\ &= \tau_2 E_2 \sigma_2' - \sigma_2 E_2 \tau_2' = 0. \end{aligned} \tag{13}$$

Thus

$$\begin{aligned} \tau_2 E_2 \sigma_2' &= \begin{pmatrix} F_2 & \lambda G_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} \sigma_{21}' & \sigma_{23}' \\ \sigma_{22}' & \sigma_{24}' \end{pmatrix} \\ &= \begin{pmatrix} F_2 \sigma_{21}' + \lambda G_2 \sigma_{22}' & F_2 \sigma_{23}' + \lambda G_2 \sigma_{24}' \\ 0 & 0 \end{pmatrix}, \end{aligned} \tag{14}$$

and (13) states that this matrix is symmetric. Hence

$$\sigma_{23} F_2 = -\sigma_{24} G_2 \lambda'. \tag{15}$$

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We also know that the matrix

$$(16)\qquad \omega_2 C_2 \bar{\omega}_2' = \tau_2 E_2 \bar{\sigma}_2' - \sigma_2 E_2 \bar{\tau}_2' = \begin{pmatrix} F_2 \bar{\sigma}_{21}' + \lambda G_2 \bar{\sigma}_{22}' & F_2 \bar{\sigma}_{23}' + \lambda G_2 \bar{\sigma}_{24}' \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \sigma_{21} F_2 + \sigma_{22} G_2 \bar{\lambda}' & 0 \\ \sigma_{23} F_2 + \sigma_{24} G_2 \bar{\lambda}' & 0 \end{pmatrix} = \begin{pmatrix} \mu & \nu \\ -\bar{\nu}' & 0 \end{pmatrix},$$

where $\nu = F_2 \bar{\sigma}_{23}' + \lambda G_2 \bar{\sigma}_{24}'$. But then ν is non-singular. By (15)

$$(17)\qquad F_2 \bar{\sigma}_{23}' = -\bar{\lambda}' G_2 \bar{\sigma}_{24}'$$

and so

$$(18)\qquad (\lambda - \bar{\lambda}') G_2 \bar{\sigma}_{24}'$$

is non-singular. We have derived the critical result that σ_{24} is non-singular.

We now define a matrix

$$(19)\qquad B = \begin{pmatrix} I & 0 \\ D & I \end{pmatrix},$$

and know that, if D has integral elements and is a p-rowed square matrix, then B is a $2p$-rowed unimodular matrix and

$$(20)\qquad B^{-1} = \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix}.$$

Then

$$(21)\qquad BC_2 B' = \begin{pmatrix} I & 0 \\ D & I \end{pmatrix} \begin{pmatrix} 0 & E_2 \\ -E_2 & 0 \end{pmatrix} \begin{pmatrix} I & D' \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & E_2 \\ -E_2 & DE_2 \end{pmatrix} \begin{pmatrix} I & D' \\ 0 & I \end{pmatrix} = \begin{pmatrix} 0 & E_2 \\ -E_2 & DE_2 - E_2 D' \end{pmatrix}.$$

But then if

$$(22)\qquad S = DE_2$$

is a symmetric matrix, we shall have

$$(23)\qquad BC_2 B' = C_2.$$

Also

$$(24)\qquad \omega_3 = \omega_2 B^{-1} = (\tau_2, \sigma_2) \begin{pmatrix} I & 0 \\ -D & I \end{pmatrix} = (\tau_2 - \sigma_2 D, \sigma_2) = (\tau_3, \sigma_2)$$

has C_2 as a principal matrix. Take

$$(25)\qquad D = \begin{pmatrix} 0 & 0 \\ 0 & I_{p-r} \end{pmatrix},$$

where I_{p-r} is the identity matrix of $p - r$ rows. Then

$$(26) \qquad \tau_3 = \tau_2 - \sigma_2 D = \begin{pmatrix} I_r & \lambda \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} \sigma_{21} & \sigma_{22} \\ \sigma_{23} & \sigma_{24} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I_{p-r} \end{pmatrix} = \begin{pmatrix} I_r & \lambda - \sigma_{22} \\ 0 & \sigma_{24} \end{pmatrix}$$

is a non-singular matrix. But then ω is equivalent to

$$(27) \qquad \omega_4 = \tau_3^{-1} \omega_2 B^{-1} = (I, \tau_4),$$

where $\tau_4 = \tau_3^{-1} \sigma_2$. Also ω_4 has been selected so that $BC_2 B' = C_2$ is a principal matrix of ω_4. Define

$$(28) \qquad \omega_5 = \omega_4 \begin{pmatrix} P & 0 \\ 0 & P \end{pmatrix} = (P, \tau_4 P).$$

Then ω_5 is equivalent to ω and has C_0 as principal matrix. So does

$$(29) \qquad \omega_0 = E^{-1} P^{-1} \omega_5 = (E^{-1}, \tau),$$

where $\tau = E^{-1} P^{-1} \tau_4 P$. The computation $\omega_0 C_0 \omega_0' = 0$ yields $\tau = \tau'$ and $i\omega_0 C_0 \bar{\omega}_0' = 2\delta$ yields δ positive definite. This completes our proof of the theorem.

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