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Ground State Solutions of Nehari–Pankov Type for a Superlinear Hamiltonian Elliptic System on $\mathbb{R}^N$

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Abstract. This paper is concerned with the following elliptic system of Hamiltonian type

$$\begin{cases} -\triangle u + V(x)u = W_v(x,u,v), & x \in \mathbb{R}^N, \\ -\triangle v + V(x)v = W_u(x,u,v), & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases}$$

where the potential V is periodic and 0 lies in a gap of the spectrum of $-\Delta + V$, $W(x,u,v)$ is periodic in x and superlinear in u and v at infinity. We develop a direct approach to finding ground state solutions of Nehari–Pankov type for the above system. Our method is especially applicable to the case when

$$W(x,u,v) = \sum_{i=1}^{k} \int_0^{|\alpha_i u + \beta_i v|} g_i(x,t)t\,dt + \sum_{j=1}^{l} \int_0^{\sqrt{u^2 + 2b_j uv + a_j v^2}} h_j(x,t)t\,dt,$$

where $\alpha_i, \beta_i, a_j, b_j \in \mathbb{R}$ with $\alpha_i^2 + \beta_i^2 \neq 0$, and $a_j > b_j^2$, $g_i(x,t)$ and $h_j(x,t)$ are nondecreasing in $t \in \mathbb{R}^+$ for every $x \in \mathbb{R}^N$ and $g_i(x,0) = h_j(x,0) = 0$.

1 Introduction

In this paper, we study the following nonlinear elliptic system of Hamiltonian type

$$\begin{cases} -\triangle u + V(x)u = W_v(x,u,v), & x \in \mathbb{R}^N, \\ -\triangle v + V(x)v = W_u(x,u,v), & x \in \mathbb{R}^N, \\ u, v \in H^1(\mathbb{R}^N), \end{cases} \tag{1.1}$$

where $N \geq 3$, $V \in C(\mathbb{R}^N, \mathbb{R})$ and $W \in C(\mathbb{R}^N \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

For the case of a bounded domain, assuming $V \equiv 0$, there are a number of papers concerned with the systems similar to (1.1). For example, see Benci and Rabinowitz [6], De Figueiredo and Ding [7], De Figueiredo and Felmer [8] and their references for superlinear systems; see Kryszewski and Szulkin [13] and the references therein for asymptotically linear systems.

A system similar to (1.1) in the whole space $\mathbb{R}^N$ was considered recently; see, for instance, [1–4, 9, 12, 16, 21, 26, 27, 29–37] and the references therein. However, most of these focused on the case $V \equiv 1$, which is not only radial but also periodic. The main

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difficulty with problems of this type is the lack of compactness in the Sobolev embedding. A common way to overcome the difficulty is by imposing a radial symmetry assumption on the nonlinearities and working on the radially symmetric function space, which possesses a compact embedding. Another common way is to avoid the indefinite character of the original functional by using the dual variational method; see for instance [1–3].

Since Kryszewski and Szulkin [14] proposed the generalized linking theorem for the strongly indefinite functionals in 1998, Li and Szulkin [15], and Bartsch and Ding [5] (see also [10]) gave several weaker versions, which provided a third way to deal with system (1.1); see [12, 17, 18, 26, 27, 29–37] and the references therein.

In this paper, we consider System (1.1) with 0 lying in a gap of the spectrum $\sigma(-\triangle+V)$ of the Schrödinger operator $-\triangle+V$. More precisely, we first make the following basic assumptions:

(V) $V\in C(\mathbb{R}^N)$, $V(x)$ is 1-periodic in each of $x_1,x_2,\dots,x_N$, and

$$\sup[\sigma(-\Delta+V)\cap(-\infty,0)]<0<\inf[\sigma(-\Delta+V)\cap(0,\infty)]; \tag{1.2}$$

(W1) $W\in C(\mathbb{R}^N\times\mathbb{R}^2,\mathbb{R}^+)$, $W(x,z)$ is 1-periodic in each of $x_1,x_2,\dots,x_N$, continuously differentiable on $z:=(u,v)\in\mathbb{R}^2$ for every $x\in\mathbb{R}^N$, and there exist constants $p\in(2,2^*)$ and $C_0>0$ such that

$$|\nabla W_z(x,z)|\le C_0\big(1+|z|^{p-1}\big)\quad\forall (x,z)\in\mathbb{R}^N\times\mathbb{R}^2;$$

(W2) $\nabla W_z(x,z)=o(|z|)$, as $|z|\to0$, uniformly in $x\in\mathbb{R}^N$;

(W3) $\lim_{|z|\to\infty}\frac{|W(x,z)|}{|z|^2}=\infty$, a.e. $x\in\mathbb{R}^N$.

Let E,E^-, and E^+ be the Hilbert spaces with $E=E^-\oplus E^+$, which are defined in Section 2. Observe that the natural functional associated with (1.1) is given by

$$\Phi(z)=\int_{\mathbb{R}^N}\big[\nabla u\cdot\nabla v+V(x)uv\big]\mathrm{d}x-\int_{\mathbb{R}^N}W(x,z)\mathrm{d}x, \tag{1.3}$$

for all $z=(u,v)\in E$, where $W(x,z)=W(x,u,v)$. Furthermore, under assumptions (V), (W1), and (W2), $\Phi\in C^1(E,\mathbb{R})$ and

$$\langle\Phi'(z),\zeta\rangle=\int_{\mathbb{R}^N}\big[\nabla u\cdot\nabla\psi+\nabla v\cdot\nabla\phi+V(x)(u\psi+v\phi)\big]\mathrm{d}x \\ -\int_{\mathbb{R}^N}\nabla W(x,z)\cdot\zeta\mathrm{d}x\quad\forall z=(u,v),\ \zeta=(\phi,\psi)\in E. \tag{1.4}$$

If $z_0=(u_0,v_0)\in E$ is a nontrivial solution of problem (1.1), then $z_0\in\mathcal{N}^-$, where

$$\mathcal{N}^-=\big\{z\in E\setminus E^-:\langle\Phi'(z),z\rangle=\langle\Phi'(z),\zeta\rangle=0\quad\forall\,\zeta\in E^-\big\}.$$

The set $\mathcal{N}^-$ was first introduced by Pankov [19], which is a subset of the Nehari manifold

$$\mathcal{N}=\big\{z\in E\setminus\{0\}:\langle\Phi'(z),z\rangle=0\big\}.$$

In general, $\mathcal{N}^-$ contains infinitely many elements of E. In fact, under assumptions (V), (W1), (W2), and (W3), for any $z\in E\setminus E^-$, there exist $t=t(z)>0$ and $\zeta=\zeta(z)\in E^-$ such that $\zeta+tz\in\mathcal{N}^-$; see Lemma 3.11.

Recently, for (1.1) with

$$W(x,u,v)=\int_0^u f(x,t)\mathrm{d}t+\int_0^v g(x,t)\mathrm{d}t,$$

Zhao et al. [34] obtained "the least energy solution" (*i.e.,* a minimizer of the corresponding energy within the set of nontrivial solutions) by variant generalized weak linking theorem and monotonicity trick developed by Schechter and Zou [20] under assumptions (V), (W1)–(W3), and the following Nehari type monotone condition

(Ne′) $\frac{f(x,t)}{|t|}$ and $\frac{g(x,t)}{|t|}$ are strictly increasing in t on $\mathbb{R}\setminus\{0\}$ for every $x\in\mathbb{R}^N$.

We must point out that "the least energy solution" (which is sometimes also called the ground state solution) in the aforementioned references is in fact a nontrivial solution z_0 that satisfies $\Phi(z_0)=\inf_{\mathcal{M}}\Phi$, where

$$\mathcal{M}=\left\{z\in E\setminus\{0\}:\Phi'(z)=0\right\}$$

is a very small subset of $\mathcal{N}^-$ that may contain only one element. In general, it is much more difficult to find a solution z_0 for (1.1) that satisfies $\Phi(z_0)=\inf_{\mathcal{N}^-}\Phi$ than one that satisfies $\Phi(z_0)=\inf_{\mathcal{M}}\Phi$.

The purpose of this paper is to find a solution z_0 for (1.1) that satisfies $\Phi(z_0)=\inf_{\mathcal{N}^-}\Phi$ under the above assumptions. Since z_0 is a solution at which Φ has least "energy" in the set $\mathcal{N}^-$ of Pankov type, we shall call it a ground state solution of Nehari–Pankov type. As a motivation, we recall a notable work of Szulkin and Weth [22] on the existence of ground state solutions of Nehari–Pankov type for strongly indefinite periodic Schrödinger equation

$$\begin{cases}-\triangle u+V(x)u=f(x,u), & x\in\mathbb{R}^N,\\ u\in H^1(\mathbb{R}^N).\end{cases}\tag{1.5}$$

Under some standard assumptions on f and the following Nehari type monotone condition

(Ne) $\frac{f(x,u)}{|u|}$ is strictly increasing in u on $\mathbb{R}\setminus\{0\}$ for every $x\in\mathbb{R}^N$.

Szulkin and Weth developed a powerful approach (the generalized Nehari manifold method) to find ground state solutions of Nehari–Pankov type on the set

$$\mathcal{N}_0=\left\{u\in E\setminus E^-:\langle\Phi'(u),u\rangle=\langle\Phi'(u),v\rangle=0\quad\forall\, v\in E^-\right\}.$$

This approach transforms, by a direct and simple reduction, the indefinite variational problem to a definite one, resulting in a new minimax characterization of the corresponding critical value. As we know, Condition (Ne) plays a very important role in the generalized Nehari manifold method.

In the recent papers [23–25], the author developed a new approach to find ground state solution of Nehari–Pankov type for (1.5). The main idea of this approach is to find a minimizing Cerami sequence for Φ outside $\mathcal{N}^-$ by using the diagonal method, which is completely different from the one of Szulkin and Weth [22].

In this paper, based on [22–25, 34], we further develop the approach in [23–25] to find ground state solution of Nehari–Pankov type for (1.1). To state our results, in addition to the aforementioned hypotheses, we make the following assumption:

(W4) For all $\theta \geq 0$, $z, \zeta \in \mathbb{R}^2$,

$$\frac{1-\theta^2}{2}\nabla W(x,z)\cdot z - \theta \nabla W(x,z)\cdot \zeta + W(x,\theta z+\zeta) - W(x,z) \geq 0.$$

We are now in a position to state the main result of this paper.

Theorem 1.1 *Assume that V and W satisfy* (V), (W1), (W2), (W3), *and* (W4). *Then* (1.1) *has a solution $z_0 \in E$ such that $\Phi(z_0) = \inf_{\mathcal{N}^-}\Phi > 0$.*

However, it is not easy to check assumption (W4). Next, we give several classes of functions satisfying (W4). Prior to this, we define one set as follows:

$$\mathcal{ND} = \{\, h \in C(\mathbb{R}^N \times \mathbb{R}^+, \mathbb{R}^+) : \begin{array}{l} h(x,t) \text{ is 1-periodic in each of } x_1, x_2, \dots, x_N \text{ and} \\ \text{nondecreasing in } t \in [0,\infty) \text{ for every } x \in \mathbb{R}^N; \\ h(x,0) \equiv 0 \text{ for } x \in \mathbb{R}^N; \\ \text{there exist constants } p \in (2,2^*) \text{ and } C_0 > 0 \text{ such} \\ \text{that } |h(x,t)| \leq C_0(1+|t|^{p-2})\ \forall (x,t) \in \mathbb{R}^N \times \mathbb{R}. \end{array} \}$$

Corollary 1.2 *Assume that V and W satisfy* (V) *and* (W3), *and that*

$$W(x,u,v) = \sum_{i=1}^{k}\int_0^{|\alpha_i u + \beta_i v|} g_i(x,t)t\,dt + \sum_{j=1}^{l}\int_0^{\sqrt{u^2+2b_j uv + a_j v^2}} h_j(x,t)t\,dt,$$

where $\alpha_i, \beta_i, a_j, b_j \in \mathbb{R}$, $\alpha_i^2 + \beta_i^2 \neq 0$ and $a_j > b_j^2$, $g_i, h_j \in \mathcal{ND}$. Then (1.1) *has a solution $z_0 \in E$ such that $\Phi(z_0) = \inf_{\mathcal{N}^-}\Phi > 0$.*

Remark 1.3 It is easy to see that the functions

$$W(x,u,v) = (u^2+uv+v^2)\ln(1+u^2+uv+v^2),$$
$$W(x,u,v) = (u+2v)^2\ln\left[1+(u+2v)^2\right] + (2u-v)^2\ln\left[1+(2u-v)^2\right],$$
$$W(x,u,v) = |u+2v|^{\sigma_1} + |3u+2v|^{\sigma_2}, \quad \sigma_1,\sigma_2 \in (2,2^*)$$

satisfy (W1), (W2), (W3), and (W4).

The remainder of this paper is organized as follows. In Section 2, we provide some preliminaries and present a variational setting for (1.1). The proofs of our theorem and corollary are given in Section 3.

2 The Variational Setting

Let $\mathcal{A} := -\Delta + V$; then $\mathcal{A}$ is self-adjoint in $L^2(\mathbb{R}^N)$ with domain $\mathfrak{D}(\mathcal{A}) = H^2(\mathbb{R}^N)$. Let $\{\mathcal{E}(\lambda) : -\infty < \lambda < +\infty\}$ be the spectral family of $\mathcal{A}$, and let $|\mathcal{A}|^{1/2}$ be the square root of $|\mathcal{A}|$. Set $\mathcal{U} = \mathrm{id} - \mathcal{E}(0) - \mathcal{E}(0-)$; then $\mathcal{U}$ commutes with $\mathcal{A}$, $|\mathcal{A}|$, and $|\mathcal{A}|^{1/2}$, and $\mathcal{A} = \mathcal{U}|\mathcal{A}|$ is the polar decomposition of $\mathcal{A}$ (see [11, Theorem 4.3.3]). By (1.2),

$$\bar{\Lambda} := \sup[\sigma(\mathcal{A}) \cap (-\infty,0)] < 0 < \underline{\Lambda} := \inf[\sigma(\mathcal{A}) \cap (0,\infty)].$$

Let $\Lambda_0 = \min\{-\bar{\Lambda}, \underline{\Lambda}\}$, then the following hold:

$$\mathcal{A} = \int_{-\infty}^{\infty} \lambda \mathrm{d}\mathcal{E}(\lambda) = \int_{-\infty}^{\bar{\Lambda}} \lambda \mathrm{d}\mathcal{E}(\lambda) + \int_{\underline{\Lambda}}^{\infty} \lambda \mathrm{d}\mathcal{E}(\lambda),$$
$$|\mathcal{A}| = \int_{-\infty}^{\infty} |\lambda| \mathrm{d}\mathcal{E}(\lambda) = \int_{\Lambda_0}^{\infty} |\lambda| \mathrm{d}[\mathcal{E}(\lambda) - \mathcal{E}(-\lambda)],$$
$$|\mathcal{A}|^{1/2} = \int_{-\infty}^{\infty} |\lambda|^{1/2} \mathrm{d}\mathcal{E}(\lambda) = \int_{\Lambda_0}^{\infty} |\lambda|^{1/2} \mathrm{d}[\mathcal{E}(\lambda) - \mathcal{E}(-\lambda)].$$

It is well known that $\mathfrak{D}(|\mathcal{A}|^{1/2})$ is a Hilbert space endowed with inner product

$$(u, v)_{\mathfrak{D}(|\mathcal{A}|^{1/2})} = \big(|\mathcal{A}|^{1/2} u, |\mathcal{A}|^{1/2} v \big)_{L^2} \quad \forall\, u, v \in \mathfrak{D}(|\mathcal{A}|^{1/2}).$$

Then

$$\|u\|^2_{\mathfrak{D}(|\mathcal{A}|^{1/2})} = \| |\mathcal{A}|^{1/2} u \|^2_{L^2} \geq \Lambda_0 \|u\|_2^2 \quad \forall\, u \in \mathfrak{D}(|\mathcal{A}|^{1/2}).$$

Obviously, $\mathfrak{D}(|\mathcal{A}|^{1/2}) = H^1(\mathbb{R}^N)$ with equivalent norm.

Set $E = \mathfrak{D}(|\mathcal{A}|^{1/2}) \times \mathfrak{D}(|\mathcal{A}|^{1/2})$; then E is a Hilbert space with the inner product

$$(z_1, z_2) = (u_1, u_2)_{\mathfrak{D}(|\mathcal{A}|^{1/2})} + (v_1, v_2)_{\mathfrak{D}(|\mathcal{A}|^{1/2})} \quad \forall\, z_i = (u_i, v_i) \in E, \quad i = 1, 2,$$

the corresponding norm is denoted by $\|\cdot\|$. By the Sobolev embedding theorem, the embedding $E \hookrightarrow L^p(\mathbb{R}^N) \times L^p(\mathbb{R}^N)$ is continuous and locally compact.

Let

$$E^- = \big\{ (u, -\mathcal{U}u) : u \in \mathfrak{D}(|\mathcal{A}|^{1/2}) \big\}, \quad E^+ = \big\{ (u, \mathcal{U}u) : u \in \mathfrak{D}(|\mathcal{A}|^{1/2}) \big\}.$$

For any $z = (u, v) \in E$, set

$$z^- = \Big(\frac{u - \mathcal{U}v}{2}, -\mathcal{U}\Big(\frac{u - \mathcal{U}v}{2} \Big) \Big), \quad z^+ = \Big(\frac{u + \mathcal{U}v}{2}, \mathcal{U}\Big(\frac{u + \mathcal{U}v}{2} \Big) \Big).$$

It is easy to check that $\mathcal{U}u \in \mathfrak{D}(|\mathcal{A}|^{1/2})$, for all $u \in \mathfrak{D}(|\mathcal{A}|^{1/2})$. Thus, $E^- \subset E$ and $E^+ \subset E$, $z^- \in E^-$ and $z^+ \in E^+$. It is obvious that $z = z^- + z^+$. On the other hand, z^- and z^+ are orthogonal with respect to the inner products $(\cdot, \cdot)_{L^2}$ and $(\cdot, \cdot)$. Thus, we have $E = E^- \oplus E^+$. By a simple calculation, one can get that

$$\frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2 \big) = \big(|\mathcal{A}|^{1/2} \mathcal{U}u, |\mathcal{A}|^{1/2} v \big)_{L^2} = (|\mathcal{A}|\mathcal{U}u, v)_{L^2} = (\mathcal{A}u, v)_{L^2}$$
$$= \int_{\mathbb{R}^N} \big[\nabla u \cdot \nabla v + V(x) uv \big] \mathrm{d}x.$$

Therefore, the functional Φ defined in (1.3) can be rewritten in a standard way:

$$\Phi(z) = \frac{1}{2}\big(\|z^+\|^2 - \|z^-\|^2 \big) - \Psi(z) \quad \forall\, z = (u, v) \in E, \tag{2.1}$$

where $\Psi(z) = \int_{\mathbb{R}^N} W(x, z) \mathrm{d}x$. Our hypotheses imply that $\Phi \in C^1(E, \mathbb{R})$, and a standard argument shows that critical points of Φ are solutions of (1.1). Moreover, by (1.4),

$$\begin{aligned} \langle \Phi'(z), \zeta \rangle &= \big(|\mathcal{A}|^{1/2} \mathcal{U}u, |\mathcal{A}|^{1/2} \psi \big)_{L^2} + \big(|\mathcal{A}|^{1/2} \mathcal{U}\phi, |\mathcal{A}|^{1/2} v \big)_{L^2} \\ &\quad - \int_{\mathbb{R}^N} \nabla W(x, z) \cdot \zeta \mathrm{d}x \\ &= (z^+, \zeta^+) - (z^-, \zeta^-) - \int_{\mathbb{R}^N} \nabla W(x, z) \cdot \zeta \mathrm{d}x \end{aligned} \tag{2.2}$$

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for all $z = (u, v)$, $\zeta = (\phi, \psi) \in E$, and $\langle \Phi'(z), z \rangle = \|z^+\|^2 - \|z^-\|^2 - \langle \Psi'(z), z \rangle$ for all $z \in E$.

3 Proofs of the Main Results

Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$. For a functional $\phi \in C^1(X, \mathbb{R})$, ϕ is said to be weakly sequentially lower semicontinuous if for any $u_n \rightharpoonup u$ in X, one has $\phi(u) \le \liminf_{n\to\infty} \phi(u_n)$, and ϕ' is said to be weakly sequentially continuous if $\lim_{n\to\infty} \langle \phi'(u_n), v \rangle = \langle \phi'(u), v \rangle$ for each $v \in X$.

Lemma 3.1 ([14,15]) *Let X be a real Hilbert space with $X = X^- \oplus X^+$ and $X^- \perp X^+$, and let $\phi \in C^1(X, \mathbb{R})$ be of the form*

$$\phi(u) = \frac{1}{2}\big(\|u^+\|^2 - \|u^-\|^2\big) - \psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Suppose that the following assumptions are satisfied:

(KS1) *$\psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semicontinuous;*

(KS2) *ψ' is weakly sequentially continuous;*

(KS3) *there exist $r > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that $\kappa := \inf \phi(S_\rho^+) > \sup \phi(\partial Q)$, where $S_\rho^+ = \{u \in X^+ : \|u\| = \rho\}$ and $Q = \{v + se : v \in X^-, s \ge 0, \|v + se\| \le r\}$.*

Then there exist a constant $c \in [\kappa, \sup \phi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying $\phi(u_n) \to c$ and $\|\phi'(u_n)\|(1 + \|u_n\|) \to 0$.

Employing a standard argument, one can easily check the following lemma.

Lemma 3.2 *Suppose that* (V), (W1), *and* (W2) *are satisfied. Then Ψ is nonnegative, weakly sequentially lower semicontinuous, and Ψ' is weakly sequentially continuous.*

Lemma 3.3 *Suppose that* (V), (W1), (W2), *and* (W4) *are satisfied. Then*

$$\Phi(z) \ge \Phi(\theta z + \zeta) + \frac{1}{2}\|\zeta\|^2 + \frac{1 - \theta^2}{2}\langle \Phi'(z), z \rangle - \theta \langle \Phi'(z), \zeta \rangle \tag{3.1}$$

for all $\theta \ge 0$, $z \in E$, $\zeta \in E^-$.

Proof By (2.1), (2.2), and (W4), one has

$$\begin{aligned} &\Phi(z) - \Phi(\theta z + \zeta) \\ &\quad = \frac{1}{2}\|\zeta\|^2 + \frac{1 - \theta^2}{2}\big(\|z^+\|^2 - \|z^-\|^2\big) + \theta(z, \zeta) \\ &\qquad - \int_{\mathbb{R}^N} \big[W(x, z) - W(x, \theta z + \zeta)\big]\,\mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\|\zeta\|^2 + \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle - \theta\langle \Phi'(z), \zeta\rangle \\
&\quad + \int_{\mathbb{R}^N}\Big[\frac{1-\theta^2}{2}\nabla W(x,z)\cdot z - \theta\nabla W(x,z)\cdot\zeta + W(x,\theta z+\zeta) - W(x,z)\Big]\mathrm{d}x \\
&\ge \frac{1}{2}\|\zeta\|^2 + \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle - \theta\langle \Phi'(z), \zeta\rangle \quad \forall\, \theta\ge 0,\ z\in E,\ \zeta\in E^-.
\end{aligned}
$$

This shows that (3.1) holds. ■

From Lemma 3.3, we have the following two corollaries.

Corollary 3.4 *Suppose that* (V), (W1), (W2), *and* (W4) *are satisfied. Then for* $z \in \mathcal{N}^-$,

$$\Phi(z) \ge \Phi(\theta z + \zeta) \quad \forall\, \theta \ge 0,\ \zeta \in E^-.$$

Corollary 3.5 *Suppose that* (V), (W1), (W2), *and* (W4) *are satisfied. Then*

$$\Phi(z) \ge \frac{\theta^2}{2}\|z\|^2 + \frac{1-\theta^2}{2}\langle \Phi'(z), z\rangle + \theta^2\langle \Phi'(z), z^-\rangle - \int_{\mathbb{R}^N} W(x,\theta z^+)\mathrm{d}x, \tag{3.2}$$

for all $z \in E$, $\theta \ge 0$.

Applying Corollary 3.4, we can prove the following lemma in the same way as [22, Lemma 2.4].

Lemma 3.6 *Suppose that* (V), (W1), (W2), *and* (W4) *are satisfied. Then*

(i) *there exists* $\rho > 0$ *such that*

$$m := \inf_{\mathcal{N}^-}\Phi \ge \kappa := \inf\big\{\Phi(z) : z \in E^+, \|z\| = \rho\big\} > 0;$$

(ii) $\|z^+\| \ge \max\{\|z^-\|, \sqrt{2m}\}$ *for all* $z \in \mathcal{N}^-$.

Lemma 3.7 *Suppose that* (V), (W1), (W2), *and* (W3) *are satisfied. Then for any* $e \in E^+$, $\sup\Phi(E^- \oplus \mathbb{R}^+ e) < \infty$, *and there is a* $R_e > 0$ *such that*

$$\Phi(z) < 0 \quad \forall z \in E^- \oplus \mathbb{R}^+ e,\ \|z\| \ge R_e.$$

The proof is standard; see [34, Lemma 4.3].

Corollary 3.8 *Suppose that* (V), (W1), (W2), *and* (W3) *are satisfied. Let* $e \in E^+$ *with* $\|e\| = 1$. *Then there is a* $r_0 > \rho$ *such that* $\sup\Phi(\partial Q) \le 0$ *for* $r \ge r_0$, *where*

$$Q = \big\{\zeta + se : \zeta \in E^-, s \ge 0,\ \|\zeta + se\| \le r\big\}. \tag{3.3}$$

Lemma 3.9 *Suppose that* (V), (W1), (W2), (W3), *and* (W4) *are satisfied. Then there exist a constant* $c \in [\kappa, \sup\Phi(Q)]$ *and a sequence* $\{z_n\} \subset E$ *satisfying*

$$\Phi(z_n) \to c, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0,$$

where Q *is defined by* (3.3).

Proof Lemma 3.9 is a direct corollary of Lemmas 3.1, 3.2, 3.6(i), and Corollary 3.8. ∎

Analogous to the proof of [23, Lemma 3.8], it is easy to show the following lemma.

Lemma 3.10 *Suppose that* (V), (W1), (W2), (W3), *and* (W4) *are satisfied. Then there exist a constant* $c_* \in [\kappa, m]$ *and a sequence* $\{z_n\} = \{(u_n, v_n)\} \subset E$ *satisfying*

$$\Phi(z_n) \to c_*, \quad \|\Phi'(z_n)\|(1 + \|z_n\|) \to 0. \tag{3.4}$$

Lemma 3.11 *Suppose that* (V), (W1), (W2), *and* (W3) *are satisfied. Then for any* $z \in E \setminus E^-$, $\mathcal{N}^- \cap (E^- \oplus \mathbb{R}^+ z) \neq \varnothing$, i.e., *there exist* $t(z) > 0$ *and* $\zeta(z) \in E^-$ *such that* $t(z)z + \zeta(z) \in \mathcal{N}^-$.

The proof is the same as that of [22, Lemma 2.6].

Lemma 3.12 *Suppose that* (V), (W1), (W2), (W3), *and* (W4) *are satisfied. Then any sequence* $\{z_n\} = \{(u_n, v_n)\} \subset E$ *satisfying*

$$\Phi(z_n) \to c \geq 0, \quad \langle \Phi'(z_n), z_n \rangle \to 0, \quad \langle \Phi'(z_n), z_n^- \rangle \to 0 \tag{3.5}$$

is bounded in E.

Proof To prove the boundedness of $\{z_n\}$, arguing by contradiction, suppose that $\|z_n\| \to \infty$. Let $\xi_n = z_n/\|z_n\|$. Then $\|\xi_n\| = 1$. By the Sobolev embedding theorem, there exists a constant $C_1 > 0$ such that $\|\xi_n\|_2 \leq C_1$. If

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B_1(y)} |\xi_n^+|^2 \mathrm{d}x = 0,$$

then by Lions' concentration compactness principle [28, Lemma 1.21], $\xi_n^+ \to 0$ in $L^p(\mathbb{R}^N)$. Fix $R > [2(1+c)]^{1/2}$. By virtue of (W1) and (W2), for $\varepsilon = 1/4(RC_1)^2 > 0$, there exists $C_\varepsilon > 0$ such that

$$W(x,z) \leq \varepsilon |z|^2 + C_\varepsilon |z|^p \quad \forall\ (x,z) \in \mathbb{R}^N \times \mathbb{R}^2.$$

Hence, it follows that

$$\begin{aligned} &\limsup_{n\to\infty} \int_{\mathbb{R}^N} W(x, Rz_n^+/\|z_n\|)\mathrm{d}x \\ &= \limsup_{n\to\infty} \int_{\mathbb{R}^N} W(x, R\xi_n^+)\mathrm{d}x \\ &\leq \limsup_{n\to\infty} R^2\varepsilon \int_{\mathbb{R}^N} |\xi_n^+|^2 \mathrm{d}x + \limsup_{n\to\infty} R^p C_\varepsilon \int_{\mathbb{R}^N} |\xi_n^+|^p \mathrm{d}x \\ &\leq \varepsilon (RC_1)^2 = \frac{1}{4}. \end{aligned} \tag{3.6}$$

Let $\theta_n = R/\|z_n\|$. Hence, by virtue of (3.2), (3.5), and (3.6), one can get

$$
\begin{aligned}
c + o(1) &= \Phi(z_n)\\
&\geq \frac{\theta_n^2}{2}\|z_n\|^2 - \int_{\mathbb{R}^N} W(x, \theta_n z_n^+)\mathrm{d}x + \frac{1-\theta_n^2}{2}\big\langle \Phi'(z_n), z_n\big\rangle + \theta_n^2\big\langle \Phi'(z_n), z_n^-\big\rangle\\
&= \frac{R^2}{2} - \int_{\mathbb{R}^N} W\big(x, Rz_n^+/\|z_n\|\big)\,\mathrm{d}x\\
&\quad + \Big(\frac{1}{2} - \frac{R^2}{2\|z_n\|^2}\Big)\big\langle \Phi'(z_n), z_n\big\rangle + \frac{R^2}{\|z_n\|^2}\big\langle \Phi'(z_n), z_n^-\big\rangle\\
&= \frac{R^2}{2} - \int_{\mathbb{R}^N} W(x, Rz_n^+/\|z_n\|)\mathrm{d}x + o(1)\\
&\geq \frac{R^2}{2} - \frac{1}{4} + o(1) > \frac{3}{4} + c + o(1).
\end{aligned}
$$

This contradiction shows that $\delta > 0$. We may assume the existence of $k_n \in \mathbb{Z}^N$ such that $\int_{B_{1+\sqrt{N}}(k_n)} |\xi_n^+|^2\mathrm{d}x > \frac{\delta}{2}$. Let $\zeta_n(x) = \xi_n(x + k_n)$. Then

$$
\int_{B_{1+\sqrt{N}}(0)} |\zeta_n^+|^2\mathrm{d}x > \frac{\delta}{2}. \tag{3.7}
$$

Now we define $\tilde{z}_n(x) = z_n(x + k_n)$; then $\tilde{z}_n/\|z_n\| = \zeta_n$ and $\|\zeta_n\| = 1$. Passing to a subsequence, we have $\zeta_n \rightharpoonup \zeta$ in E, $\zeta_n \to \zeta$ in $L^2_{\mathrm{loc}}(\mathbb{R}^N)$, and $\zeta_n \to \zeta$ a.e. on $\mathbb{R}^N$. Obviously, (3.7) implies that $\zeta \neq 0$. Hence, it follows from (3.5), (W3), and Fatou's lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{c + o(1)}{\|z_n\|^2} = \lim_{n\to\infty} \frac{\Phi(z_n)}{\|z_n\|^2}\\
&= \lim_{n\to\infty}\Big[\frac{1}{2}\big(\|\xi_n^+\|^2 - \|\xi_n^-\|^2\big) - \int_{\mathbb{R}^N} \frac{W(x, z_n)}{\|z_n\|^2}dx\Big]\\
&= \lim_{n\to\infty}\Big[\frac{1}{2}\big(\|\xi_n^+\|^2 - \|\xi_n^-\|^2\big) - \int_{\mathbb{R}^N} \frac{W(x + k_n, \tilde{z}_n)}{|\tilde{z}_n|^2}|\zeta_n|^2 dx\Big]\\
&\leq \frac{1}{2} - \liminf_{n\to\infty} \int_{\mathbb{R}^N} \frac{W(x, \tilde{z}_n)}{|\tilde{z}_n|^2}|\zeta_n|^2 dx = -\infty,
\end{aligned}
$$

which is a contradiction. Thus, $\{z_n\}$ is bounded in E. ∎

Lemma 3.13 ([25, Lemma 2.3]) *Suppose that $t \mapsto h(x,t)$ is nondecreasing on $\mathbb{R}$ and $h(x,0) = 0$ for any $x \in \mathbb{R}^N$. Then*

$$
\Big(\frac{1-\theta^2}{2}\tau - \theta\sigma\Big) h(x,\tau)|\tau| \geq \int_{\theta\tau+\sigma}^{\tau} h(x,s)|s|\mathrm{d}s \quad \forall\ \theta \geq 0,\ \ \tau,\ \sigma \in \mathbb{R}. \tag{3.8}
$$

Lemma 3.14 *Suppose that $W(x,u,v) = \int_0^{|\alpha u + \beta v|} g(x,t)t\mathrm{d}t$, where $\alpha, \beta \in \mathbb{R}$ with $\alpha^2 + \beta^2 \neq 0$ and $g \in \mathcal{ND}$. Then W satisfies* (W1), (W2), *and* (W4).

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Proof It is easy to see that W satisfies (W1) and (W2). Next, we show that W also satisfies (W4). Let $g(x,t)=0$ for $t<0$. Note that

$$|\theta a+b|\geq\theta|a|+\frac{ab}{|a|}\quad\forall\theta\geq 0,a,b\in\mathbb{R}. \tag{3.9}$$

For any $x\in\mathbb{R}^N$, it follows from (3.8) and (3.9) that

$$\begin{aligned}&\frac{1-\theta^2}{2}\nabla W(x,z)\cdot z-\theta\nabla W(x,z)\cdot\zeta+W(x,\theta z+\zeta)-W(x,z)\\&=\Big[\frac{1-\theta^2}{2}(\alpha u+\beta v)^2-\theta(\alpha u+\beta v)(\alpha\phi+\beta\psi)\Big]g(x,|\alpha u+\beta v|)\\&\quad-\int_{|\theta(\alpha u+\beta v)+(\alpha\phi+\beta\psi)|}^{|\alpha u+\beta v|}g(x,t)t\,\mathrm{d}t\\&\geq\Big[\frac{1-\theta^2}{2}(\alpha u+\beta v)^2-\theta(\alpha u+\beta v)(\alpha\phi+\beta\psi)\Big]g(x,|\alpha u+\beta v|)\\&\quad-\int_{\theta|\alpha u+\beta v|+(\alpha u+\beta v)(\alpha\phi+\beta\psi)/|\alpha u+\beta v|}^{|\alpha u+\beta v|}g(x,t)|t|\mathrm{d}t\geq 0\end{aligned}$$

for all $\theta\geq 0$, $z=(u,v)$, $\zeta=(\phi,\psi)\in\mathbb{R}^2$. This shows that (W4) holds. ∎

Lemma 3.15 *Suppose that* $W(x,u,v)=\int_0^{\sqrt{u^2+2buv+av^2}}h(x,t)t\,\mathrm{d}t$, *where* $a,b\in\mathbb{R}$ *with* $a>b^2$ *and* $h\in\mathcal{ND}$. *Then* W *satisfies* (W1), (W2), *and* (W4).

Proof It is easy to see that W satisfies (W1) and (W2). Next, we show that W also satisfies (W4). Let $h(x,t)=0$ for $t<0$ and for $z=(u,v)$, $\zeta=(\phi,\psi)\in\mathbb{R}^2$, let

$$A=\begin{pmatrix}1&b\\b&a\end{pmatrix},\quad zAz^{\top}=(u,v)\begin{pmatrix}1&b\\b&a\end{pmatrix}\begin{pmatrix}u\\v\end{pmatrix}=u^2+2buv+av^2$$

and

$$zA\zeta^{\top}=(u,v)\begin{pmatrix}1&b\\b&a\end{pmatrix}\begin{pmatrix}\phi\\\psi\end{pmatrix}=u\phi+b(\phi v+u\psi)+av\psi.$$

Then $W(x,u,v)=W(x,z)=\int_0^{\sqrt{zAz^{\top}}}h(x,t)t\,\mathrm{d}t$. By virtue of Lemma 3.13, one has

$$\Big[\frac{1-\theta^2}{2}\sqrt{zAz^{\top}}-\frac{\theta(zA\zeta^{\top})}{\sqrt{zAz^{\top}}}\Big]h\big(x,\sqrt{zAz^{\top}}\big)\sqrt{zAz^{\top}}\geq\int_{\theta\sqrt{zAz^{\top}}+zA\zeta^{\top}/\sqrt{zAz^{\top}}}^{\sqrt{zAz^{\top}}}h(x,\tau)|\tau|\mathrm{d}\tau, \tag{3.10}$$

for all $\theta\geq 0$, $z,\zeta\in\mathbb{R}^2$. It is easy to verify that

$$zA\zeta^{\top}\leq\sqrt{zAz^{\top}}\sqrt{\zeta A\zeta^{\top}}\quad\forall z,\zeta\in\mathbb{R}^2,$$

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which, together with (3.10), implies

$$\begin{aligned}
&\frac{1-\theta^2}{2}\nabla W(x,z)\cdot z - \theta\nabla W(x,z)\cdot\zeta + W(x,\theta z+\zeta) - W(x,z)\\
&\quad= \frac{1-\theta^2}{2}h\big(x,\sqrt{zAz^\top}\big)zAz^\top - \theta\big(zA\zeta^\top\big)h\big(x,\sqrt{zAz^\top}\big)\\
&\qquad+\int_{\sqrt{zAz^\top}}^{\sqrt{(\theta z+\zeta)A(\theta z+\zeta)^\top}} h(x,\tau)|\tau|\mathrm{d}\tau\\
&\quad= \Big[\frac{1-\theta^2}{2}\sqrt{zAz^\top} - \frac{\theta\big(zA\zeta^\top\big)}{\sqrt{zAz^\top}}\Big]h\big(x,\sqrt{zAz^\top}\big)\sqrt{zAz^\top}\\
&\qquad+\int_{\sqrt{zAz^\top}}^{\sqrt{\theta^2\left(zAz^\top\right)+\zeta A\zeta^\top+2\theta\left(zA\zeta^\top\right)}} h(x,\tau)|\tau|\mathrm{d}\tau\\
&\quad\ge \Big[\frac{1-\theta^2}{2}\sqrt{zAz^\top} - \frac{\theta\big(zA\zeta^\top\big)}{\sqrt{zAz^\top}}\Big]h\big(x,\sqrt{zAz^\top}\big)\sqrt{zAz^\top}\\
&\qquad+\int_{\sqrt{zAz^\top}}^{\theta\sqrt{zAz^\top}+zA\zeta^\top/\sqrt{zAz^\top}} h(x,\tau)|\tau|\mathrm{d}\tau\\
&\quad\ge 0
\end{aligned}$$

for all $\theta \ge 0, z, \zeta \in \mathbb{R}^2$. This shows that (W4) holds. ∎

Proof of Theorem 1.1 Applying Lemmas 3.10 and 3.12, we deduce that there exists a bounded sequence $\{z_n\} = \{(u_n, v_n)\} \subset E$ satisfying (3.4). The rest of the proof is standard. ∎

Employing Theorem 1.1 and Lemmas 3.14 and 3.15, we have Corollary 1.2 immediately.

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