Note on the Researches of Maclaurin on Circular Cubics.

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1. In the works known to us which contain historical or bibliographical information about circular cubics no sufficient indication of the researches of Maclaurin is found. Yet many of the classic propositions connected with these curves are due to this eminent geometer, who investigated constructions for unicursal circular cubics, for certain non-unicursal circular cubics, and for the special circular cubics now known as *the trisectrix of Maclaurin*, *the oblique cissoid*, and *the strophoid*. His name does not appear even in the list of writings on the strophoid published by Tortolini and Günther.

It seems to us useful, therefore, to indicate what researches he made on these curves, not always, however, reproducing his demonstrations, but replacing them by analytical proofs which are easier to follow. He pointed out also the connection between results obtained by him and later discoveries.

2. Maclaurin deals with circular cubics in his *Geometria Organica* (1720, pp. 33–37). He considers first the unicursal circular cubics, for which he gives the following construction :—

Consider two points O and B and a straight line AM (Fig. 1). Through B draw any straight line BM, and through M draw ML perpendicular to BM. Through O draw OH perpendicular to ML. Then the locus of H is a unicursal circular cubic.

Take the parallel through O to AM as x-axis, and the perpendicular as y-axis. Let $OK = b$ and B be $(-a, -\beta)$. Then if M is (u, b), the equation of ML is

$$y - b = -\frac{a+u}{\beta+b}(x-u),$$

and the equation of OH is

$$y = \frac{\beta+b}{a+u}x.$$

Eliminating u between these equations, we find the equation to the locus of H,

$$y(x^2+y^2) = by^2 - axy + (\beta + b)x^2 \dots\dots\dots\dots\dots (1).$$

This equation represents all the unicursal circular cubics.

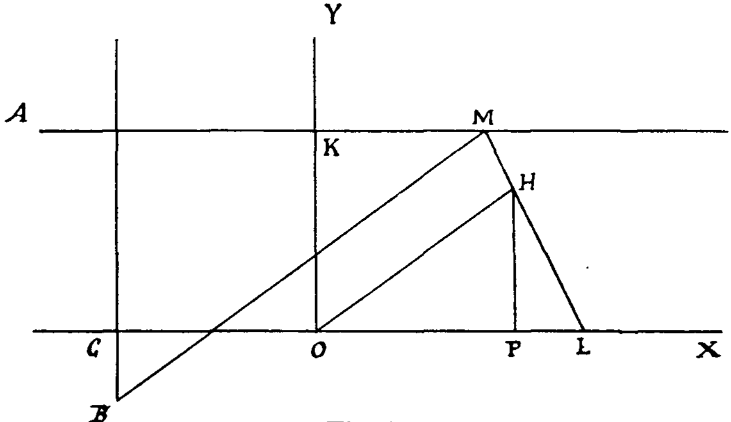

Fig. 1.

3. The method of constructing unicursal circular cubics by considering them as cissoidals of a circle* was also given by

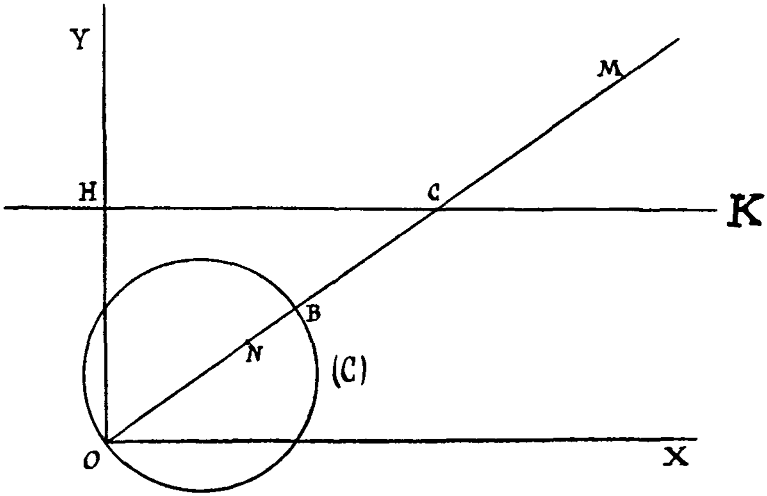

Fig. 2.

* See, for example, our Treatise on Special Curves, Vol. I., p. 20.

Maclaurin (*l.c.* p. 35). He has shown that by drawing through a given point O (Fig. 2) on the circumference of a circle any straight line OB, and taking on this line, measuring from the point C where it cuts a given line HK, two segments CM and CN equal to OB, two points M and N are found whose loci are the same as those defined in the preceding paragraph. It might be thought that M and N describe two branches of the same curve, but that is not so.

Take OX and OY parallel and perpendicular respectively to HK. Then if the centre of the given circle is (a_1, β_1), $\mathrm{OH} = c$, and $\angle \mathrm{XOB} = \theta$,

$$\mathrm{OB} = \rho_1 = 2a_1 \cos\theta + 2\beta_1 \sin\theta,$$
$$\mathrm{OC} = \rho_2 = c/\sin\theta.$$

Therefore the polar equation to the locus of N is

$$\rho = \rho_2 - \rho_1 = \frac{c}{\sin\theta} - 2(a_1\cos\theta + \beta_1\sin\theta),$$

whence the cartesian equation is

$$y(x^2 + y^2) = cx^2 - 2a_1xy + (c - 2\beta_1)y^2.$$

Similarly the locus of M is

$$y(x^2 + y^2) = cx^2 + 2a_1xy + (c + 2\beta_1)y^2.$$

Each of these equations can represent all the unicursal circular cubics. The first expresses that the cubic is the cissoidal of the circle (C) and the line HK; the second that it is the cissoidal of the circle (C′) and the same line, where (C) and (C′) are symmetric with respect to O.

The locus of N coincides with the cubic found in (2) when the centre (a_1, β_1) of the circle (C) and the line HK are taken so that

$$a_1 = a/2, \;\; \beta_1 = \beta/2, \;\; c = b + \beta.$$

The construction given and these relations can be established directly by pure geometry, and Maclaurin's method does not differ essentially from this.

4. Maclaurin also gives the following construction for the cubics :—

Consider a quadrilateral ABCD, right-angled at A and C. If A and C are fixed points and B describes a circle which passes through A, then D describes a unicursal circular cubic.

Take A as origin, AC as y-axis, and let $AC = a$. Then the equations to AB, BC are

$$x = ky,\ x = k'(y - a).$$

Let the equation to the circle be

$$x^2 + y^2 - 2\alpha_2 x - 2\beta_2 y = 0.$$

Since B is on the circle

$$ak'(1 + k^2) - 2(k' - k)(k\alpha_2 + \beta_2) = 0.$$

The equations to AD, DC are

$$y = -kx,\ y - a = -k'x,$$

and the locus of D is given by the result of eliminating k and k' between the last three equations, *i.e.* by

$$y(x^2 + y^2) = (a - 2\beta_2)x^2 + 2\alpha_2 xy + ay^2,$$

which represents all the unicursal circular cubics. This equation is identical with equation (1), § 2, when

$$\alpha_2 = -\alpha/2,\ \beta_2 = -\beta/2,\ a = b.$$

Maclaurin deduced this construction as a corollary from that given in § 2.

5. Maclaurin met again the cubics considered when investigating the pedals of the parabola (*Geometria Organica*, p. 101), and remarked that if the pole is outside the parabola, the pedal has a node; that if the pole is on the parabola, the pedal (cissoid) has a cusp; and that when the pole is inside the parabola, the pedal has an isolated point. This proposition is well known.

6. The method employed by Barrow (*Lectiones geometricae*, 1669, p. 69) for constructing the curve now called the strophoid was generalised by Maclaurin (*Geometria Organica*, p. 36), in the following way:—

Consider two points O and A (Fig. 3) and a straight line CD. Through the point O draw any straight line OB, and take on this line two segments BM, BN equal to AB. Let us find the locus of M and N as OB varies.

Take O as origin and OX, OY perpendicular and parallel respectively to CD. Let A be (α, β) and $OC = a$. Then if OM or $ON = \rho$ and $\angle COB = \theta$,

$$\rho = \frac{a}{\cos\theta} \mp AB = \frac{a}{\cos\theta} \mp \sqrt{\{(\alpha - a)^2 + (\beta - a\tan\theta)^2\}}.$$

Let $x=\rho\cos\theta$, $y=\rho\sin\theta$, and we have for the cartesian equation to the locus of M or N

$$x(x^2+y^2)=2a(x^2+y^2)+(\alpha^2+\beta^2-2a\alpha)x-2\alpha\beta y \ldots\ldots(2).$$

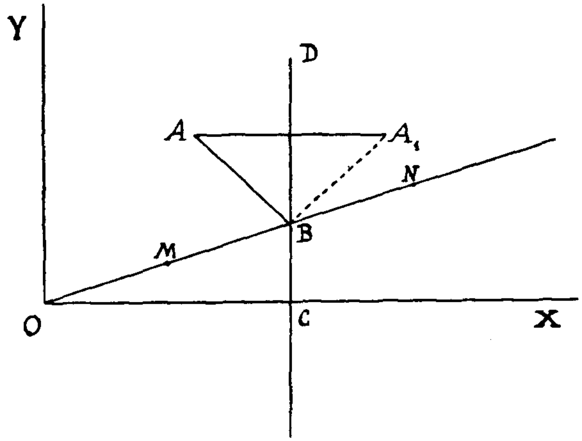

Fig. 3.

We shall make some remarks on the question just considered.

In the first place, it is identical with that solved later by M. Lagrange in the volume corresponding to 1900 of the *Nouvelles Annales de Mathématiques.*

We remark next that equation (2) is a particular case of the equation

$$x(x^2+y^2)=\mathrm{A}(x^2+y^2)-\mathrm{B}x-\mathrm{C}y, \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

which represents the cubics known as Van-Rees' focals,* because they were met by him when investigating the focal lines of a right cone on an elliptic base (*Correspondancé Mathematique de Quetelet*, t. V., p. 361).

It is not possible to construct all the focals of Van-Rees by Maclaurin's method. For, comparing equations (2) and (3), we find for determining α, β, a, the equations

$$\mathrm{A}=2a,\ \mathrm{B}=2a\alpha-\alpha^2-\beta^2,\ \mathrm{C}=2\alpha\beta,$$

which give

$$a=\frac{\mathrm{A}}{2},\ \ \beta=\frac{\mathrm{C}}{\mathrm{A}},\ \ \alpha=a\pm\frac{1}{2\mathrm{A}}\sqrt{\mathrm{A}^4-4\mathrm{C}^2-4\mathrm{A}^2\mathrm{B}}.$$

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* See our Treatise on Special Curves, Vol. I., p. 45.

Consequently α is imaginary when

$$A^4 - 4C^2 - 4A^2B < 0.$$

The case considered by Maclaurin is the case where the focal is composed of an oval and an infinite branch, and in this case the last equation gives for α two real values which correspond to the points A and A_1, symmetrically placed with respect to CD. These points evidently give the same locus.

7. Maclaurin has specially noted the case (*l.c.* p. 38) where the point A is on the line CD, that is, when $\alpha = a$ or $A^4 - 4C^2 - 4A^2B = 0$, and he remarks that the cubic has then a node. This is the case in which his construction reduces to that given by Barrow, and the curve is then a strophoid.

This cubic has again been discussed by Maclaurin in his Treatise on Fluxions (No. 316), where he finds it in seeking the locus of the point of intersection of two *straight* lines which pass through two fixed points and turn about these points in the same direction, so that the angle which one makes with a given line is double the angle which the other makes with a second given line.

8. Maclaurin gave a mechanical method for drawing the strophoid (*Geometria Organica*, p. 38). Take two rods, OP and PQ (Fig. 4), rigidly connected at P so that the angle QPO is equal

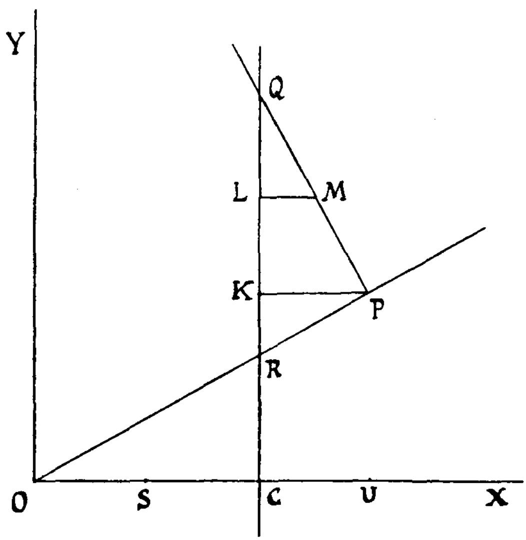

Fig. 4.

to the angle QCO between two given lines. Let this apparatus be moved so that one of the rods passes through O, and so that a point Q on the other, such that $PQ = OC$ moves on the line CQ. Then the point P describes a strophoid. For, since the triangles ORC, RPQ are equal, $RP = RC$, and therefore (§§6, 7) the locus of P is a strophoid.

9. This construction is an extension of that given by Newton for the right cissoid. That great geometer proved that if the angles OCQ, OPQ are right angles, M, the mid-point of PQ, describes that curve. The proposition of Newton can be deduced from that of Maclaurin, as we proceed to show.

Take O as origin, OX perpendicular to CQ, and OY parallel to CQ. The equation to the strophoid described by P is (§§6, 7)

$$y^2 = \frac{x(x-a)^2}{2a-x},$$

where $OC = a$.

If P is (x, y), the equation to OP is $xY = yX$, and the equation to PQ is

$$xX + yY = x^2 + y^2.$$

Substituting $X = a$ in this equation, we find

$$QC = \frac{x^2 + y^2 - ax}{y}.$$

Let now M be (x_1, y_1). Draw ML parallel to OC, and we have

$$x_1 = \tfrac{1}{2}(x+a), \;\; y_1 = \tfrac{1}{2}(KC + QC) = \frac{2y^2 + x^2 - ax}{2y}.$$

Eliminating x and y between these equations and the equation to the strophoid, we obtain

$$y_1^2 = \frac{(a-2x_1)^3}{4(2x_1 - 3a)},$$

which represents a cissoid having its cusp at S, the mid-point of OC. The axis of this cissoid coincides with the line OC, and the asymptote passes through the point U, whose distance from C is equal to SC.

10. This construction can be generalised as follows:—

Suppose that

$$QM = k \,.\, QP,$$

k denoting any real number. Then

$$QL = k \,.\, QK, \;\; LM = k \,.\, KP.$$

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Hence if M is (x_1, y_1)

$$x_1 = a + k(x-a), \quad y_1 = CQ - k(CQ - y),$$

whence
$$y_1 = \frac{x(x-a)[a-k(2a-x)]}{(2a-x)y}.$$

But
$$y^2 = \frac{x(x-a)^2}{2a-x},$$

and therefore
$$y_1^2 = \frac{x[a-k(2a-x)]^2}{2a-x}.$$

Eliminating x between this equation and $x_1 = a + k(x-a)$, we obtain

$$x_1(x_1^2 + y_1^2) = (k+1)a(x_1^2+y_1^2) + k(k-2)a^2x_1 - k^2(k-1)a^3.$$

This is the equation to the locus of M, and transferring the origin to the double point $(ka, 0)$, we obtain the form

$$x_1(x_1^2+y_1^2) = a[(1-2k)x_1^2 + y_1^2].$$

This equation represents *the unicursal circular cubics with one axis of symmetry.* All these curves can therefore be constructed by Newton's method for the cissoid.

11. The unicursal circular cubic called the Trisectrix of Maclaurin was considered by the famous mathematician in the first volume of his *Treatise on Fluxions.* No mention of it is made in the *Geometria Organica.* He defines it as the locus of the vertex M of the triangle MOC when OM and MC turn round the points O and C, so that the supplement of the angle MCO is equal to three times the angle MOC. Maclaurin did not apply it to the trisection of an angle, but this application is an immediate consequence of the definition. The curve called by some authors Burton's Trisectrix is the same as Maclaurin's.