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AN ELEMENTARY PROOF OF THE FROBENIUS FACTORIZATION THEOREM FOR DIFFERENTIAL EQUATIONS

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Consider the 1st order linear system

$$x' = A_n(t)x, \qquad t \in I \text{ (interval)}, \tag{1}$$

$A_n(t) = (a_{ij}(t))_{i,j=1}^n$, when $A_n(t)$ is real valued and continuous on I, $a_{i,i+1}(t) \neq 0$, $a_{ij}(t) \equiv 0$ for $j \geq i+2$, $t \in I$ and $i=1, \ldots, n-1$. In what follows, components c^j of a vector c will be identifiable by the superscript. The purpose of this note is to give a simple induction proof of the following theorem of Frobenius [1].

FACTORIZATION THEOREM. *If the system* (1) *possesses n solutions* $\mu_1(t), \ldots, \mu_n(t)$ *such that*

$$W_k(\mu) \equiv \det(\mu_j^i(t)) \neq 0, \qquad i, j = 1, \ldots, k, \qquad k \leq n, \tag{2}$$

$t \in I$, *then* (1) *is equivalent to the single differential equation of* n^{th} *order for* $\mu = x^1$ *of the form*

$$D(\alpha_{n-1} \cdots D(\alpha_2 D(\alpha_1 D(\alpha_0 \mu))) \cdots) = 0 \tag{3}$$

where

$$\alpha_j = \frac{W_j^2(\mu)}{a_{j,j+1} W_{j-1}(\mu) W_{j+1}(\mu)}, \qquad j = 0, \ldots, n-1, \tag{4}$$

$D = d/dt$, $W_{-1} \equiv W_0 \equiv a_{01} \equiv 1$.

A rather long and complicated proof of the Factorization Theorem is given by Hartman [2, pp. 51–54].

Suppose $\mu_1, \ldots, \mu_n$ are linearly independent solutions of (1) with $\mu_1^1(t) \neq 0$ on I. The change of variable $x = Uw$ in (1) where $w = (w^0, w^1, \ldots, w^{n-1})$,

$$U = \begin{pmatrix} \mu_1^1(t) & 0 \\ \cdot & \\ \cdot & I_{n-1} \\ \cdot & \\ \mu_1^n(t) & \end{pmatrix},$$

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and I_{n-1} is the identity matrix of order $n-1$, has the simple consequence of transforming (1) into a similar system which is essentially of only the $(n-1)^{st}$ dimension. That is, the vectors $(v_1, \ldots, v_{n-1})$, where

$$v_{j-1}^i = -(\mu_1^{i+1}/\mu_1^1)\mu_j^1+\mu_j^{i+1}, \qquad j = 2, \ldots, n, \tag{5}$$

$i=1, \ldots, n-1$, are linearly independent solutions of the system

$$v' = B_{n-1}(t)v, \tag{6}$$

where if $B_{n-1}(t)=(b_{ij}(t))$, then

$$b_{ij}(t) = a_{i+1,j+1}(t), \qquad i = 1, \ldots, n-1, \qquad j = 2, \ldots, n-1. \tag{7}$$

Furthermore,

$$W_{k+1}(\mu_1, \ldots, \mu_{k+1}) = \mu_1^1 W_k(v_1, \ldots, v_k), \qquad k = 1, \ldots, n-1. \tag{8}$$

Proof of Theorem. The proof is by induction on the dimension of the system. If $n=1$, the result is immediate. Suppose $n>1$, since $W_1(\mu)=\mu_1^1(t)\neq 0$, the change of variable is applicable. Furthermore, (2) and (8) imply

$$W_k(v) \neq 0, \qquad k = 1, \ldots, n-1.$$

Hence, the induction hypothesis applied to (6) implies

$$D(\beta_{n-2} \cdots D(\beta_2 D(\beta_1 D(\beta_0 v)))\cdots) = 0 \tag{9}$$

where

$$\beta_j = \frac{W_j^2(v)}{b_{j,j+1}W_{j-1}(v)W_{j+1}(v)}.$$

However, (7) and (8) imply

$$\beta_j = \frac{W_j^2(v)(\mu_1^1)^2}{b_{j,j+1}W_{j-1}(v)W_{j+1}(v)(\mu_1^1)^2} = \frac{W_{j+1}^2(\mu)}{a_{j+1,j+2}W_j(\mu)W_{j+2}(\mu)} = \alpha_{j+1} \tag{10}$$

$j=0, \ldots, n-2$. Furthermore

$$D(\beta_0 v) = D(\alpha_1 D(\alpha_0 \mu)), \tag{11}$$

since $v^1=-(\mu_1^2/\mu_1^1)x^1+x^2$ and $(x^1)'=a_{11}x^1+a_{12}x^2$. Substituting (11) and (10) into (9), we obtain (3). Hence the induction is complete.

Bibliography

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2. P. Hartman, *Ordinary differential equations*, Wiley, New York, 1964.

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