LMS J. Comput. Math. 13 (2010) 130–143 © 2010 Authors
doi:10.1112/S1461157008000430

A lower bound for Garsia's entropy for certain Bernoulli convolutions

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Abstract

Let $\beta \in (1, 2)$ be a Pisot number and let H_β denote Garsia's entropy for the Bernoulli convolution associated with β. Garsia, in 1963, showed that $H_\beta < 1$ for any Pisot β. For the Pisot numbers which satisfy $x^m = x^{m-1} + x^{m-2} + \ldots + x + 1$ (with $m \geqslant 2$), Garsia's entropy has been evaluated with high precision by Alexander and Zagier for $m = 2$ and later by Grabner, Kirschenhofer and Tichy for $m \geqslant 3$, and it proves to be close to 1. No other numerical values for H_β are known. In the present paper we show that $H_\beta > 0.81$ for all Pisot β, and improve this lower bound for certain ranges of β. Our method is computational in nature.

1. *Introduction and summary*

Representations of real numbers in non-integer bases were introduced by Rényi [**19**] and first studied by Rényi and Parry [**16, 19**]. Let β be a real number > 1. A *β-expansion* of the real number $x \in [0, 1]$ is an infinite sequence of integers $(a_1, a_2, a_3, \ldots)$ such that $x = \sum_{n \geqslant 1} a_n \beta^{-n}$. The reader is referred to Lothaire [**15**, Chapter 7] for more on these topics. For the purposes of this paper, we assume $1 < \beta < 2$ and $a_i \in \{0, 1\}$.

Let μ_β denote the *Bernoulli convolution* parameterized by β on $I_\beta := [0, 1/(\beta - 1)]$, that is,

$$\mu_\beta(E) = \mathbb{P}\left\{(a_1, a_2, \ldots) \in \{0, 1\}^{\mathbb{N}} : \sum_{k=1}^{\infty} a_k \beta^{-k} \in E\right\}$$

for any Borel set $E \subseteq I_\beta$, where $\mathbb{P}$ is the product measure on $\{0, 1\}^{\mathbb{N}}$ with $\mathbb{P}(a_1 = 0) = \mathbb{P}(a_1 = 1) = 1/2$. Since $\beta < 2$, it is obvious that $\operatorname{supp}(\mu_\beta) = I_\beta$.

Bernoulli convolutions have been studied for decades (see for example Peres, Schlag and Solomyak [**17**] and Solomyak [**22**]), but there are still many open problems in this area. The most significant property of μ_β is the fact that it is either absolutely continuous or purely singular (see Jessen and Wintner [**12**]); Erdős showed that if β is a Pisot number, then it is singular (see [**5**]). No other β with this property have been found so far.

Recall that a number $\beta > 1$ is called a *Pisot number* if it is an algebraic integer whose Galois conjugates $h \neq \beta$ are less than 1 in modulus. Such is the golden ratio $\tau = (1 + \sqrt{5})/2$ and, more generally, the *multinacci numbers* τ_m, the positive real root satisfying $x^m = x^{m-1} + x^{m-2} + \ldots + x + 1$ with $m \geqslant 2$. The set of Pisot numbers is typically denoted by S. It has been proved by Salem that S is a closed subset of $(1, \infty)$ (see [**20**]). Moreover, Siegel has proved that the smallest Pisot number is the real cubic unit satisfying $x^3 = x + 1$ (see [**21**]). Amara [**2**] gave a complete description of the set of all limit points of the Pisot numbers in $(1, 2)$. In particular, we have the following theorem.

Theorem 1 (Amara). *The limit points of S in $(1, 2)$ are the following:*

$$\varphi_1 = \psi_1 < \varphi_2 < \psi_2 < \varphi_3 < \chi < \psi_3 < \varphi_4 < \ldots < \psi_r < \varphi_{r+1} < \ldots < 2,$$

Received 24 November 2008; revised 13 November 2009.

2000 Mathematics Subject Classification 26A30 (primary), 11R06 (secondary).

Research of K. G. Hare supported in part by NSERC of Canada. Computational support provided in part by the Canadian Foundation for Innovation, and the Ontario Research Fund.

where

$$\begin{cases} \text{the minimal polynomial of } \varphi_r \text{ is } \Phi_r(x) = x^{r+1} - 2x^r + x - 1, \\ \text{the minimal polynomial of } \psi_r \text{ is } \Psi_r(x) = x^{r+1} - x^r - \ldots - x - 1, \\ \text{the minimal polynomial of } \chi \text{ is } \mathcal{X}(x) = x^4 - x^3 - 2x^2 + 1. \end{cases}$$

A description of the Pisot numbers approaching these limit points was given by Talmoudi [**23**]. Regular Pisot numbers are defined as the Pisot roots of the polynomials in Table 1. Pisot numbers that are not regular Pisot numbers are called irregular Pisot numbers. For each of these limit points (φ_r, ψ_r or χ), there exists an ϵ (dependent on the limit point) such that all Pisot numbers in an ϵ-neighborhood of this limit point are these regular Pisot numbers. The Pisot root of the defining polynomial approaches the limit point as n tends to infinity. It should be noted that these polynomials are not necessarily minimal, and may contain some cyclotomic factors. Also, they are only guaranteed to have a Pisot number root for sufficiently large n.

Computationally, Boyd [**3, 4**] has given an algorithm that will find all Pisot numbers in an interval, where, in the case of limit points, the algorithm can detect the limit points and compensate for them.

Garsia [**9**] introduced a new notion associated with a Bernoulli convolution. Namely, put

$$D_n(\beta) = \left\{ x \in I_\beta : x = \sum_{k=1}^{n} a_k \beta^{-k} \text{ with } a_k \in \{0, 1\} \right\}$$

and, for $x \in D_n(\beta)$,

$$p_n(x) = \#\left\{ (a_1, \ldots, a_n) \in \{0, 1\}^n : x = \sum_{k=1}^{n} a_k \beta^{-k} \right\}. \tag{1}$$

Finally, put

$$H_\beta^{(n)} = - \sum_{x \in D_n(\beta)} \frac{p_n(x)}{2^n} \log \frac{p_n(x)}{2^n}$$

and

$$H_\beta = \lim_{n \to \infty} \frac{H_\beta^{(n)}}{n \log \beta}$$

(it was shown in [**9**] that the limit always exists). The value H_β is called *Garsia's entropy*.

Obviously, if β is transcendental or algebraic but not satisfying an algebraic equation with coefficients $\{-1, 0, 1\}$, then all the sums $\sum_{k=1}^{n} a_k \beta^{-k}$ are distinct, whence $p_n(x) = 1$ for any $x \in D_n(\beta)$, and $H_\beta = \log 2 / \log \beta > 1$.

TABLE 1. *Regular Pisot numbers.*

Limit points	Defining polynomials
φ_r	$\Phi_r(x)x^n \pm (x^r - x^{r-1} + 1)$
	$\Phi_r(x)x^n \pm (x^r - x + 1)$
	$\Phi_r(x)x^n \pm (x^r + 1)(x - 1)$
ψ_r	$\Psi_r(x)x^n \pm (x^{r+1} - 1)$
	$\Psi_r(x)x^n \pm (x^r - 1)/(x - 1)$
χ	$\mathcal{X}(x)x^n \pm (x^3 + x^2 - x - 1)$
	$\mathcal{X}(x)x^n \pm (x^4 - x^2 + 1)$

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However, if β is Pisot, then it was shown in [9] that $H_\beta < 1$: which means in particular that β does satisfy an equation with coefficients $\{0, \pm 1\}$. Furthermore, Garsia also proved that if $H_\beta < 1$, then μ_β is singular.

In 1991 Alexander and Zagier [1] managed to evaluate H_β for the golden ratio $\beta = \tau$ with an astonishing accuracy. It turned out that H_τ is close to 1: in fact, $H_\tau \approx 0.9957$. Grabner, Kirschenhofer and Tichy [10] extended this method to the multinacci numbers; in particular, $H_{\tau_3} \approx 0.9804$, $H_{\tau_4} \approx 0.9867$ and so on. They also showed that H_{τ_m} is strictly increasing for $m \geqslant 3$, and $H_{\tau_m} \to 1$ as $m \to \infty$ exponentially fast.

The method suggested in [1] has, however, its limitations and apparently cannot be extended to non-multinacci Pisot parameters β. Consequently, no numerical value for H_β is known for any non-multinacci Pisot β, not even a lower bound.

The main goal of this paper is to present a universal lower bound for H_β for β a Pisot number in $(1, 2)$. We prove that $H_\beta > 0.81$ for all such β (Theorem 9) and improve this bound for certain ranges of β (see discussion in Remark 7 and Proposition 10).

2. *The maximal growth exponent*

Denote by $\mathcal{E}_n(x; \beta)$ the set of all 0–1 words of length n which may act as prefixes of β-expansions of x. We first prove a simple characterization of this set.

LEMMA 2. *We have*

$$\mathcal{E}_n(x; \beta) = \left\{ (a_1, \ldots, a_n) \in \{0,1\}^n \;\middle|\; 0 \leqslant x - \sum_{k=1}^{n} a_k \beta^{-k} \leqslant \frac{\beta^{-n}}{\beta - 1} \right\}.$$

Proof. Let $(a_1, \ldots, a_n) \in \mathcal{E}_n(x; \beta)$; then the fact that there exists a β-expansion of x beginning with this word implies

$$\sum_{1}^{n} a_k \beta^{-k} \leqslant x \leqslant \sum_{1}^{n} a_k \beta^{-k} + \frac{\beta^{-n}}{\beta - 1},$$

the second inequality following from $\sum_{n+1}^{\infty} a_k \beta^{-k} \leqslant \beta^{-n}/(\beta - 1)$.

The converse follows from the fact that if $0 \leqslant y \leqslant 1/(\beta - 1)$, where $y = \beta^n (x - \sum_{k=1}^{n} a_k \beta^{-k})$, then y has a β-expansion $(a_{n+1}, a_{n+2}, \ldots)$. □

The following lemma will play a central role in this paper.

LEMMA 3. *Suppose there exists $\lambda \in (1, 2)$ such that $\#\mathcal{E}_n(x; \beta) = O(\lambda^n)$ for all $x \in I_\beta$. Then*

$$H_\beta \geqslant \log_\beta \frac{2}{\lambda}. \tag{2}$$

Proof. Let $(a_1, a_2, \ldots)$ be a β-expansion of x. Denote by $p_n(a_1, \ldots, a_n)$ the number of 0–1 words $(a'_1, \ldots, a'_n)$ such that $\sum_{k=1}^{n} a_k \beta^{-k} = \sum_{k=1}^{n} a'_k \beta^{-k}$. Then, as was shown by Lalley [14, Theorems 1 and 2],

$$\sqrt[n]{p_n(a_1, \ldots, a_n)} \to 2\beta^{-H_\beta}, \quad \mathbb{P}\text{-almost everywhere } (a_1, a_2, \ldots) \in \{0,1\}^{\mathbb{N}}. \tag{3}$$

Since $p_n(a_1, \ldots, a_n) \leqslant \#\mathcal{E}_n(x; \beta)$ for $x = \sum_{k=1}^{n} a_k \beta^{-k}$, we have $\sqrt[n]{p_n(a_1, \ldots, a_n)} \leqslant \varepsilon_n \lambda$ with $\varepsilon_n \to 1$, which, together with (3), implies (2). □

Define the *maximal growth exponent* as follows:

$$\mathfrak{M}_\beta := \sup_{x \in I_\beta} \limsup_{n \to \infty} \sqrt[n]{\#\mathcal{E}_n(x; \beta)}.$$

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It follows from Lemma 3 that

$$H_\beta \geqslant \log_\beta \frac{2}{\mathfrak{M}_\beta}. \tag{4}$$

Computing $\mathfrak{M}_\beta$ explicitly for a given Pisot β looks like a difficult problem (unless β is multinacci; see § 6), so our goal is to obtain good upper bounds for $\mathfrak{M}_\beta$ for various ranges of β. To do that, we will need the following simple, but useful, claim.

Proposition 4. *If $\#\mathcal{E}_{n+r}(x;\beta) \leqslant R \cdot \#\mathcal{E}_n(x;\beta)$ for all $n \geqslant n_0$ for some $n_0 \geqslant 1$ and some $r \geqslant 2$, then $\mathfrak{M}_\beta \leqslant \sqrt[r]{R}$.*

Proof. By induction,

$$\#\mathcal{E}_{n_0+rk}(x;\beta) \leqslant \#\mathcal{E}_{n_0}(x;\beta)R^k \leqslant 2^{n_0}R^k.$$

Let $n \geqslant n_0$, and choose k_n such that $n_0 + r(k_n - 1) \leqslant n < n_0 + rk_n$. Then

$$\#\mathcal{E}_n(x;\beta) \leqslant \#\mathcal{E}_{n_0+rk_n}(x;\beta) \leqslant 2^{n_0}R^{k_n}.$$

The result follows from

$$\lim_{n\to\infty} (2^{n_0}R^{k_n})^{1/n} = \lim_{n\to\infty} 2^{n_0/n}R^{k_n/n} = R^{1/r} = \sqrt[r]{R}$$

by noticing that $n_0/n \to 0$ and $k_n/n \to 1/r$ as $n \to \infty$. □

Example 5. For the examples in this paper, we give only four digits of precision. In fact, much higher precision was used in the computations (about 50 digits). Let us consider a toy example showing how to apply (4) to $\beta = \beta_* \approx 1.6737$, the largest root of $x^5 - 2x^4 + x^3 - x^2 + x - 1$ (which is a Pisot number).

Let us first determine $\#\mathcal{E}_2(x;\beta_*)$, dependent upon x. After that, we will determine $\max_{x\in I_{\beta_*}} \#\mathcal{E}_2(x;\beta_*)$. For ease of notation, we will denote $m_n(\beta) = \max_{x\in I_\beta} \#\mathcal{E}_n(x;\beta)$. Hence, in this case, we are determining $m_2(\beta_*)$. Consider the values of x such that $x = (a_1/\beta) + (a_2/\beta^2) + \ldots$ for the initial string (a_1, a_2). We see that

$$\frac{a_1}{\beta} + \frac{a_2}{\beta^2} \leqslant x \leqslant \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{1}{\beta^3} + \frac{1}{\beta^4} + \ldots = \frac{a_1}{\beta} + \frac{a_2}{\beta^2} + \frac{1/\beta^3}{1 - 1/\beta}.$$

This gives us upper and lower bounds for possible initial strings of (a_1, a_2) (see Table 2).

We next partition possible values of x in $I_\beta = [0, 1.4845]$ based on these upper and lower bounds (see Table 3). This immediately shows that $m_2(\beta_*) = 2$. Hence, by induction, $\#\mathcal{E}_{n+2}(x;\beta_*) \leqslant 2\#\mathcal{E}_n(x;\beta_*)$, whence, by Proposition 4, $\mathfrak{M}_{\beta_*} \leqslant \sqrt{2}$. By (4), $H_{\beta_*} > \frac{1}{2}\log_{\beta_*} 2 \approx 0.6729$.

Obviously, this bound is rather crude, and in the rest of the paper we will refine this method to obtain better bounds. One thing we need to do is show how we would use this for an entire range of β values, instead of just for a specific value. For instance, in the example above, we could show that $m_2(\beta) = 2$ for all $\beta > \tau = (1 + \sqrt{5})/2$. In addition, we will want to show how we would perform this calculation for algebraic β, where we can take advantage of the algebraic nature of β.

3. *The algorithm*

Let us consider our toy example of $\beta = \beta_*$ again. We see that for each initial string (a_1, a_2), we got a lower and an upper bound for possible $x = a_1\beta^{-1} + a_2\beta^{-2} + \ldots$. For example, for $(a_1, a_2) = (1, 0)$ these were approximately 0.5975 and 1.1275, respectively. We then used these lower and upper bounds to partition I_β into ranges. We next show that if the relative order of these lower and upper bounds is not changed, then the partitioning of I_β into ranges can be done in exactly the same way.

Put

$$(a_1, \ldots, a_k)_L = \sum_1^k a_j \beta^{-j} \quad \text{and} \quad (a_1, \ldots, a_k)_U = \sum_1^k a_j \beta^{-j} + \frac{\beta^{-k}}{\beta - 1},$$

that is, $[(a_1, \ldots, a_k)_L, (a_1, \ldots, a_k)_U]$ is the interval of all possible values of x whose β-expansion starts with $(a_1, \ldots, a_k)$. For example, $(1, 0)_L = 0.5975\ldots$ and $(1, 0)_U = 1.1275\ldots$. This says that if

$$(0, 0)_L < (0, 1)_L < (0, 0)_U < (1, 0)_L < (0, 1)_U < (1, 1)_L < (1, 0)_U < (1, 1)_U,$$

then we have Table 4 as the equivalent table to Table 3. For fixed β, these $(a_1, a_2, \ldots, a_k)_L$ and $(a_1, a_2, \ldots, a_k)_U$ are called *critical points for* β or simply *critical points*.

For each inequality, there are precise values of β where the inequality will hold. For example, knowing that $\beta > 1$, we get that

$$(0, 0)_U < (1, 0)_L \iff \frac{\beta^{-3}}{1 - \beta^{-1}} < \frac{1}{\beta} \iff \frac{1 + \sqrt{5}}{2} < \beta.$$

So, if $\beta > \tau = 1.618\ldots$, then $(0, 0)_U < (1, 0)_L$.

This observation means that we need to determine for which values of β we have $(a_1, a_2)_{L/U} = (a_1', a_2')_{L/U}$. We will call these values of β the *transitions points* which will affect $m_n(\beta)$.

There are some immediate observations we can make that reduces the number of equations to be checked.

- $(a_1, a_2)_L = (a_1', a_2')_L$ and $(a_1, a_2)_U = (a_1', a_2')_U$ have the same set of solutions.
- $(a_1, a_2)_L = (a_1, a_2)_U$ has no solutions.

TABLE 2. *Upper and lower bounds for x for initial strings of length 2 of its β-expansion.*

(a_1, a_2)	Lower bound	Upper bound
$(0, 0)$	0	0.5300
$(0, 1)$	0.3570	0.8870
$(1, 0)$	0.5975	1.1275
$(1, 1)$	0.9545	1.4845

TABLE 3. *Initial strings (a_1, a_2), depending on $x \in (0, 1.4875)$.*

Range (approx.)	Possible initial string of expansion
$x \in (0.0, 0.3570)$	$(0, 0)$
$x \in (0.3570, 0.5300)$	$(0, 0), (0, 1)$
$x \in (0.5300, 0.5975)$	$(0, 1)$
$x \in (0.5975, 0.8870)$	$(0, 1), (1, 0)$
$x \in (0.8870, 0.9545)$	$(1, 0)$
$x \in (0.9545, 1.1275)$	$(1, 0), (1, 1)$
$x \in (1.1275, 1.4845)$	$(1, 1)$

TABLE 4. *Upper and lower bounds for x for initial strings of length 2 of its β-expansion.*

Range	Possible initial string of β-expansion of x
$x \in ((0, 0)_L, (0, 1)_L)$	$(0, 0)$
$x \in ((0, 1)_L, (0, 0)_U)$	$(0, 0), (0, 1)$
$x \in ((0, 0)_U, (1, 0)_L)$	$(0, 1)$
$\vdots$	$\vdots$

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- If $a_1 \leqslant a_1'$ and $a_2 \leqslant a_2'$, then none of

$$\begin{aligned}(a_1, a_2)_L &= (a_1', a_2')_L,\\ (a_1, a_2)_L &= (a_1', a_2')_U,\\ (a_1, a_2)_U &= (a_1', a_2')_U\end{aligned}$$

have solutions in I_β.

The first two observations were used when finding all transition points. The last observation was made by one of the referees after all of the computations were completed, and hence was not used as a means of eliminating equations to check.

In our length-2 example again, we need to check (after elimination by the three observations above) when

$$\begin{array}{llll}(0,0)_U = (0,1)_L, & (0,0)_U = (1,0)_L, & (0,0)_U = (1,1)_L, & (0,1)_L = (1,0)_L,\\ (0,1)_L = (1,0)_U, & (1,0)_U = (1,1)_L, & (0,1)_U = (1,0)_L, & (0,1)_U = (1,1)_L.\end{array}$$

Solving all of these equations, we see that the only transition points in $(1, 2)$ for length 2 are $\sqrt{2} \approx 1.4142$ and $\tau \approx 1.6180$.

So, given that we know $m_n(\beta_*) = 2$, and that we have a transition point at $\tau = 1.618\ldots$, we can say for all $\beta \in (\tau, 2)$ that $m_2(\beta) = 2$. Using a similar method, we can show that for $\beta \in (\sqrt{2}, \tau)$ that $m_2(\beta) = 3$, and that for $\beta \in (1, \sqrt{2})$ that $m_2(\beta) = 4$.

It is worth noting that these results do not say what happens when $\beta = \sqrt{2}$ or $\beta = \tau$. The transition points will need to be checked separately.

There is one not so obvious, but important, observation that should be made at this point. It is possible for an inequality to hold for β, where β is in a disjoint union of intervals.

For example, we have

$$(0, 1, 1, 1, 1)_L < (1, 0, 0, 0, 1)_U$$

for $\beta \in (1, \sigma) \cup (\tau, 2)$, where $\sigma^3 - \sigma^2 - 1 = 0$, with $\sigma \approx 1.4656$. This means that it is possible for $m_n(\beta)$ to not be a decreasing function with respect to β. For example, $m_5(1.81) = 3$, $m_5(1.85) = 4$ and $m_5(1.88) = 3$. This phenomenon appears to become more common for larger values of n.

4. *Numerical computations*

In this section we will talk about the specific computations, and how they were done. The process started with length $n = 2$, and then progressively worked on $n = 3, 4, 5, \ldots$ up to $n = 14$. We used this process to find the global minimum for all $\beta \in (1.6, 2)$ minus a finite set of transition points. The code for finding transition points, numerical lower bounds and symbolic lower bounds can be found in the home page of the first author [**11**].

- For each length in order, find all solutions β to

$$(a_0, a_1, \ldots, a_{n-1})_{L/U} = (a_0', a_1', \ldots, a_{n-1}')_{L/U}$$

subject to the conditions mentioned in the previous section.

- For each of these solutions, check to see if the transition point is a Pisot number. If so, we will have to check this transition point using the methods of § 5.
- Use these transition points to partition $(1, 2)$ into subintervals, upon which $m_n(\beta)$ is constant.
- For the mid point of each of these subintervals, compute $m_n(\beta)$.

To compute $m_n(\beta)$, we first consider all 0–1 sequences $w_1, w_2, \ldots$ of length n. For each of these sequences, find their upper and lower bounds, say $\{\alpha_1, \alpha_2, \ldots\} = \{w_{1L}, w_{1U}, w_{2L}, w_{2U}, \ldots\}$. Here the α_i are reordered such that $\alpha_i < \alpha_{i+1}$ for all i. We then loop through each interval (α_i, α_{i+1}) and check how many of the w_i are valid on this interval. We keep track of the interval with the maximal set of valid w_i.

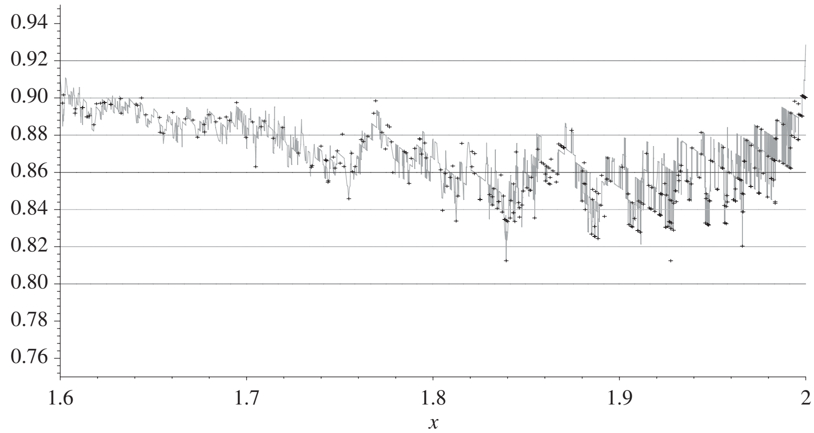

FIGURE 1. *Lower bound for H_β, for Pisot $\beta \in (1.6, 2.0)$, and Pisot transition points.*

It should be noted that the number of times we needed to run this algorithm was rather big. At level 14, we had slightly more than 300 000 tests where we needed to find the maximal set.

These calculations were done in Maple on 22 separate 4-CPU, 2.8-GHz machines each with 8 GB of RAM. These calculations were managed using the N1 Grid Engine. This cluster was capable of performing 88 simultaneous computations.

After this, we looked at all of these subintervals between transition points, and calculated the lower bounds for H_β at the end points, to find a global minimum. This gives rise to the main result of the paper.

THEOREM 6. *If $\beta > 1.6$, and β is not a transition point for $n \leqslant 14$, then $H_\beta > 0.81$.*

REMARK 7. This theorem is weaker than necessary for most values of β. For specific ranges of values of β, we actually get a number of stronger results.

- Most $\beta \in (1.6, 2.0)$ have $H_\beta > 0.82$ (99.9%), and a majority (51.4%) have $H_\beta > 0.87$. Here 'most' is a bit misleading. Almost every β has $H_\beta = \log 2/\log \beta$. Of those that do not, there is no result that shows that they should be evenly distributed (and they most likely are not). So, by 'most' we mean that for some finite collection of intervals, that make up 99.9% of $(1.6, 2.0)$, that *all* β in this finite collection of intervals have $H_\beta > 0.82$.
- The minimum occurs near $\tau_3 \approx 1.8392$ (see Figures 1 and 2).
- For $\beta \in (1.6, 1.7)$, we have $H_\beta > 0.87$ (Figure 3) and, for β near 2.0, we have $H_\beta > 0.9$ (Figure 4).

5. *Calculations for symbolic β*

In the previous section, we showed for all but a finite number of Pisot numbers β in $(1.6, 2)$ that $H_\beta > 0.81$. To extend the result to all such β in $(1, 2)$, there are still some Pisot numbers that will need to be checked individually.

These include the finite set of Pisot numbers less that 1.6 (of which there are 12), and the finite set of Pisot numbers that are also transition points (of which there are 427). In particular, we get the following theorem.

THEOREM 8. *For all Pisot numbers $\beta < 1.6$ and all Pisot transition points (for $n \leqslant 14$), we have $H_\beta > 0.81$.*

Combined, this theorem and Theorem 6 yield the following theorem.

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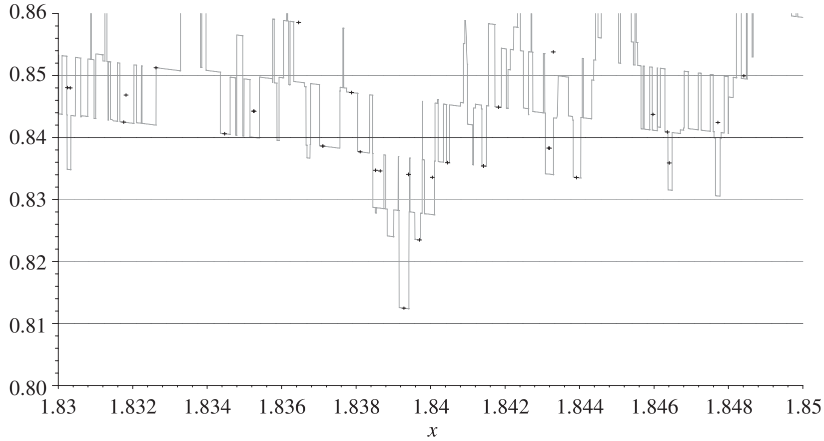

FIGURE 2. *Lower bound for H_β, for Pisot $\beta \in (1.83, 1.85)$, and Pisot transition points.*

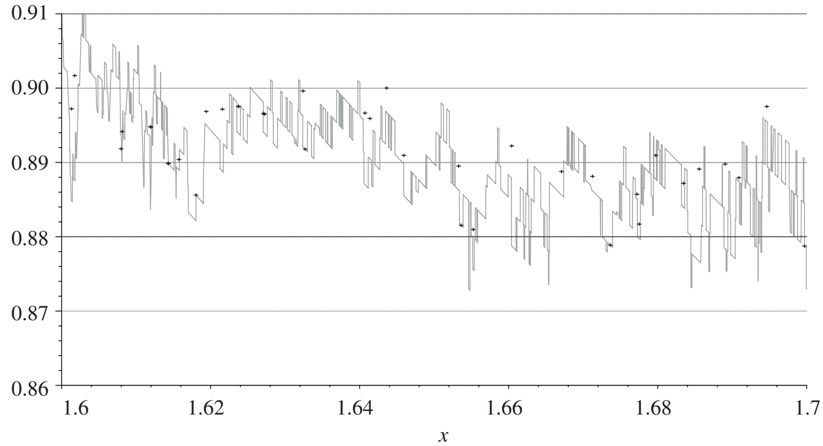

FIGURE 3. *Lower bound for H_β, for Pisot $\beta \in (1.6, 1.7)$, and Pisot transition points.*

THEOREM 9. *For any Pisot β we have $H_\beta > 0.81$.*

As a corollary, we obtain a result on small Pisot numbers.

PROPOSITION 10. *All Pisot $\beta < 1.7$ have Garsia entropy $H_\beta > 0.87$.*

There are actually a lot of advantages to doing a symbolic check as compared to the numerical techniques of the previous section. Some of these include not requiring high-precision arithmetic and the combining of equivalent strings, both of which have speed and memory advantages. These are described in the example below.

To illustrate the (computer-assisted) proof of Theorem 8, consider as an example $\beta = \tau$, the golden ratio. As before, we wish to find the values of x that satisfy

$$\frac{a_1}{\tau} + \frac{a_2}{\tau^2} \leqslant x \leqslant \frac{a_1}{\tau} + \frac{a_2}{\tau^2} + \frac{1/\tau^3}{1 - 1/\tau}.$$

But now we can find exact symbolic values for these ranges. In particular, we notice that $((1/\tau^3)/(1 - 1/\tau)) = \tau - 1$. Secondly, as $(1/\tau) = \tau - 1$ and $(1/\tau^2) = 2 - \tau$, we obtain Table 6.

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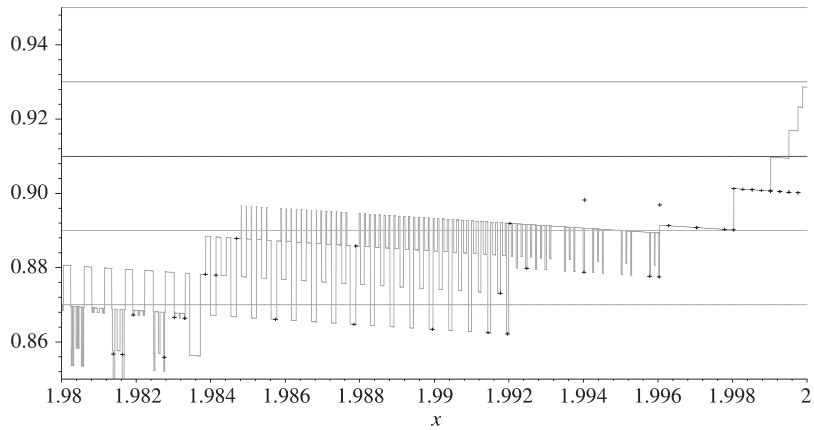

FIGURE 4. *Lower bound for H_β, for Pisot $\beta \in (1.98, 2.0)$, and Pisot transition points.*

So, in particular, it *is* possible for x to start with both $(0,0)$ and $(1,0)$. But, if this is the case, then $x = (0,0,1,1,1,\ldots) = (1,0,0,0,\ldots) = \tau - 1$. So, it is not possible for x to have an infinite number of expansions starting with $(0,0)$ and an infinite number of expansions starting with $(1,0)$. Similar arguments can be used for the other critical point, $x = 1$.

So, we can discard the critical points and subdivide the possible values of x into the ranges given in Table 7.

This immediately shows $m_2(\tau) = 2$. Hence, by induction, $\#\mathcal{E}_{n+2}(x;\tau) \leqslant 2\#\mathcal{E}_n(x;\tau)$, whence $\mathfrak{M}_\tau \leqslant \sqrt{2}$. By (4), $H_\tau > \frac{1}{2}\log_\tau 2 = 0.7202100$.

The main advantage of this method comes when we have longer strings. In particular, it is easy to see that $(1,0,0) = (0,1,1)$ (see Table 8). This allows us to compress information.

This gives that for $x \in (\tau - 1, 4 - 2\tau)$ we have the initial string of $(0,1,0)$, $(0,1,1)$, $(1,0,0)$, and if $x \in (3\tau - 4, 1)$ we have the initial string of $(1,0,1)$, $(0,1,1)$, $(1,0,0)$.

TABLE 5. *Lower bounds for Garsia's entropy for all Pisot numbers < 1.6.*

Minimal polynomial of β	Pisot number	Length	Lower bound for H_β
$x^3 - x - 1$	1.3247	17	0.88219
$x^4 - x^3 - 1$	1.3803	16	0.87618
$x^5 - x^4 - x^3 + x^2 - 1$	1.4433	15	0.89257
$x^3 - x^2 - 1$	1.4656	15	0.88755
$x^6 - x^5 - x^4 + x^2 - 1$	1.5016	14	0.90307
$x^5 - x^3 - x^2 - x - 1$	1.5342	15	0.89315
$x^7 - x^6 - x^5 + x^2 - 1$	1.5452	13	0.90132
$x^6 - 2x^5 + x^4 - x^2 + x - 1$	1.5618	15	0.90719
$x^5 - x^4 - x^2 - 1$	1.5701	15	0.88883
$x^8 - x^7 - x^6 + x^2 - 1$	1.5737	14	0.90326
$x^7 - x^5 - x^4 - x^3 - x^2 - x - 1$	1.5900	15	0.89908
$x^9 - x^8 - x^7 + x^2 - 1$	1.5912	14	0.90023

TABLE 6. *Upper and lower bounds for initial strings of length 2 for $x = a_1\tau^{-1} + a_2\tau^{-2} + \ldots$.*

(a_1, a_2)	Lower bound	Upper bound
$(0,0)$	0	$\tau - 1 \approx 0.618$
$(0,1)$	$2 - \tau \approx 0.382$	1
$(1,0)$	$\tau - 1 \approx 0.618$	$2\tau - 2 \approx 1.236$
$(1,1)$	1	$\tau \approx 1.618$

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Our implementation does not maintain a separate entry for $(0,1,1)$ and $(1,0,0)$, as they are equivalent. Instead, the algorithm stores only one of these two strings, and indicates that this has weight 2. For the general Pisot β, this is checked by noticing that $(a_1, a_2, \ldots, a_n)$ is equivalent to the same word as $(b_1, b_2, \ldots, b_n)$ if and only if

$$a_n x^{n-1} + \ldots + a_1 \equiv b_n x^{n-1} + b_{n-1} x^{n-2} + \ldots + b_1 \equiv c_{d-1} x^{d-1} + \ldots + c_d \pmod{p(x)}$$

for some c_i, with $p(x)$ the minimal polynomial for β, of degree d. Given the large amount of overlapping that we see for large lengths, this will have major cost savings, in both memory and time.

6. *The maximal growth exponent for the multinacci family and discussion*

In this section we will compute the maximal growth exponent for the multinacci family and compare our lower bound (4) with the actual values.

Let, as above, τ_m denote the largest root of $x^m - x^{m-1} - \ldots - x - 1$ (hence $\tau = \tau_2$). Define the *local dimension* of the Bernoulli convolution μ_β as follows:

$$d_\beta(x) = \lim_{h \to 0} \frac{\log \mu_\beta(x-h, x+h)}{\log h}$$

(if the limit exists). As was shown in Lalley [**14**], $d_\beta(x) \equiv H_\beta$ for μ_β-almost everywhere $x \in I_\beta$ for any Pisot β.

Notice that it is well known that the limit in question exists if it does so along the subsequence $h = c\beta^{-n}$ for any fixed $c > 0$ (see for example [**6**]). We choose $c = (\beta - 1)^{-1}$, so

$$d_\beta(x) = -\lim_{n \to \infty} \frac{1}{n} \log_\beta \mu_\beta\left(x - \frac{\beta^{-n}}{\beta - 1}, x + \frac{\beta^{-n}}{\beta - 1}\right). \tag{5}$$

Let $\beta = \tau_m$ for some $m \geqslant 2$.

Lemma 11. *Suppose that β is multinacci, and put*

$$\varepsilon_\beta(x) = \lim_{n \to \infty} \sqrt[n]{\#\mathcal{E}_n(x; \beta)}.$$

Table 7. *Initial string of τ-expansion of x, depending on x.*

Range	Possible initial string of the τ-expansion
$x \in (0, 2-\tau)$	$(0,0)$
$x \in (2-\tau, \tau-1)$	$(0,0)$, $(0,1)$
$x \in (\tau-1, 1)$	$(0,1)$, $(1,0)$
$x \in (1, 2\tau-2)$	$(1,0)$, $(1,1)$
$x \in (2\tau-2, \tau)$	$(1,1)$

Table 8. *Upper and lower bounds for initial string of length 3 for $x = a_1\tau^{-1} + a_2\tau^{-2} + a_3\tau^{-3} + \ldots$.*

$a_1a_2a_3$	Lower bound	Upper bound
$(0,0,0)$	0	$5-3\tau \approx 0.1459$
$(0,0,1)$	$2\tau-3 \approx 0.2361$	$2-\tau \approx 0.3820$
$(0,1,0)$	$2-\tau \approx 0.3820$	$4-2\tau \approx 0.7639$
$(0,1,1) = (1,0,0)$	$\tau-1 \approx 0.6180$	1
$(1,0,1)$	$3\tau-4 \approx 0.8541$	$2\tau-2 \approx 1.2361$
$(1,1,0)$	1	$3-\tau \approx 1.3820$
$(1,1,1)$	$2\tau-2 \approx 1.2361$	$\tau \approx 1.6180$

This limit exists if and only if $d_\beta(x)$ exists and, in this case,

$$d_\beta(x) = \log_\beta \frac{2}{\varepsilon_\beta(x)}.$$

Proof. Let $x = \sum_{k=1}^\infty a_k\beta^{-k}$ and consider $(a_1, \ldots, a_n)$, the first n terms of this sequence. We see that

$$\begin{aligned}(a_1, \ldots, a_n)_L &= \sum_{k=1}^n a_k\beta^{-k} \\ &\geqslant \sum_{k=1}^n a_k\beta^{-k} + \sum_{k=n+1}^\infty (a_k - 1)\beta^{-k} \\ &\geqslant \sum_{k=1}^\infty a_k\beta^{-k} - \sum_{k=n+1}^\infty \beta^{-k} \\ &= x - \frac{\beta^n}{\beta - 1}\end{aligned}$$

and

$$\begin{aligned}(a_1, \ldots, a_n)_U &= \sum_{k=1}^n a_k\beta^{-k} + \sum_{k=n+1}^\infty \beta^{-k} \\ &\leqslant \sum_{k=1}^\infty a_k\beta^{-k} + \sum_{k=n+1}^\infty \beta^{-k} \\ &= x + \frac{\beta^n}{\beta - 1}.\end{aligned}$$

Further, this is true, regardless of which representation $(a_1, a_2, \ldots)$ of x that we take. Hence, if $(a_1, \ldots, a_n) \in \mathcal{E}_n(x, \beta)$, then, for *all* $a'_{n+1}, a'_{n+2}, \ldots \in \{0, 1\}$, we have

$$\begin{aligned}\sum_{k=1}^n a_k\beta^{-k} + \sum_{k=n+1}^\infty a'_k\beta^{-k} &\in ((a_1, a_2, \ldots, a_n)_L, (a_1, a_2, \ldots, a_n)_U) \\ &\subseteq \left(x - \frac{\beta^{-n}}{\beta - 1}, x + \frac{\beta^{-n}}{\beta - 1}\right).\end{aligned}$$

This in turn implies that

$$\mu_\beta\left(x - \frac{\beta^{-n}}{\beta - 1}, x + \frac{\beta^{-n}}{\beta - 1}\right) \geqslant 2^{-n}\#\mathcal{E}_n(x; \beta). \tag{6}$$

Now put

$$\widetilde{\mathcal{E}}_n(x; \beta) = \left\{(\tilde{a}_1, \ldots, \tilde{a}_n) \in \{0, 1\}^n \;\middle|\; -\frac{\beta^{-n}}{\beta - 1} \leqslant x - \sum_{k=1}^n \tilde{a}_k\beta^{-k} \leqslant \frac{2\beta^{-n}}{\beta - 1}\right\}.$$

Our next goal is to prove the inequality

$$\mu_\beta\left(x - \frac{\beta^{-n}}{\beta - 1}, x + \frac{\beta^{-n}}{\beta - 1}\right) \leqslant 2^{-n}\#\widetilde{\mathcal{E}}_n(x; \beta). \tag{7}$$

Let $y \in (x - (\beta^{-n}/(\beta - 1)), x + (\beta^{-n}/(\beta - 1)))$ have an expansion $y = \sum_{k=1}^\infty \tilde{a}_k\beta^{-k}$. It suffices to show that $(\tilde{a}_1, \ldots, \tilde{a}_n) \in \widetilde{\mathcal{E}}_n(x; \beta)$.

By noticing that $-(\beta^{-n}/(\beta - 1)) \leqslant x - y \leqslant \beta^{-n}/(\beta - 1)$, we get first that

$$-\frac{\beta^{-n}}{\beta - 1} \leqslant x - y \leqslant x - \sum_{k=1}^\infty \tilde{a}_k\beta^{-k} \leqslant x - \sum_{k=1}^n \tilde{a}_k\beta^{-k}$$

and further that

$$
\begin{aligned}
& x - y \leqslant \frac{\beta^{-n}}{\beta - 1} \\
\Longrightarrow\quad & x - \sum_{k=1}^{\infty} \tilde{a}_k \beta^{-k} \leqslant \frac{\beta^{-n}}{\beta - 1} \\
\Longrightarrow\quad & x - \sum_{k=1}^{n} \tilde{a}_k \beta^{-k} - \sum_{k=n+1}^{\infty} \tilde{a}_k \beta^{-k} \leqslant \frac{\beta^{-n}}{\beta - 1} \\
\Longrightarrow\quad & x - \sum_{k=1}^{n} \tilde{a}_k \beta^{-k} \leqslant \sum_{k=n+1}^{\infty} \tilde{a}_k \beta^{-k} + \frac{\beta^{-n}}{\beta - 1} \\
\Longrightarrow\quad & x - \sum_{k=1}^{n} \tilde{a}_k \beta^{-k} \leqslant \sum_{k=n+1}^{\infty} \beta^{-k} + \frac{\beta^{-n}}{\beta - 1} \\
\Longrightarrow\quad & x - \sum_{k=1}^{n} \tilde{a}_k \beta^{-k} \leqslant 2\frac{\beta^{-n}}{\beta - 1}.
\end{aligned}
$$

Hence, $(\tilde{a}_1, \ldots, \tilde{a}_n) \in \widetilde{E}_n(x; \beta)$, as required.

Combining (6) and (7), we obtain

$$
2^{-n} \#\mathcal{E}_n(x; \beta) \leqslant \mu_\beta \left(x - \frac{\beta^{-n}}{\beta - 1}, x + \frac{\beta^{-n}}{\beta - 1} \right) \leqslant 2^{-n} \#\widetilde{\mathcal{E}}_n(x; \beta),
$$

whence

$$
\begin{aligned}
\log_\beta 2 - \frac{1}{n} \log_\beta \#\widetilde{\mathcal{E}}_n(x; \beta) &\leqslant -\frac{1}{n} \log_\beta \mu_\beta \left(x - \frac{\beta^{-n}}{\beta - 1}, x + \frac{\beta^{-n}}{\beta - 1} \right) \\
&\leqslant \log_\beta 2 - \frac{1}{n} \log_\beta \#\mathcal{E}_n(x; \beta). \qquad (8)
\end{aligned}
$$

Notice that (8) in fact holds for any β. Now we use the fact that β is multinacci. It follows from [**6**, Lemma 2.11] that for a multinacci β one has $\sqrt[n]{p_n(x)} \sim \sqrt[n]{p_n(x')}$ provided $|x - x'| \leqslant C\beta^{-n}$ for any fixed $C > 0$ and any $x, x' \in D_n(\beta)$ which are not end points of I_β. (Here $p_n(x)$ is given by (1).)

Observe that

$$
\begin{aligned}
\#\mathcal{E}_n(x; \beta) &= \sum_{\substack{y \in D_n(\beta): \\ 0 \leqslant y - x \leqslant \frac{\beta^{-n}}{\beta - 1}}} p_n(y), \\
\#\widetilde{\mathcal{E}}_n(x; \beta) &= \sum_{\substack{y \in D_n(\beta): \\ -\frac{\beta^{-n}}{\beta - 1} \leqslant y - x \leqslant \frac{2\beta^{-n}}{\beta - 1}}} p_n(y).
\end{aligned}
$$

TABLE 9. *Lower bounds and the actual values for H_{τ_m}.*

m	$\log_{\tau_m}(2/\mathfrak{M}_{\tau_m})$	H_{τ_m}
2	0.9404	0.9957
3	0.8531	0.9804
4	0.8450	0.9869
5	0.8545	0.9926

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In view of the Garsia separation lemma (see [8, Lemma 1.51]), each sum runs along a finite set whose cardinality is bounded by some constant (depending on β) for all n.

Hence, $\sqrt[n]{\#\mathcal{E}_n(x;\beta)} \sim \sqrt[n]{\#\widetilde{\mathcal{E}}_n(x;\beta)}$ for all $x \in (0, (1/(\beta-1)))$, and (8) together with (5) yield the claim of the lemma. □

Consequently, for a multinacci β,

$$\inf_{x \in I_\beta^*} d_\beta(x) = \log_\beta \frac{2}{\mathfrak{M}_\beta}, \tag{9}$$

where $I_\beta^* = \{x \in (0, (1/(\beta-1))) : d_\beta(x) \text{ exists}\}$. In [6, Theorem 1.5], Feng showed that

$$\inf_{x \in I_{\tau_m}^*} d_{\tau_m}(x) = \begin{cases} \log_\tau 2 - \dfrac{1}{2}, & m = 2, \\ \dfrac{m}{m+1} \log_{\tau_m} 2, & m \geqslant 3. \end{cases}$$

This immediately gives us the explicit formulae for the maximal growth exponent for the multinacci family, namely,

$$\mathfrak{M}_{\tau_m} = \begin{cases} \sqrt{\tau}, & m = 2, \\ 2^{1/(m+1)}, & m \geqslant 3. \end{cases}$$

In fact, one can easily obtain the values x at which $\mathfrak{M}_\beta$ is attained. More precisely, for $\beta = \tau$ the maximum growth is attained at x with the β-expansion $(1000)^\infty$, that is, at $x = (5+\sqrt{5})/10$ (this was essentially proved by Pushkarev [18], via multizigzag lattice techniques).

For $m \geqslant 3$ the maximal growth point is x with the β-expansion $(10^m)^\infty$. These claims can be easily verified via the matrix representation for $p_n(x)$ given in [6], and we leave it as an exercise for the interested reader. (Recall that the growth exponent for $p_n(x)$ is the same as for $\#\mathcal{E}_n(x;\beta)$ for the multinacci case.)

Finally, since we know the exact values of the maximal growth exponent for this family, we can assess how far our estimate (that is, the smallest value of the local dimension) is from the actual value of H_β (which is the average value of $d_\beta(x)$ for μ_β-almost everywhere x). Table 9 is the comparison table.

We see that for $m \geqslant 3$ our bounds are far below H_β; moreover, our method cannot in principle produce a uniform lower bound for all β better than 0.845. However, as a first approximation it still looks pretty good.

Remark 12. We believe that (9) holds for all Pisot $\beta \in (1,2)$. If this were the case, then (4) would effectively yield a lower bound for the infimum of the local dimension of μ_β. This may prove useful, as, similarly to the entropy, no lower bound for d_β is known for the non-multinacci β. Furthermore, if one could compute the exact value of $\mathfrak{M}_\beta$, this would yield the exact value of $\inf_{x \in I_\beta^*} d_\beta(x)$.

7. *Acknowledgements and additional comments*

The authors would like to thank the two referees for many useful suggestions. In addition, we would like to communicate a question asked to us from one of the referees, that the authors feel would make an interesting question for possible future research.

> In § 6, besides the multinacci, could you say something on $\beta = (a + \sqrt{a^2+4})$, with an integer $a \geqslant 2$? (Maybe using results from Komatsu [13].) Or, more generally, on numbers β that are roots of a polynomial $X^n - a_{n-1}X^{n-1} - \ldots - a_1 X - a_0$, where $a_{n-1} \geqslant \ldots \geqslant a_1 \geqslant 1$?

We would also like to mention the recent paper by Feng and the second author [7], in which the average growth exponent for β-expansions is studied for the Pisot parameters β.

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