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ULRICH ABEL
Technische Hochschule Mittelhessen, Department MND, Wilhelm-Leuschner-Straße 13, 61169 Friedberg, Germany
e-mail: *Ulrich.Abel@mnd.thm.de*

107.36 Similarities and circle-preserving bijections of the plane

1. *Introduction*

A *similarity* of the complex plane $\mathbb{C}$ is a map of the form $z \to az + b$, or $z \to a\bar{z} + b$, where a and b are complex numbers with $a \neq 0$. Each similarity is a bijection of $\mathbb{C}$ onto itself, and maps a line onto a line, and a circle onto a circle. In addition, it is known that the converse is true: if f is a bijection of $\mathbb{C}$ onto itself that maps each line onto a line, and each circle onto a circle, then f is a similarity of $\mathbb{C}$. The sole purpose of this Note is to use this converse to provide an opportunity for students to experience and, more importantly, engage in, a substantial proof of a single result. So, instead of providing the details, we break the proof into a number of simpler (and, we hope, manageable) steps, and invite readers to formally justify these steps for themselves.

The *circle* $\{z : |z - a| = r\}$ is denoted by $C(a, r)$, and the (open) *disc* $\{z : |z - a| < r\}$ by $D(a, r)$, where (in each case) $a \in \mathbb{C}$ and $r > 0$. Although we are assuming that f maps each circle onto a circle, we are not assuming that f maps each disc onto a disc; in fact, we shall prove that this must be so. Note also that we are not assuming that f is continuous, and again we shall prove that this is so.

2. *The converse result*

We now give our sketch of the proof that if f is a bijection of $\mathbb{C}$ onto itself that maps each line onto a line, and each circle onto a circle, then f is a similarity of $\mathbb{C}$. It is important to recall that any three points in $\mathbb{C}$ are either *collinear* (they lie on a line), or *concyclic* (they lie on a circle), but not both. Also, as f is a bijection of $\mathbb{C}$ onto itself, f^{-1} exists and is also a bijection of $\mathbb{C}$ onto itself.

We are assuming that the image of a line is a line, but this is not the same as saying that each line is the image of a line (informally, 'a line goes to a line' is not the same as 'a line comes from a line'). However, this latter statement is true, as we shall now show. Let L' be any line, take any three points w_1, w_2 and w_3 on L', and let $z_j = f^{-1}(w_j)$. If the z_j are concyclic, they lie on a circle, and then the w_j also lie on a circle. As the w_j are collinear this is false; thus the z_j are collinear and so lie on a line L. It is now clear that $f(L) = L'$; thus any given line L' is the image of some line L. A similar argument holds for circles, so we have proved that

(1) if L is a line then $f^{-1}(L)$ is a line; if C is a circle then $f^{-1}(C)$ is a circle.

Next, the reader should verify that as f is a bijection, for any subsets A and B of $\mathbb{C}$ we have

$$f(A \cap B) = f(A) \cap f(B).$$

In particular, A and B are disjoint if, and only if, $f(A)$ and $f(B)$ are disjoint. This shows that

(2a) the lines L and L' are parallel if, and only if, the lines $f(L)$ and $f(L')$ are parallel;

(2b) a line L lies outside a circle C if, and only if, $f(L)$ lies outside $f(C)$;

(2c) a line L is tangent to a circle C if, and only if, $f(L)$ is tangent to $f(C)$;

(2d) a line L meets a circle C at two points if, and only if, $f(L)$ meets $f(C)$ at two points.

The disc $D(a, r)$ is the 'inside' of the circle $C(a, r)$ and the points z in $D(a, r)$ are characterized by the fact that every line through z meets $C(a, r)$ in two points. If z is inside a circle C, then every line through $f(z)$ meets $f(C)$ in two points, and this shows that the inside of C maps to the inside of $f(C)$; equivalently,

(3) if f maps $C(u, r)$ to $C(v, R)$, then it also maps $D(u, r)$ to $D(v, R)$.

Next, two distinct points u and v on a circle C lie on a line through the centre of C if, and only if, the tangents to C at u and v are parallel. Thus, by considering two pairs of parallel tangents, we see that

(4) f maps the centre of a circle C to the centre of the circle $f(C)$.

Now take any z_0 in $\mathbb{C}$, and any positive ε, let $w_0 = f(z_0)$, and consider the circle $C(w_0, \varepsilon)$. By (1), there is a circle C that maps onto $C(w_0, \varepsilon)$, and by (4), the circle C has centre z_0. Thus $C = C(w_0, \delta)$, say. Now (3) shows that f maps $D(z_0, \delta)$ onto $D(w_0, \varepsilon)$, which means that if $|z - z_0| < \delta$ then $|f(z) - f(z_0)| < \varepsilon$; thus

(5) f is continuous at each point of $\mathbb{C}$.

The next series of steps shows that f is a similarity. As $f(0) \neq f(1)$, we can define a similarity h of $\mathbb{C}$ onto itself by

$$h(z) = \frac{z - f(0)}{f(1) - f(0)}.$$

It is clear that the map $z \to h(f(z))$, which we denote by hf, is a bijection that maps lines to lines, and circles to circles, and also satisfies $hf(0) = 0$ and $hf(1) = 1$. Thus if $\mathbb{R}$ is the real line then $hf(\mathbb{R}) = \mathbb{R}$. Now it is sufficient to show that hf is a similarity for if this is so, then f (which is the composition $h^{-1}(hf)$ of similarities) is also similarity. Thus, by relabelling hf as f, it is sufficient to assume from now on that

(6) $f(0) = 0, f(1) = 1$, and $f(\mathbb{R}) = \mathbb{R} = f^{-1}(\mathbb{R})$.

If we now consider (i) any two distinct complex numbers u and v, (ii) the (unique) circle C for which u and v are the endpoints of a diameter of C, and (iii) the tangents to C at u and v, we see that the tangents to $f(C)$ at $f(u)$ and $f(v)$ are parallel. Thus $f(u)$ and $f(v)$ are the ends of a diameter of $f(C)$, and this with (4) shows that

(7) for all complex numbers u and v,

$$f\left(\frac{u + v}{2}\right) = \frac{f(u) + f(v)}{2}$$

(informally, f maps 'mid-points to mid-points').

Since $f(0) = 0$ and $f(1) = 1$, (7) shows that $f(-1) = -1, f(2) = 2$, and so on or, more generally, that $f(n) = n$ for every integer n. This then shows that that $f(n/2) = n/2$ for every integer n, and so on, so that $f(n/2^m) = n/2^m$ for all integers n and all positive integers m. Then, as f is continuous, we conclude that

(8) $f(x) = x$ for all real x.

Now consider any real number a, and let C be the circle with diameter $[a, a + 1]$. The vertical line through a is the tangent to C at a, and this is mapped to the tangent to $f(C)$ at $f(a)$. However, (8) implies that $f(a) = a$ and $f(C) = C$, so we see that f maps each vertical line to itself. Now consider *any* circle C in $\mathbb{C}$, and let T_1 and T_2 be the two vertical tangents to C. Then, as $f(T_j) = T_j$, we see that T_1 and T_2 are also tangents to $f(C)$, and this means that $f(C)$ has the same radius as C. Now consider any two complex numbers u and v and let C be the circle centre u that passes through v. If we now use the fact that $f(C)$ is the circle with centre $f(u)$ that passes through $f(v)$, and that C and $f(C)$ have the same radius, we find that f is an *isometry*:

(9) for all complex u and v, $|f(u) - f(v)| = |u - v|$.

Now consider any complex number $z = x + iy$, and let $f(z) = a + ib$. Since

$$|f(z) - t|^2 = |f(z) - f(t)|^2 = |z - t|^2$$

for all real t, we find that

(10) for each complex number $z, f(z)$ is either z or $\bar{z}$.

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Finally, we need to show that we cannot have $f(z) = z$ and $f(w) = \bar{w}$ for some non-real z and w: note that (10) by itself does not exclude this possibility. Now by (10), $f(i) = i$ or $f(i) = -i$. If $f(i) = i$ and z is above the real axis, then

$$|f(z) - i| = |f(z) - f(i)| = |z - i| < |\bar{z} - i|$$

so that $f(z) \neq \bar{z}$. Thus $f(z) = z$. Similar considerations apply when z is below the real axis, so that if $f(i) = i$ then $f(z) = z$ for all z. Similar considerations also apply when $f(i) = -i$, so, finally, we conclude that

(11) if $f(i) = i$ then $f(z) = z$ for all z; if $f(i) = -i$ then $f(z) = \bar{z}$ for all z.

Thus f is either $z \to z$ or $z \to \bar{z}$, and our proof is complete.

3. *Two functional equations*

The two functional equations

$$f(z + w) = f(z) + f(w) \tag{I}$$

and

$$f\left(\frac{z + w}{2}\right) = \frac{f(z) + f(w)}{2} \tag{II}$$

are not equivalent to each other since (I) implies that $f(0) = 0$, and (II) is satisfied by any constant function f. However, it is true that a map f of $\mathbb{C}$ onto itself satisfies (I) for all z and w if, and only if, it satisfies (II) for all z and w and, in addition, $f(0) = 0$ (and we leave the reader to provide the proof). Obviously, the function f given by $f(x) = ax$ satisfies (I) but, perhaps surprisingly, there are solutions of (I) that are not continuous anywhere (and so not of this form); see, for example, [1, p. 96] and [2, pp. 108–112]. However, it is known that any *continuous* solution of (I) is of the form $f(z) = az$; thus it appears that the continuity of f (or some equivalent fact) is an essential step in the argument above.

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A. F. BEARDON
D.P.M.M.S.,
University of Cambridge,
Wilberforce Rd.,
Cambridge CB3 0WB
e-mail: *afb@dpmms.cam.ac.uk*