AUTOMATIC CONTINUITY OF POSITIVE FUNCTIONALS
ON TOPOLOGICAL INVOLUTION ALGEBRAS

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This paper surveys the known results on automatic continuity of
positive functionals on topological *-algebras and then shows
how two theorems on Banach *-algebras extend to complete
metrizable topological *-algebras. The two theorems concerned
are Loy's theorem on separable Banach *-algebras A with centre
Z such that AZ is of countable codimension and Varopoulos'
result on Banach *-algebras with bounded approximate identity.
Both theorems have the conclusion that all positive functionals
on such algebras are continuous. The extension of the second
theorem requires the algebra to be locally convex and the
approximate identity to be 'uniformly bounded'. Neither
extension requires the algebra to be LMC. This means that the
proof of the first theorem is quite different from the
corresponding Banach algebra result (which used spectral theory).
The proof of the second is closer to the previously known LMC
version, but actually neater by being more general. It is also
shown that the well-known estimate of $|f(a^*ba)|$ for a positive
functional f on a Banach *-algebra may be obtained without the
usual use of spectral theory. The paper concludes with a list of
open questions.

1. Introduction

The purpose of this paper is to give analogues for complete metrizable
algebras of the following two theorems on Banach algebras.

THEOREM 1.1 (Loy [*12*, Theorem 2.1]). *Let A be a separable Banach algebra with involution and let Z be the centre of A . If AZ is of countable (that is, at most countable) codimension in A , then every positive functional on A is continuous.*

THEOREM 1.2 (Varopoulos [*21*]). *Let A be a Banach algebra with involution. If A has a bounded approximate identity, then every positive functional on A is continuous.*

Theorem 1.1 is the end product of a series of results. It is based on an application of Souslin techniques to earlier results on (not necessarily separable) algebras (Theorem 1.4 below). Theorem 1.2 may be found in Sinclair's book [*19*, Theorem 13.11]. Both these theorems hold for Banach algebras whose involutions are not necessarily continuous, whereas our proofs will require continuous involutions. However, it should be noted that involutions on semisimple Banach algebras are automatically continuous [*19*, Corollary 6.13], so this may be regarded as only a minor loss. Apart from this, and an adjustment to the notion of 'bounded approximate identity', our results (Theorems 3.6 and 4.3 below) are verbatim extensions of these theorems. Our proofs, however, are substantially different.

Let us establish some terminology. All topological vector spaces will be Hausdorff. All algebras will be over the complex field $\mathbf{C}$. Involutions will not be assumed continuous unless specifically stated. By an *F-space* we shall mean a complete metrizable (not necessarily locally convex) topological vector space. By an *F-algebra* we shall mean a topological algebra which is an F-space. The multiplication is assumed to be separately continuous, but a theorem of Arens [*1*] ([*22*, Theorem 7.2]) then makes it jointly continuous; that is, the map $(x, y) \mapsto xy$ from $A \times A$ to A is continuous. If X, Y are subspaces of A , then XY denotes the subspace generated by the appropriate set of products; that is,

$$XY = \left\{ \sum_{i=1}^{n} x_i y_i : n = 1, 2, 3, \ldots; x_i \in X, y_i \in Y \ (1 \leq i \leq n) \right\} .$$

We write A^2 for AA . By an *F*-algebra* we shall mean an F-algebra with a continuous involution. The *positive cone* of a *-algebra A is the set

$$A^+ = \left\{ \sum_{i=1}^{n} x_i^* x_i \ : \ n = 1, \ 2, \ 3, \ \ldots; \ x_i \in A \ (1 \leq i \leq n) \right\} \ .$$

A *positive functional* on A is a linear map $f : A \to \mathbb{C}$ such that $f(x^*x) \geq 0$ for all $x \in A$; that is, such that $f(A^+) \subseteq \mathbb{R}^+$.

With these notations, we summarize the other previously known results about automatic continuity of positive functionals on topological *-algebras. First, a trivial argument with Hamel bases shows the following.

THEOREM 1.3. *A necessary condition for an F-algebra A with involution to have no discontinuous positive functionals is that A^2 be closed and of finite codimension in A .*

In the other direction, we have the following.

THEOREM 1.4 ([19, Theorem 13.7]). *Let A be a Banach algebra with involution, and let Z be the centre of A . If Z^2A is closed and of finite codimension in A , then every positive functional on A is continuous.*

THEOREM 1.5. *Let A be a complete metrizable algebra with involution. If A^2 is closed and of finite codimension and A^+ is closed, then every positive functional on A is continuous.*

This is an easy corollary of a result of Namioka [15, Corollary 5.5]. In the following extension of it, due to Neumann [16, (3.6) (ii)], the involution need not even satisfy the identity $(xy)^* = y^*x^*$ $(x, y \in A)$.

THEOREM 1.6. *Let A be a complete metrizable algebra with involution such that A^2 is of countable codimension in A and A^+ is complete. (It is sufficient, in fact, to assume that every increasing Cauchy sequence in A^+ converges there.) Then A^2 is closed and of finite codimension in A and every positive functional on A is continuous.*

THEOREM 1.7 (Dao-Shing Shah [18]). *If A is an F*-algebra with identity, then every positive functional on A is continuous.*

Actually, Shah proves this only for LMC algebras, but his method

extends easily to the general case (see [7, Theorem (I.8.5)], [6, Theorem 11.1] or [17]).

If we drop the hypothesis of metrizability, even Shah's theorem fails easily (see [14, remark following Proposition 12.2]), since the question we should be asking is whether positive functionals are automatically *bounded*. This automatic boundedness can be proved for complete LMC algebras with continuous involution and identity ([7, Theorem (I.8.6)], [9]), but it fails in general.   In [8, Addendum], an example is given of a complete, commutative, locally convex algebra with identity, continuous involution and jointly continuous multiplication, on which there is an unbounded positive functional.

Finally, we mention two very recent results of Johnson [11, Theorems 3.5 and 3.7].   For F^*-algebras, these read as follows.

THEOREM 1.8. *Let A be an F^*-algebra such that $A^2 = A$.   Suppose either:*

(a)   *A is commutative and* LMC *and the set of elements with bounded spectrum is dense in A ; or*

(b)   *there is an approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$ for A , which is contained in the centre of A and is such that the set $\{e_\lambda x : \lambda \in \Lambda\}$ is bounded, for every $x \in A$.*

Then every positive functional on A is continuous.

2. Results in general F^*-algebras

In this section, we present the best result on F^*-algebras we have been able to reach, so far, without invoking the hypothesis of separability (Theorem 2.5). We begin with a result which is basic to the rest of the paper. For Banach algebras with involution, it may be proved using Ford's square root lemma (see [19, Lemma 13.2 (iv)]). In general topological algebras Ford's lemma is not available, so we must use Shah's theorem, which avoids the need to consider spectra of elements, at the expense of requiring us to assume that the involution is continuous; hence the prevalence of this assumption in the present paper.

THEOREM 2.1. *Let A be an F^*-algebra and f a positive functional*

on A . Then, for every $a, b \in A$, the functional $x \mapsto f(axb)$ on A is continuous.

Proof.   Using the identity

$$4axb = (b+a^*)^*x(b+a^*) + i(b+ia^*)^*x(b+ia^*)$$
$$- (b-a^*)^*x(b-a^*) - i(b-ia^*)^*x(b-ia^*) ,$$

we see that it is sufficient to prove that the functional $f_a : x \mapsto f(a^*xa)$ is continuous, for each $a \in A$.

Let $A^\sim$ be the algebra $A \oplus \mathbb{C}e$ obtained by adjoining an identity e to A in the usual way, with the obvious topology. Then f_a extends to a positive functional on $A^\sim$ by

$$f_a(x+\lambda e) = f(a^*xa) + \lambda f(a^*a) \quad (x \in A, \ \lambda \in \mathbb{C}) .$$

By Shah's theorem (Theorem 1.7 above), f_a is continuous on $A^\sim$, and hence on A .

An interesting corollary to this is that we may obtain the usual estimate for the norm of f when A is a Banach algebra [19, Lemma 13.2 (iv)], without using Ford's square root lemma. En route to this, we first obtain a result for general F^*-algebras.

THEOREM 2.2.   Let f be a positive functional on an F^*-algebra A . If $x \in A$ is such that $(xx^*)^n \to 0$ (equivalently $(x^*x)^n \to 0$), as $n \to \infty$, then

$$|f(a^*xa)| \le f(a^*a) \quad (a \in A) .$$

Proof.   We need the following algebraic lemma.

LEMMA 2.3.   If f is a positive functional on a $*$-algebra A , then

$$|f(xyz)|^{2^{n+1}} \le f(xx^*)^{2^n-1} f\big(x(yy^*)^{2^n}x^*\big) f(z^*z)^{2^n}$$

for $n = 0, 1, 2, \ldots$ and every $x, y, z \in A$.

Proof.   We start with the fact that f must satisfy the Cauchy-Schwarz inequality

$$|f(a^*b)|^2 \le f(a^*a)f(b^*b) \quad (a, b \in A) .$$

Putting $a^* = xy$, $b = z$ in this gives

(1) $$|f(xyz)|^2 \le f(xyy^*x^*)f(z^*z) \ ,$$

which proves the lemma for $n = 0$. Furthermore, (1) implies

(2) $$f\big(x(yy^*)^{2^n}x^*\big)^2 \le f\big(x(yy^*)^{2^{n+1}}x^*\big)f(xx^*) \ ,$$

for all n . Now suppose the lemma holds for n . Then

$$|f(xyz)|^{2^{n+2}} \le \Big[f(xx^*)^{2^n-1}f\big(x(yy^*)^{2^n}x^*\big)f(z^*z)^{2^n}\Big]^2$$

$$\le f(xx^*)^{2^{n+1}-1}f\big(x(yy^*)^{2^{n+1}}x^*\big)f(z^*z)^{2^{n+1}} \ ,$$

by (2). Thus the lemma is proved, by induction.

    Proof of Theorem 2.2 continued. By Theorem 2.1, we have

(3) $$f\big(a^*(xx^*)^{2^n}a\big) \to 0$$

as $n \to \infty$ and, by the lemma,

(4) $$|f(a^*xa)|^{2^n} \le f(a^*a)^{2^n-1}f\big(a^*(xx^*)^{2^{n-1}}a\big) \quad (n = 1, 2, 3, \ldots) \ .$$

If $f(a^*a) = 0$, then (4) implies $f(a^*xa) = 0$, as desired. Otherwise, (3) and (4) give

$$\left|\frac{f(a^*xa)}{f(a^*a)}\right|^{2^n} \to 0$$

as $n \to \infty$, so

$$\left|\frac{f(a^*xa)}{f(a^*a)}\right| < 1 \ .$$

This completes the proof.

    COROLLARY 2.4 ([*19*, Lemma 13.2 (iv)]). *If f is a positive functional on a Banach algebra A with continuous involution, then*

$$|f(a^*ba)| \le f(a^*a)\nu(b^*b)^{\frac{1}{2}} \quad (a, b \in A) \ ,$$

where $\nu(x)$ denotes the spectral radius of x . Equivalently [19, Lemma 13.2 (iii)],

$$|f(a^*ha)| \le f(a^*a)\nu(h) \quad (a \in A, h = h^* \in A) \ .$$

    Proof. This is immediate from Theorem 2.2, since $(b^*b)^n \to 0$ as

$n \to \infty$ when $\nu(b^*b) < 1$.

Returning to F^*-algebras, we can develop Theorem 2.1 further, as follows.

THEOREM 2.5. *If f is a positive functional on an F^*-algebra A with centre Z , then the mapping*

$$(x,\, y,\, z) \mapsto f(xyz) \; : \; A \times A \times Z \to \mathbb{C}$$

is continuous.

Proof. Since $xyz = xzy = zxy$, Theorem 2.1 implies that

$$x \mapsto f(xyz) \; , \quad y \mapsto f(xyz) \; , \quad z \mapsto f(xyz)$$

are continuous maps from A to $\mathbb{C}$, for all values of the other two variables. It only remains to apply a 'separate continuity implies joint continuity' lemma similar to the theorem of Arens cited above. The following has sufficient generality for our purpose. Its proof is a straightforward generalization of that of Arens' original theorem (see [1] or [22, Theorem 7.2]), and is therefore omitted.

LEMMA 2.6. *Let E_1, E_2, ..., E_n be F-spaces and let F be any topological vector space. Let $T : E_1 \times \ldots \times E_n \to F$ be a multilinear map, separately continuous in each variable. (That is, each of the maps*

$$x_i \mapsto T(x_1,\, \ldots,\, x_n) \; : \; E_i \to F \quad \left(x_j \in E_j \; (j \neq i),\; 1 \leq i \leq n\right)$$

is continuous.) Then T is (jointly) continuous.

3. Separable F^*-algebras

When the F^*-algebra is separable, we may improve on the results of the previous section by using the theory of analytic spaces. We first recall the basic definitions and results we shall need. For details, we refer the reader to [3], [13], [10] or, with slightly different terminology, [2].

DEFINITION 3.1. A topological space is *analytic* if it is the continuous image of a complete, separable metric space.

THEOREM 3.2 ([3, Theorem 5.5]). *Let E be an F-space and F an analytic (linear) subspace of countable codimension. Then F is closed*

and of finite codimension.

　　THEOREM 3.3 (The Pettis Lemma [3, Theorem 5.1]). *Let E be a topological vector space and let X be a subset of E which is analytic and non-meagre in E . Then X − X (that is, {x−y : x, y ∈ X}) is a neighbourhood of zero in E .*

　　THEOREM 3.4 (The Analytic Graph Theorem [3, Theorem 5.2]). *Let E be an F-space, F any topological vector space and T : E → F a linear mapping whose graph is an analytic subspace of E × F . Then T is continuous.*

　　We shall express our use of analytic space techniques in the form of a general lemma (which may be of independent interest). Part *(i)* is very similar to the main result of [4] and to [12, Theorem 1.3 (i)], but we include it, for completeness.

　　LEMMA 3.5. *(i) Let E_1, ..., E_n, E be separable F-spaces. Let $T : E_1 \times ... \times E_n \to E$ be a continuous multilinear mapping such that the linear span H of $T(E_1 \times ... \times E_n)$ is of countable codimension in E . Then H is closed and of finite codimension and there is a positive integer K such that every element of H is expressible in the form*

$$\sum_{i=1}^{K} T(x_{i1}, ..., x_{in}) \text{ for some } x_{ij} \in E_j (1 \le i \le K, 1 \le j \le n) .$$

　　(ii) If, further, F is another F-space and S : E → F is a linear map such that $ST : E_1 \times ... \times E_n \to F$ is (jointly) continuous, then S is continuous.

　　Proof. Let H_k be the set of elements of H expressible in the form

$$\sum_{i=1}^{k} T(x_{i1}, ..., x_{in}) \text{ for some } x_{ij} \in E_j (1 \le i \le k, 1 \le j \le n) . \text{ Then}$$

each H_k is analytic, being a continuous image of the complete, separable metric space $(E_1 \times ... \times E_n)^k$, and H is analytic, being a continuous image of the complete, separable metric space consisting of the disjoint union $\bigcup_{k=1}^{\infty} (E_1 \times ... \times E_n)^k$. By Theorem 3.2, H is closed and of finite

codimension in E . So H is a complete metric space. Moreover,
$H = \bigcup\limits_{k=1}^{\infty} H_k$ so, by Baire's Category Theorem, H_k is non-meagre in H , for
some k . Applying the Pettis Lemma to this H_k , we see that $H_k - H_k$ is
a zero-neighbourhood in H . Since H_k is a linear subspace, this means
that $H_k - H_k = H$; that is, $H_{2k} = H$. This proves (i) with $K = 2k$.

To prove (ii), we first consider the restriction $S|H$ of S to H .
We observe that, since ST is continuous, the map

$$\left(x_{i1}, \ldots, x_{in}\right)_{i=1}^{K} \mapsto \sum_{i=1}^{K} ST\left(x_{i1}, \ldots, x_{in}\right) : \left(E_1 \times \ldots \times E_n\right)^{K} \to F$$

is continuous, so its graph G is a closed subset of
$\left(E_1 \times \ldots \times E_n\right)^{K} \to F$. Therefore G is a complete separable metric space.
Now the graph $\mathrm{Gr}(S|H)$ of $S|H$ is the image of G under the continuous
map

$$\left[\left(x_{i1}, \ldots, x_{in}\right)_{i=1}^{K}, y\right] \mapsto \left[\sum_{i=1}^{K} T\left(x_{i1}, \ldots, x_{in}\right), y\right] .$$

Therefore $\mathrm{Gr}(S|H)$ is analytic. By the Analytic Graph Theorem, Theorem
3.4, $S|H$ is continuous. Since H is closed and of finite codimension,
it follows that S is continuous.

THEOREM 3.6. *Let A be a separable F^*-algebra with centre Z such
that AZ is of countable codimension in A . Then every positive
functional on A is continuous.*

Proof. Let f be a positive functional on A . Since AZ is of
countable codimension, so is $A^2 Z$. We apply the previous lemma with
$n = 3$, $E_1 = E_2 = E = A$, $E_3 = Z$, $F = \mathbb{C}$, $S = f$ and T the
multiplication map (so that Theorem 2.5 just says that ST is continuous).
The conclusion is that $A^2 Z$ is closed and of finite codimension and that
f is continuous. (A similar argument shows that AZ is closed and of
finite codimension.)

COROLLARY 3.7. *Let A be a commutative, separable F^*-algebra.*

Then a necessary and sufficient condition for every positive functional on
A to be continuous is that A^2 be closed and of finite codimension
(equivalently, that A^2 be of countable codimension).

   Proof. By Theorems 3.6 and 1.3.

4. F^*-algebras with approximate identities

       In extending Varopoulos' theorem (Theorem 1.2 above) to F^*-algebras,
we shall follow his approach of first proving a version of Cohen's
Factorization Theorem and then combining it with the fact that the map
$x \longmapsto f(axb)$ is continuous (our Theorem 2.1). The required version of
Cohen's Theorem was first proved by Craw [5] for Fréchet (that is,
complete, metrizable, LMC) algebras; another proof being given, using
modules, by Summers [20]. We shall show here that Cohen's Theorem
extends, in a particularly neat way, to complete, metrizable, locally
convex (but not necessarily LMC) algebras. We follow Summers' approach.

       Let A be a locally convex F-algebra. Its topology may be defined
by a countable set of seminorms $\{p_i : i = 1, 2, 3, \ldots\}$, and these may be
chosen so that $p_i(x) \leq p_j(x)$ whenever $i \leq j$ and $x \in A$. Since the
multiplication is jointly continuous, we may further adjust the choice of
defining seminorms so that

(5) $p_i(xy) \leq p_{i+1}(x)p_{i+1}(y)$ $(i = 1, 2, 3, \ldots; x, y \in A)$.

By a *uniformly bounded left approximate identity* in A , we shall mean a
net $\{e_\lambda\}_{\lambda \in \Lambda}$ such that $e_\lambda a \to a$ for every $a \in A$ and there exists $K > 0$
such that the set

$$\left\{\left(K^{-1}e_\lambda\right)^n : \lambda \in \Lambda, n = 1, 2, 3, \ldots\right\}$$

is bounded in A . (Note that this agrees with Craw's definition in the
LMC case.) By a *left Fréchet A-module*, we shall mean a left A-module X
which is an F-space (not necessarily locally convex), such that the
multiplication map $(a, x) \longmapsto ax : A \times X \to X$ is separately (and hence
jointly) continuous. We shall say that X is *essential* if A has a
uniformly bounded left approximate identity $\{e_\lambda\}_{\lambda \in \Lambda}$ such that $e_\lambda x \to x$

for all $x \in X$. We shall let d denote a translation-invariant metric defining the topology on X , and write $|x| = d(x, 0)$ $(x \in X)$. (Note that $|\cdot|$ is subadditive, but that we do not have $|\alpha x| = |\alpha||x|$ $(\alpha \in \mathbb{C}, x \in X)$.)

THEOREM **4.1.** *Let X be an essential Fréchet A-module. Then, for every $x \in X$ and every $\epsilon > 0$, there exist $a \in A$, $y \in X$ such that $x = ay$, $y \in \overline{Ax}$ and $d(x, y) < \epsilon$.*

Proof. Let $\{e_\lambda\}_{\lambda \in \Lambda}$ be a uniformly bounded left approximate identity and p_i $(i = 1, 2, 3, \ldots)$ seminorms, as described above. Let $A^\sim$ denote the algebra $A \oplus \mathbb{C}e$ obtained by adjoining an identity e to A in the usual way. To simplify notation, we shall write $a + \xi$ for $(a, \xi e)$ $(a \in A, \xi \in \mathbb{C})$. The seminorms p_j on A extend naturally to $A^\sim$ by

$$p_j(a+\xi) = p_j(a) + |\xi| \quad (a \in A, \xi \in \mathbb{C}) .$$

Let K be the uniform bound for $\{e_\lambda\}_{\lambda \in \Lambda}$, as above, and let $\rho > 0$ be chosen so that $\rho/(1+\rho) < 1/(2K)$. Then, for each $\lambda \in \Lambda$, $(1+\rho-\rho e_\lambda)$ is invertible in $A^\sim$. To show this, we must check that our hypotheses are sufficient for the usual argument to work. We have

$$(6) \qquad (1+\rho-\rho e_\lambda)(1+\rho)^{-1} \sum_{n=1}^{N} \left(\rho(1+\rho)^{-1}e_\lambda\right)^n = 1 - \left(\rho(1+\rho)^{-1}e_\lambda\right)^{N+1} .$$

Since $\left\{\left(K^{-1}e_\lambda\right)^n : \lambda \in \Lambda, n = 1, 2, 3, \ldots\right\}$ is bounded, we have, for each j , a constant M_j such that

$$(7) \qquad p_j\left(\left(K^{-1}e_\lambda\right)^n\right) < M_j \quad (\lambda \in \Lambda, n = 1, 2, 3, \ldots) .$$

Hence

$$p_j\left(\left(\rho(1+\rho)^{-1}e_\lambda\right)^{N+1}\right) < 2^{-N-1}M_j \quad (j = 1, 2, 3, \ldots) ,$$

and so the right hand side of (6) tends to 1 as $N \to \infty$. Now the partial sums of $\sum_{n=1}^{\infty} \left(\rho(1+\rho)^{-1}e_\lambda\right)^n$ form a Cauchy sequence, since

$$p_j\left[\sum_{n=p}^{q}\left[\rho(1+\rho)^{-1}e_\lambda\right]^n\right] \leq \sum_{n=p}^{q} 2^{-n}M_j \quad (p \leq q, \; j = 1, \; 2, \; 3, \; \ldots) \; .$$

Hence the infinite sum converges and

$$\left(1+\rho-\rho e_\lambda\right)^{-1} = (1+\rho)^{-1}\sum_{n=1}^{\infty}\left[\rho(1+\rho)^{-1}e_\lambda\right]^n \; .$$

(Notice that in this well-known argument we have not required the seminorms to be submultiplicative, but we have used the local convexity of A .) We note that

(8) $$p_j\left[\left(1+\rho-\rho e_\lambda\right)^{-1}\right] \leq (1+\rho)^{-1}\sum_{n=1}^{\infty} 2^{-n}M_j = (1+\rho)^{-1}M_j \; .$$

We shall choose a sequence $e_1, \; e_2, \; \ldots$ of elements of $\{e_\lambda : \lambda \in \Lambda\}$ inductively, beginning with e_1 which is chosen so that

(9) $$|\rho x - \rho e_1 x| < \varepsilon/2 \; .$$

Suppose $e_1, \; \ldots, \; e_n$ have been chosen. We write

$$t_n = \left(1+\rho-\rho e_1\right) \; \ldots \; \left(1+\rho-\rho e_n\right) \; ,$$

$$h_n = t_n^{-1} \; ,$$

$$a_n = h_n - (1+\rho)^{-n} \in A \; ,$$

$$y_n = t_n x \; .$$

Then we choose e_{n+1} so that

(10) $$p_n\left(e_{n+1}a_n - a_n\right) < 2^{-n}$$

and

(11) $$|\rho t_n x - \rho t_n e_{n+1} x| < \varepsilon 2^{-n-1} \; .$$

We can obtain (10) because $e_\lambda a_n \to a_n$ and (11) because $\rho t_n e_\lambda x \to \rho t_n x$, by continuity of scalar and module multiplication and the fact that X is essential. Since the seminorms p_j are increasing, (10) implies

(12) $$p_j\left(e_{n+1}a_n - a_n\right) < 2^{-n} \quad (1 \le j \le n) \ .$$

We prove that $\left(a_n\right)_{n=1}^{\infty}$ is a Cauchy sequence. It follows from the definition of $\left(a_n\right)$ that

$$a_{n+1} - a_n = \rho\left(1+\rho-\rho e_{n+1}\right)^{-1}\left\{(1+\rho)^{-n-1}e_{n+1}+e_{n+1}a_n - a_n\right\} \ .$$

Hence, using (5),

$$p_j\left(a_{n+1}-a_n\right) \le \rho p_{j+1}\left(\left(1+\rho-\rho e_{n+1}\right)^{-1}\right)\left\{(1+\rho)^{-n-1}p_{j+1}\left(e_{n+1}\right)+p_{j+1}\left(e_{n+1}a_n - a_n\right)\right\} \ ,$$

for all j . Applying (8), (12) and the inequality $p_j\left(e_\lambda\right) < K M_j$ $(\lambda \in \Lambda$, $j = 1, 2, 3, \ldots)$, which is a special case of (7), we see that

$$p_j\left(a_{n+1}-a_n\right) \le \rho(1+\rho)^{-1}M_{j+1}\left\{(1+\rho)^{-n-1}K M_{j+1}+2^{-n}\right\}$$

for $1 \le j \le n-1$. This shows that $\left(a_n\right)$ is a Cauchy sequence: we let $a = \lim a_n$. (Note that we have again used the local convexity of A .)

We also have that $\left(y_n\right)$ is Cauchy, since

(13) $$\left|y_{n+1}-y_n\right| = \left|\rho t_n x-\rho t_n e_{n+1}x\right| < \varepsilon 2^{-n-1} \quad (n = 1, 2, 3, \ldots) \ ,$$

by the definitions of y_n and t_n and inequality (11). We let $y = \lim y_n$.

The conclusions of the theorem are now clear. Since $h_n y_n = x$ and $y_n = t_n x$ for all n , we have $x = ay$ and $y \in \overline{Ax}$. Finally,

$$d(x, y) \le \left|x-y_1\right| + \sum_{n=1}^{\infty} \left|y_{n+1}-y_n\right|$$

$$\le \varepsilon 2^{-1} + \sum_{n=1}^{\infty} \varepsilon 2^{-n-1} \ , \text{ by (9) and (13),}$$

$$= \varepsilon \ .$$

COROLLARY 4.2. *Let* A *be a locally convex* F-*algebra with uniformly bounded left approximate identity. Then, for every sequence* $x_n \to 0$ *in*

A , there exists $a \in A$ and a sequence $y_n \to 0$ in A such that $x_n = ay_n$
for all n .

Proof. Let $c_0(A)$ denote the left Fréchet A-module of all zero-
convergent sequences in A with the topology given by the seminorms

$$P_j\big(\big(a_n\big)\big) = \sup_n p_j\big(a_n\big)$$

corresponding to the topology-defining seminorms p_j on A . Then (5)
implies

$$P_j(ax) \le p_{j+1}(a)P_{j+1}(x) \quad \big(a \in A, \, x \in c_0(A)\big) \ ,$$

and it is easy to check that $c_0(A)$ is essential. The corollary now
follows by appling Theorem 4.1 with $X = c_0(A)$.

THEOREM 4.3. *Let A be a locally convex F^*-algebra with uniformly*
bounded left approximate identity. Then every positive functional on A
is continuous.

Proof. Let $x_n \to 0$ in A . Then, by Corollary 4.2, we may write
$x_n = ay_n$ with $a \in A$ and $y_n \to 0$ in A . By a second application of
Corollary 4.2, we can write $y_n^* = b^*z_n^*$ with $b^* \in A$ and $z_n^* \to 0$ in A .
Thus, since $*$ is continuous, we have $x_n = az_n b$ with $z_n \to 0$. By
Theorem 2.1, $f\big(x_n\big) \to 0$; hence the result.

5.   Problems

In this final section, we list a few questions which we believe to be
open.

5.1. Is it true that if A is an F^*-algebra with A^2 closed and of
finite codimension, then every positive functional on A is continuous?
Is this true under any of the following additional hypotheses on A :

(i) A is Banach;

(ii) A is separable;

(iii) A is Banach and separable;

(iv) A is commutative (but not separable)?

5.2. Does Theorem 4.1 (and hence Corollary 4.2 and Theorem 4.3) hold when A is not locally convex?

5.3. A possible approach to 5.1 would be to try to obtain something like the conclusion of Corollary 4.2, that is, simultaneous factorization of a sequence, from a hypothesis such as $A^2 = A$ which is weaker than the existence of a uniformly bounded left approximate identity. Specifically, we ask the following questions.

(i) Suppose that A is a separable Banach algebra such that, for every $x \in A$, there exist $a, b \in A$ with $x = ab$ (and we may also require that $\|a\|\|b\| \leq K\|x\|$ for some fixed K). Is it then true that every sequence $x_n \to 0$ in A may be written as $x_n = ay_n$ for some $a \in A$ and $y_n \to 0$ in A ?

(ii) Suppose that A is a separable Banach algebra with $A^2 = A$. Then every $x \in A$ may be written in the form $x = \sum_{i=1}^{k} a_i b_i$ for some $a_1, \ldots, a_k, b_1, \ldots, b_k \in A$, where k is a positive integer independent of x (see [4], [12] or Lemma 3.5 *(i)* above). We also know that we may arrange that $\sum_{i=1}^{k} \|a_i\|\|b_i\| \leq K\|x\|$ for some number K independent of x [*12*]. Does it follows that every sequence $x_n \to 0$ in A may be written in the form $x_n = \sum_{i=1}^{k} a_i y_{in}$, for some $a_i, y_{in} \in A$ $(1 \leq i \leq k$, $n = 1, 2, 3, \ldots)$ with $y_{in} \to 0$ as $n \to \infty$ $(1 \leq i \leq k)$?

(iii) Do (i) and (ii) hold even when the sequence $\left(x_n\right)$ is replaced by a pair $\left(x_1, x_2\right)$?

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