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ϵ-FAMILIES OF OPERATORS IN TRIEBEL-LIZORKIN AND TENT SPACES

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ABSTRACT. In this paper, we study the boundedness of ϵ-families of operators on Triebel-Lizorkin with wide range of parameters. We also prove that ϵ-families of operators are bounded from Triebel-Lizorkin spaces into (generalized) tent spaces, and obtain a characterization of certain Triebel-Lizorkin spaces in terms of tent spaces. In particular, the boundedness of fractional operators in Triebel-Lizorkin, and a sharp version of $T1$ theorem for generalized Calderón-Zygmund operators on Triebel-Lizorkin spaces can be considered as applications of (proofs of) these results.

1. **Introduction.** In this paper, we study the boundedness of ϵ-families of operators in Triebel-Lizorkin and (generalized) tent spaces. Christ and Journé introduced ϵ-families of operators in [CJ], where a $T1$ theorem for Carleson measures was proved. A discrete version of ϵ-families of operators has been studied in [HJTW] for certain Triebel-Lizorkin and Besov spaces under the framework of the Frazier-Jawerth theory of smooth atoms and molecules. In this paper, we discuss a different approach, which relies on the method introduced in [CDMS] and the main result of the paper on the relation between Triebel-Lizorkin spaces and tent spaces.

Tent spaces were systematically treated in [CMS], where boundedness of convolution operators were discussed. We give a slightly more general definition of tent spaces, which is similar in nature to the definition of Triebel-Lizorkin spaces. We then obtain the boundedness of ϵ-families of operators from Triebel-Lizorkin spaces into corresponding (generalized) tent spaces (Theorem 1.3). In the case of $1 < p < \infty$, the proof is based on an extrapolation theorem for vector-valued singular operators (Theorem A, see [GR]). For the cases of $p \leq 1$ or ∞, we make a use of the technique developed by Fefferman and Stein in their fundamental work on H^p spaces [FS], and take an advantage of the atomic decomposition for tent spaces. As a consequence, we obtain a characterization of Triebel-Lizorkin spaces in terms of tent spaces.

Finally, we briefly discuss two applications of our results: the boundedness of fractional operators, and a sharp version of $T1$ theorem for generalized Calderón-Zygmund operators (see [T] and [HS]) in Triebel-Lizorkin spaces.

Let $\epsilon > 0$, and denote $[\epsilon]$ the largest integer strictly less than ϵ. A family $S = \{S_t\}_{t>0}$ of operators is called an *ϵ-family* of operators on $\mathbb{R}^n$ if, for each $t > 0$, the kernel $K_t(x, y)$

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of S_t satisfies the following conditions:

$$|K_t(x,y)| \le C\frac{t^\epsilon}{(t+|x-y|)^{n+\epsilon}}, \tag{1.1}$$

for $x, y \in \mathbb{R}^n$, and

$$|K_t(x,y) - P^{[\epsilon]}_{y-z}K_t(x,z)| \le C\Big(\frac{|y-z|}{t+|x-y|}\Big)^\epsilon \frac{t^{\epsilon-[\epsilon]}}{(t+|x-y|)^{n+\epsilon-[\epsilon]}}, \tag{1.2}$$

for $2|y-z| \le t+|x-y|$, where

$$P^{[\epsilon]}_{y-z}K_t(x,z) = \sum_{|\gamma|\le[\epsilon]} \frac{(y-z)^\gamma}{\gamma!} D_2^\gamma K_t(x,z) \tag{1.3}$$

is the Taylor polynomial of degree $[\epsilon]$ of $K_t(x,y)$ with respect to the second variable in terms of $y-z$ at point z. Here the common multi-index notation is used, *i.e.*, $\gamma = (\gamma_1, \gamma_2, \ldots, \gamma_n)$ is an n-tuple of nonnegative integer, $|\gamma| = \gamma_1 + \gamma_2 + \cdots + \gamma_n$, $x^\gamma = x_1^{\gamma_1}x_2^{\gamma_2}\cdots x_n^{\gamma_n}$ for $x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$, $D^\gamma = \partial_{x_1}^{\gamma_1}\partial_{x_2}^{\gamma_2}\cdots\partial_{x_n}^{\gamma_n}$, and D_2 stands for the derivative with respect to the second variable. It is easy to observe that by replacing ϵ by any number less that ϵ but bigger that $[\epsilon]$ we can always assume that ϵ is not an integer.

If S is a family of operators, we shall denote the adjoint of S by $S^* = \{S_t^*\}_{t>0}$. Since we only consider homogeneous function spaces, we shall say that an ϵ-family of operators S has the reproducing property if $S(x^\gamma) = 0$ for all $|\gamma| \le [\epsilon]$ and there is an ϵ-family of operators R with the same property as S such that

$$\int_0^\infty R_t^* S_t \frac{dt}{t} = I \tag{1.4}$$

holds in the distribution sense, where I denotes the identity operator. We call that such a pair (S, R) is a reproducing couple.

A well-known example of reproducing couples is given by the Calderón reproducing formula, that is, for any given $\epsilon > 0$, there exists a Schwartz function $\phi \in \mathcal{S}$ so that ϕ is radial, $\mathrm{spt}(\phi) \subset \{x \in \mathbb{R} : |x| < 1\}$, $\int_{\mathbb{R}^n} x^\gamma \phi(x)\,dx = 0$ for all $|\gamma| \le [\epsilon]$, and $\int_0^\infty |\hat\phi(t\xi)|^2 \frac{dt}{t} = 1$ for $\xi \ne 0$, where $\hat\phi$ is the Fourier transform of ϕ. If we define $Q_t^\phi f = \phi_t * f$, where $\phi_t(x) = t^{-n}\phi(x/t)$, then Q^ϕ is clearly an ϵ-family of operators, and (Q^ϕ, Q^ϕ) is a reproducing couple.

A variant of the last example is the following continuous version of the φ-transform formula [FJ]. There exist Schwartz functions ψ, $\varphi \in \mathcal{S}$ such that both $\hat\varphi$, $\hat\psi \in C^\infty$ with compact support and vanishing near the origin, and $\int_0^\infty \hat\psi(t\xi)\hat\varphi(t\xi)\frac{dt}{t} = 1$ for $\xi \ne 0$. Thus, the reproducing property (1.4) is valid with $S = Q^\varphi$ and $R = Q^\psi$. After the first version of this paper was submitted, we learned that Han [H1] was able to show the existence of nonconvolution reproducing couples.

Let ω be a weight function on $\mathbb{R}^n$, then ω is called a *doubling weight* if there is a constant C so that $|2B|_\omega \le C|B|_\omega < \infty$ for all balls B in $\mathbb{R}^n$, where $2B$ is the ball with the same center as B but twice the radius, and $|B|_\omega = \int_B \omega(y)\,dy$. We also recall that a weight function ω is said to satisfy *the reverse Hölder condition of order* q ($1 \le q \le \infty$), and write $\omega \in \mathcal{RH}_q$, if there exists a constant C such that

$$\Big(\frac{1}{|B|}\int_B \omega(x)^q\,dx\Big)^{1/q} \le C\Big(\frac{1}{|B|}\int_B \omega(x)\,dx\Big), \quad 1 < q < \infty, \tag{1.5}$$

for all balls B in $\mathbb{R}^n$, where $|B|$ denotes the Lebesgue measure of the ball B. We notice that condition (1.5) is trivial for $q = 1$, and it is a convention that condition (1.5) is void for $q = \infty$.

For $1 \le p < \infty$, we denote $p' = p/(p-1)$. We recall that a weight function ω is a Muckenhoupt's $\mathcal{A}_p$ weight if there is a constant C such that

$$\Big(\frac{1}{|B|}\int_B \omega(y)\,dy\Big)\Big(\frac{1}{|B|}\int_B \omega(y)^{1-p'}\,dy\Big)^{p-1} \le C, \quad 1 < p < \infty \tag{1.6}$$

or

$$\frac{1}{|B|}\int_B \omega(y)\,dy \le C\operatorname{essinf}_B \omega, \quad p = 1, \tag{1.7}$$

for all balls B in $\mathbb{R}^n$. It is easy to see that $\omega \in \mathcal{A}_p$ if and only if $\omega^{1-p'} \in \mathcal{A}_{p'}$. Also, it is clear, by Hölder's inequality, that if $\omega \in \mathcal{A}_p$ then $\omega \in \mathcal{A}_q$ for all $q \ge p$. A non-trivial fact is that any $\mathcal{A}_p$ weight ω has the strong doubling property: for each ball B and measurable subset E of B,

$$\frac{|B|_\omega}{|E|_\omega} \le C\Big(\frac{|B|}{|E|}\Big)^p, \tag{1.8}$$

with a constant C independent of B and E (see [Tor], for instance).

Let $\phi \in \mathcal{S}$ be the Schwartz function in Calderón's reproducing formula, which has ℓ moment conditions (*i.e.*, $\int_{\mathbb{R}^n} x^\gamma \phi(x)\,dx = 0$ for all $|\gamma| \le \ell$). We denote $Q_t = Q_t^\phi$. Let ω be a weight function. For $\alpha \in \mathbb{R}$ with $|\alpha| < \ell + 1$, and $0 < p, q \le \infty$, the weighted Triebel-Lizorkin space $\dot F_p^{\alpha,q}(\omega)$ is defined as the space of tempered distributions modulo polynomials, $f \in \mathcal{S}'/\mathcal{P}$, with respect to the norm (or quasi-norm)

$$\|f\|_{\dot F_p^{\alpha,q}(\omega)} = \Big\|\Big(\int_0^\infty (t^{-\alpha}|Q_t f|)^q \frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\omega)}, \quad 0 < p,\ q < \infty; \tag{1.9}$$

and with the sup-norm $\sup\{t^{-\alpha}|Q_t f(x)| : t > 0\}$ instead of $L^q(\frac{dt}{t})$ norm when $q = \infty$, or

$$\|f\|_{\dot F_\infty^{\alpha,q}(\omega)} = \Big\|\sup_{B\ni x}\Big(\frac{1}{|B|}\int_B \int_0^{r(B)} \big(t^{-\alpha}|Q_t f(x)|\big)^q \frac{dt}{t}\,dx\Big)^{1/q}\Big\|_{L^\infty(\omega)}, \quad 0 < q < \infty, \tag{1.10}$$

where the supremum is taking over all balls $B \subset \mathbb{R}^n$ such that $x \in B$, and $r(B)$ is the radius of the ball B. We recall that, by definition, $\|g\|_{L^\infty(\omega)} = \operatorname{esssup}\{g(x)/\omega(x) : x \in \mathbb{R}^n\}$. It is well-known, and also will be implied by our results, that spaces $\dot F_p^{\alpha,q}(\omega)$ are well defined in the sense that they do not depend on (up to equivalent norms) the choices of ϕ. More

$$A^{\alpha,\infty}f(x) = \sup_{(y,t)\in\Gamma(x)} t^{-\alpha}|f(y,t)|, \quad q=\infty, \tag{1.14}$$

and

$$C^{\alpha,q}f(x) = \sup_{B\ni x}\Big(\frac{1}{|B|}\iint_{\hat{B}}\big(t^{-\alpha}|f(y,t)|\big)^q\frac{dy\,dt}{t}\Big)^{1/q}, \quad 1\le q<\infty, \tag{1.15}$$

where the supremum is taking over all of balls $B\subset\mathbb{R}^n$ which contain x, and $\hat{B}$ is the tent over the ball B, that is, $\hat{B}=\{(x,t)\in\mathbb{R}^{n+1}_+ : B(x,t)\subset B\}$.

The (generalized) tent space $T^{\alpha,q}_p(\omega)$ is the space of all measurable functions (including Borel measures, if $q=1$) f on $\mathbb{R}^{n+1}_+$ so that $\|f\|_{T^{\alpha,q}_p(\omega)}=\|A^{\alpha,q}f\|_{L^p(\omega)}<\infty$ if $1\le p,q<\infty$; or $\|f\|_{T^{\alpha,q}_\infty(\omega)}=\|C^{\alpha,q}f\|_{L^\infty(\omega)}<\infty$, if $p=\infty$ and $1\le q<\infty$. When $q=\infty$, the (generalized) tent space $T^{\alpha,\infty}_p(\omega)$ (with $1\le p<\infty$) is defined as the space of all continuous functions f on $\mathbb{R}^{n+1}_+$ so that $\|f\|_{T^{\alpha,\infty}_p(\omega)}=\|A^{\alpha,\infty}f\|_{L^p(\omega)}<\infty$, and $\|f_\varepsilon-f\|_{T^{\alpha,\infty}_p(\omega)}\longrightarrow 0$ as $\varepsilon\longrightarrow 0$, where $f_\varepsilon(x,t)=f(x,t+\varepsilon)$. For $1<p,q<\infty$, it is clear that the dual space of $T^{\alpha,q}_p(\omega)$ is $T^{-\alpha,q'}_{p'}(\omega^{1-p'})$. The duality of spaces $T^{\alpha,q}_1(\omega)$ and $T^{-\alpha,q'}_\infty(\omega)$ can be shown exactly in the same way as in [CMS] (see also [D]), more precisely, we have

$$\iint_{\mathbb{R}^{n+1}_+}|f(x,t)g(x,t)|\frac{dx\,dt}{t}\le C\int_{\mathbb{R}^n}A^{\alpha,q}f(x)C^{-\alpha,q'}g(x)\,dx, \tag{1.16}$$

for all $f\in T^{\alpha,q}_1(\omega)$ and $g\in T^{-\alpha,q'}_\infty(\omega)$.

We now state the main result of this paper.

THEOREM 1.3. *Suppose Hypothesis 1.1. Then there is a constant C so that*

$$\|Sf\|_{T^{\alpha,q}_p(\omega)}\le C\|f\|_{\dot{F}^{\alpha,q}_p(\omega)}, \tag{1.17}$$

for all $f\in\dot{F}^{\alpha,q}_p(\omega)$. Conversely,

$$\Big\|\int_0^\infty S^*_t g(\cdot,t)\frac{dt}{t}\Big\|_{\dot{F}^{\alpha,q}_p(\omega)}\le C\|g\|_{T^{\alpha,q}_p(\omega)}, \tag{1.18}$$

for all $g\in T^{\alpha,q}_p(\omega)$.

We remark that inequality (1.18) is an extension of a result in [CMS], where convolution operators were studied for tent spaces $T^{0,q}_p$.

As an immediate consequence of the last theorem, we have the following characterization of Triebel-Lizorkin spaces by means of tent spaces, which is an analogue of the relation between the Luzin area function and Littlewood-Paley g-function.

COROLLARY 1.4. *Let $1\le p,q\le\infty$ (p and q are not both ∞) and $\alpha\in\mathbb{R}$. Suppose that $\omega\in\mathcal{A}_p$ if $1<p<\infty$, and $\omega\in\mathcal{A}_1\cap\mathcal{RH}_{q'}$ if $p=1$ or $p=\infty$. Then a tempered distribution $f\in\mathcal{S}'/\mathcal{P}$ belongs to $\dot{F}^{\alpha,q}_p(\omega)$ if and only if $Sf\in T^{\alpha,q}_p(\omega)$ for some ϵ-family of operators S with $\epsilon>|\alpha|$, provided S has the reproducing property.*

The paper is organized as follows. In Section 2, we list a few lemmas which is needed throughout of the paper, the proofs will be omitted for some well-known statements.

Theorem 1.3 is proved for the case of $1 < p < \infty$ in Section 3. While in Section 4 we generalize some results of Littlewood-Paley-Stein theory for Triebel-Lizorkin spaces, and in Section 5, we use the results in Section 4 to complete the proof for both Theorems 1.2 and 1.3 in the cases of $p \leq 1$ or $p = \infty$. Finally, in Section 6, we point out two applications of our results. In what follows, the letter C will denote constants, which may be different from line to line, as long as the dependency of the constants are clear in the context.

2. **Some lemmas.** We need to consider the operators

$$(S_tQ_s)f(x) = \int_{\mathbb{R}^n} K_{t,s}(x,y)f(y)\,dy, \tag{2.1}$$

for $t, s > 0$, and their adjoints $(S_tQ_s)^*$. Let $K_{t,s}(x,y)$ and $K^*_{t,s}(x,y)$ be the kernels of S_tQ_s and $(S_tQ_s)^*$, respectively, then

$$K_{t,s}(x,y) = \int_{\mathbb{R}^n} K_t(x,z)\phi_s(y-z)\,dz, \tag{2.2}$$

and

$$K^*_{t,s}(x,y) = \int_{\mathbb{R}^n} K_t(y,z)\phi_s(x-z)\,dz. \tag{2.3}$$

The first two lemmas are well-known (see [CDMS], [HH] and [HS]).

LEMMA 2.1. *The kernel $K_{t,s}(x,y)$ satisfies the following estimates*

$$|K_{t,s}(x,y)| \leq C\Big(\frac{s}{t}\Big)^{\epsilon} \frac{t^{2\epsilon-[\epsilon]}}{(t+|x-y|)^{n+2\epsilon-[\epsilon]}}, \quad \text{for } 0 < s \leq t, \tag{2.4}$$

and

$$|K_{t,s}(x,y)| \leq C\Big(\frac{t}{s}\Big)^{\epsilon} \frac{s^{2\epsilon-[\epsilon]}}{(s+|x-y|)^{n+2\epsilon-[\epsilon]}}. \quad \text{for } 0 < t \leq s. \tag{2.5}$$

The same estimates are also valid for the kernel $K^*_{t,s}(x,y)$.

LEMMA 2.2. *For fixed s, $t > 0$,*

$$\int_{\mathbb{R}^n} \big(|K_{t,s}(x,y)| + |K_{t,s}(y,x)|\big)|f(y)|\,dy \leq C\Big(\frac{s}{t}\Big)^{\epsilon} Mf(x), \quad \text{if } 0 < s \leq t, \tag{2.6}$$

$$\int_{\mathbb{R}^n} \big(|K_{t,s}(x,y)| + |K_{t,s}(y,x)|\big)|f(y)|\,dy \leq C\Big(\frac{t}{s}\Big)^{\epsilon} Mf(x), \quad \text{if } 0 < t \leq s, \tag{2.7}$$

where M is the Hardy-Littlewood maximal operator. The same estimates are also valid for the kernel $K^*_{t,s}(x,y)$.

The next lemma is actually equivalent to the statement of Theorem 1.2 for the case of $1 \leq p, q \leq \infty$, the proof is very similar to the proof given in [HH], and hence omitted.

precisely, if ϕ_1 and ϕ_2 are two Schwartz functions which satisfy Calderón's reproducing formula, but with ℓ_1 and ℓ_2 moment conditions respectively, then spaces $\dot{F}_p^{\alpha,q}(\omega)$ defined by using ϕ_1 and ϕ_2 are equivalent for $|\alpha| < \min\{\ell_1, \ell_2\}$. Of course, one can also use the φ-transform formula to define the Triebel-Lizorkin spaces by setting $Q_t = Q_t^{\varphi}$, which has all of moment conditions.

The Besov spaces $\dot{B}_p^{\alpha,q}(\omega)$ are defined in a similar way (see [Tr]). In particular, $\dot{F}_{\infty}^{\alpha,\infty}(\omega) = \dot{B}_{\infty}^{\alpha,\infty}(\omega)$, by definition.

Before stating our results we first list our general hypothesis for the results concerning Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\omega)$ and tent spaces $T_p^{\alpha,q}(\omega)$.

HYPOTHESIS 1.1. *Let $\epsilon > 0$ be given, and suppose that the definition function ϕ of Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}(\omega)$ has at least $[\epsilon]$ moment conditions. We assume that*

(1) $|\alpha| < \epsilon$;

(2) $\max\{\frac{n}{n+\epsilon}, \frac{n}{n+\epsilon+\alpha}\} < p \leq \infty$, $1 \leq q \leq \infty$, and p and q are not both ∞;

(3) $\omega \in \mathcal{A}_p$ if $1 < p < \infty$, and $\omega \in \mathcal{A}_1 \cap \mathcal{RH}_{(q/p)'}$ if $p \leq 1$ or $p = \infty$;

(4) S is an ϵ-family of operators on $\mathbb{R}^n$ with $S(x^{\gamma}) = 0$ for all $|\gamma| \leq [\epsilon]$.

The following theorem is a motivation for this paper:

THEOREM 1.2. *Suppose Hypothesis 1.1, then there are constants C so that*

$$\left\| \left(\int_0^{\infty} (t^{-\alpha} |S_t f|)^q \frac{dt}{t} \right)^{1/q} \right\|_{L^p(\omega)} \leq C \|f\|_{\dot{F}_p^{\alpha,q}(\omega)}, \tag{1.11}$$

for all $f \in \dot{F}_p^{\alpha,q}(\omega)$ with $p \neq \infty$, and

$$\left\| \sup_{B \ni x} \left(\frac{1}{|B|} \int_B \int_0^{r(B)} \left(t^{-\alpha} |S_t f(x)| \right)^q \frac{dt}{t} \, dx \right)^{1/q} \right\|_{L^{\infty}(\omega)} \leq C \|f\|_{\dot{F}_{\infty}^{\alpha,q}(\omega)}, \tag{1.12}$$

for all $f \in \dot{F}_{\infty}^{\alpha,q}(\omega)$.

For $1 \leq p, q \leq \infty$ (and $|\alpha| < 1$), an (unweighted) discrete version of Theorems 1.2 have been proved previously in [HJTW], and hence the proof for this case is omitted, see also [HH]. For the case of $p \leq 1$, the theorem seems to be new, and the proof will be given in Section 5. We also remark that it follows from the duality argument that the converse inequalities of (1.11) and (1.12) hold if, in addition, $1 \leq p, q \leq \infty$ and S has the reproducing property.

We now recall some notation for tent spaces. Let $\mathbb{R}_+^{n+1} = \mathbb{R}^n \times (0, \infty)$ be the usual upper-half space of $\mathbb{R}^{n+1}$. The standard cone $\Gamma(x)$ at $x \in \mathbb{R}^n$ is defined by

$$\Gamma(x) = \{(y,t) \in \mathbb{R}_+^{n+1} : |x - y| < t\}.$$

Let $\alpha \in \mathbb{R}$, and let ω be a weight function, for a given function f on $\mathbb{R}_+^{n+1}$, we define

$$A^{\alpha,q} f(x) = \left(\iint_{\Gamma(x)} \left(t^{-\alpha} |f(y,t)| \right)^q \frac{dy\,dt}{t^{n+1}} \right)^{1/q}, \quad 1 \leq q < \infty, \tag{1.13}$$

LEMMA 2.3. *Let $K_{t,s}(x,y)$ be a kernel function which satisfies estimates (2.4) and (2.5). Then, for $|\alpha| < \epsilon$, $1 < p < \infty$, $1 \le q \le \infty$, and $\omega \in A_p$, there exists a constant C such that*

$$\Big\| \iint_{\mathbb{R}^{n+1}_+} K_{t,s}(x,y) g(y,s) \frac{dy\,ds}{s} \Big\|_{L_p^{\alpha,q}(\omega)} \le C \|g\|_{L_p^{\alpha,q}(\omega)}, \tag{2.8}$$

for all measurable function $g(x,t)$ on $\mathbb{R}^{n+1}_+$, where

$$\|g\|_{L_p^{\alpha,q}(\omega)} = \Big\| \Big(\int_0^\infty \big(t^{-\alpha} |g(\cdot,t)| \big)^q \frac{dt}{t} \Big)^{1/q} \Big\|_{L^p(\omega)}. \tag{2.9}$$

The last lemma of the section is an important kernel estimates, which will be used in the proof of Theorem 1.3.

LEMMA 2.4. *Let $K_{t,s}(x,y)$ be the kernel defined by (2.2), then there is a number δ with $0 < \delta < \min\{1, \epsilon - |\alpha|\}$ so that, if $4|y-y'| \le |x-y|$ and $|x-z| \le t$ for some $x \in \mathbb{R}^n$, then*

$$|K_{t,s}(z,y) - K_{t,s}(z,y')| \le C \Big(\frac{s}{t}\Big)^{\epsilon-\delta} \frac{t^{2\epsilon-[\epsilon]-\delta} |y-y'|^\delta}{(t+|z-y|)^{n+2\epsilon-[\epsilon]}}, \quad \textit{for } 12s \le t + |z-y|, \tag{2.10}$$

$$|K_{t,s}(z,y) - K_{t,s}(z,y')| \le C \Big(\frac{t}{s}\Big)^{\epsilon-\delta} \frac{s^{2\epsilon-[\epsilon]-\delta} |y-y'|^\delta}{(s+|z-y|)^{n+2\epsilon-[\epsilon]}}, \quad \textit{for } 12s \ge t + |z-y|. \tag{2.11}$$

PROOF. If $12s \le t+|z-y|$ and $|y-y'| \le 2s$, then the support of $\phi_s(y-\cdot)-\phi_s(y'-\cdot)$ is contained in the ball $\{w : |w-y| < 4s\}$, and so

$$|y-w| < 4s \le \frac{1}{3}(t+|z-y|) \le \frac{1}{3}(t+|z-w|+|w-y|).$$

Therefore, $2|y-w| \le t+|z-w|$, and $t+|z-y| \le \frac{3}{2}(t+|z-w|)$. It then follows from the fact that $\int x^\gamma \phi(x)\,dx = 0$ for all $|\gamma| \le [\epsilon]$, and the estimate (1.2) that

$$\begin{aligned}
|K_{t,s}(z,y) - K_{t,s}(z,y')| &= \Big| \int_{\mathbb{R}^n} \big(K_t(z,w) - P^{[\epsilon]}_{w-y} K_t(z,y)\big)\big(\phi_s(y-w) - \phi_s(y'-w)\big)\,dw \Big| \\
&\le \int_{\mathbb{R}^n} |K_t(z,w) - P^{[\epsilon]}_{w-y} K_t(z,y)|\,|\phi_s(y-w) - \phi_s(y'-w)|\,dw \\
&\le C \frac{t^{\epsilon-[\epsilon]}|y-y'|}{s^{n+1}} \int_{|w-y|<4s} \frac{|w-y|^\epsilon}{(t+|z-w|)^{n+2\epsilon-[\epsilon]}}\,dw \\
&\le C \frac{t^{\epsilon-[\epsilon]}|y-y'|}{s^{n+1}(t+|z-y|)^{n+2\epsilon-[\epsilon]}} \int_{|w-y|<4s} |w-y|^\epsilon\,dw \\
&= C \frac{t^{\epsilon-[\epsilon]}|y-y'|}{s^{1-\epsilon}(t+|z-y|)^{n+2\epsilon-[\epsilon]}} \\
&\le C \Big(\frac{s}{t}\Big)^{\epsilon-\delta} \frac{t^{2\epsilon-[\epsilon]-\delta}|y-y'|^\delta}{(t+|z-y|)^{n+2\epsilon-[\epsilon]}}.
\end{aligned}$$

If $12s \le t + |z-y|$ and $|y-y'| > 2s$, then the supports of $\phi_s(y-\cdot)$ and $\phi_s(y'-\cdot)$ are disjoint. We also note that, by assumptions $4|y-y'| \le |x-y|$ and $|x-z| \le t$, thus, if $|w-y'| < s$ we have

$$|w-y| \le |w-y'| + |y-y'| < s + |y-y'| < \frac{1}{3}(t+|z-y|) \le \frac{1}{3}(t+|z-w|+|w-y|).$$

Hence, $2|y-w| \le t+|z-w|$ and $t+|z-y| \le \frac{3}{2}(t+|z-w|)$ if either $|w-y| < s$ or $|w-y'| < s$. Therefore, since $\int |\phi(x)|\,dx$ is finite,

$$\begin{aligned}|K_{t,s}(z,y) - K_{t,s}(z,y')| &\le \int_{|y-w|<s} |K_t(z,w) - P^{[\epsilon]}_{w-y}K_t(z,y)|\,|\phi_s(y-w)|\,dw \\ &\quad + \int_{|y'-w|<s} |K_t(z,w) - P^{[\epsilon]}_{w-y}K_t(z,y)|\,|\phi_s(y'-w)|\,dw \\ &\le C\Big(\frac{s}{t}\Big)^{\epsilon} \frac{t^{2\epsilon-[\epsilon]}}{(t+|z-y|)^{n+2\epsilon-[\epsilon]}} \le C\Big(\frac{s}{t}\Big)^{\epsilon-\delta} \frac{t^{2\epsilon-[\epsilon]-\delta}|y-y'|^{\delta}}{(t+|z-y|)^{n+2\epsilon-[\epsilon]}}.\end{aligned}$$

If $t+|z-y| \le 12s$, then $|y-y'| < 3s$, since, by assumptions,

$$4|y-y'| \le |x-y| \le |x-z| + |z-y| < t + |z-y| < 12s.$$

Applying the hypothesis that $S(x^{\gamma}) = 0$ for all $|\gamma| \le [\epsilon]$, we have

$$\begin{aligned}&K_{t,s}(z,y) - K_{t,s}(z,y') \\ &\quad = \int_{\mathbb{R}^n} K_t(z,w)\big(\phi_s(y-w) - P^{[\epsilon]}_{z-w}\phi_s(y-z) + P^{[\epsilon]}_{z-w}\phi_s(y'-z) - \phi_s(y'-w)\big)\,dw \\ &\quad = \int_{|z-w|\le 4s} \cdots + \int_{4s<|z-w|} \cdots = I + II.\end{aligned}$$

Since

$$\begin{aligned}&\Big|\big(\phi_s(y-w) - P^{[\epsilon]}_{z-w}\phi_s(y-z)\big) - \big(\phi_s(y'-w) - P^{[\epsilon]}_{z-w}\phi_s(y'-z)\big)\Big| \\ &\qquad\qquad \le C\min\Big\{\frac{|z-w|^{[\epsilon]+1}|y-y'|}{s^{n+[\epsilon]+2}}, \frac{|y-y'|}{s^{n+1}}\Big\},\end{aligned}$$

by using the estimate (1.1),

$$\begin{aligned}|I| &\le C\frac{t^{\epsilon}|y-y'|}{s^{n+[\epsilon]+2}} \int_{|z-w|\le 4s} \frac{|z-w|^{[\epsilon]+1}}{(t+|z-w|)^{n+\epsilon}}\,dw \\ &\le C\frac{t^{\epsilon-\delta}|y-y'|}{s^{n+[\epsilon]+2}} \int_{|z-w|\le 4s} \frac{|z-w|^{[\epsilon]+1}}{(t+|z-w|)^{n+\epsilon-\delta}}\,dw \\ &\le C\Big(\frac{t}{s}\Big)^{\epsilon-\delta} \frac{s^{2\epsilon-[\epsilon]-\delta}|y-y'|^{\delta}}{s^{n+2\epsilon-[\epsilon]}} \\ &\le C\Big(\frac{t}{s}\Big)^{\epsilon-\delta} \frac{s^{2\epsilon-[\epsilon]-\delta}|y-y'|^{\delta}}{(s+|z-y|)^{n+2\epsilon-[\epsilon]}},\end{aligned}$$

and

$$\begin{aligned}|II| &\le C\frac{t^{\epsilon}|y-y'|}{s^{n+1}}\int_{4s<|z-w|}\frac{1}{(t+|z-w|)^{n+\epsilon}}\,dw\\ &\le C\frac{t^{\epsilon-\delta}|y-y'|}{s^{n+1}}\int_{4s<|z-w|}\frac{1}{(t+|z-w|)^{n+\epsilon-\delta}}\,dw\\ &\le C\frac{t^{\epsilon-\delta}|y-y'|}{s^{n+1+\epsilon-\delta}}\\ &\le C\Big(\frac{t}{s}\Big)^{\epsilon-\delta}\frac{s^{2\epsilon-[\epsilon]-\delta}|y-y'|^{\delta}}{(s+|z-y|)^{n+2\epsilon-[\epsilon]}}.\end{aligned}$$

where we have used $|y-y'|\le 3s$ and $s+|z-y|\le 7s$ in both of the last inequalities of two chains of inequalities above. This completes the proof. ∎

3. **Proof of Theorem 1.3 (Case $1<p<\infty$).** We first note that, for $1<p,q<\infty$, (1.7) and (1.8) are equivalent by the duality argument. For instance, suppose (1.17), we take any $f\in \dot F_{p'}^{-\alpha,q'}(\omega^{1-p'})$, then

$$\begin{aligned}\Big|\int_{\mathbb{R}^n}\Big(\int_0^\infty S_t g(y,t)\frac{dt}{t}\Big)f(y)\,dy\Big| &= \Big|\iint_{\mathbb{R}^{n+1}_+} g(x,t)S_tf(y)\frac{dy\,dt}{t}\Big|\\ &\le C\int_{\mathbb{R}^n}A^{\alpha,q}(g)(x)A^{-\alpha,q'}(Sf)(x)\,dx\\ &\le C\|g\|_{T_p^{\alpha,q}(\omega)}\|Sf\|_{T_{p'}^{-\alpha,q'}(\omega^{1-p'})}\\ &\le C\|g\|_{T_p^{\alpha,q}(\omega)}\|f\|_{\dot F_{p'}^{-\alpha,q'}(\omega^{1-p'})},\end{aligned}$$

by (1.17), which shows (1.18).

We now prove (1.17) for the case of $1<p<\infty$. We first observe that if $q=1$ and $1<p<\infty$, then for any $h\in L^{p'}(\omega^{1-p'})$ such that $\|h\|_{L^{p'}(\omega^{1-p'})}\le 1$, we have

$$\begin{aligned}\int_{\mathbb{R}^n}A^{\alpha,1}(Sf)(x)h(x)\,dx &= \iint_{\mathbb{R}^{n+1}_+}\big(t^{-\alpha}|S_tf(y)|\big)\Big(\frac{1}{t^n}\int_{|x-y|<t}h(x)\,dx\Big)\frac{dy\,dt}{t}\\ &\le C\int_{\mathbb{R}^n}\Big(\int_0^\infty\big(t^{-\alpha}|S_tf(y)|\big)\frac{dt}{t}\Big)Mh(y)\,dy\\ &\le C\Big\|\Big(\int_0^\infty(t^{-\alpha}|S_tf|)^q\frac{dt}{t}\Big)^{1/q}\Big\|^q_{L^p(\omega)}\|Mh\|_{L^{p'}(\omega^{1-p'})}.\end{aligned}$$

Thus, (1.17) follows from weighted norm inequality for the maximal function in this case.

Next, we consider the case of $q=\infty$ and $1<p<\infty$. Let $\varepsilon>0$, we shall prove that there exists a constant C independent of ε so that

$$\int_{\mathbb{R}^n}\Big(\sup_{(y,t):\varepsilon\le|x-y|<t}t^{-\alpha}|S_tf(y)|\Big)g(x)\,dx\le C\Big\|\sup_{s>0}s^{-\alpha}|Q_sf|\Big\|_{L^p(\omega)}, \tag{3.1}$$

for all $g(x)\ge 0$ with $\|g\|_{L^{p'}(\omega^{1-p'})}\le 1$. Then the theorem will follow from this by the monotone convergence theorem for the present case.

We note that if $|x-y|<t$ and $z\in\mathbb{R}^n$ then $t+|x-z|\le 2(t+|y-z|)$. Hence, by using Lemma 2.1, we have

$$\begin{aligned}
&\sup_{(y,t):\varepsilon\le|x-y|<t}\int_0^\infty\Big(\frac{t}{s}\Big)^{-\alpha}|K_{t,s}(y,z)|\frac{ds}{s}\\
&\qquad\le C\sup_{(y,t):\varepsilon\le|x-y|<t}\left(\int_0^t\Big(\frac{s}{t}\Big)^{\epsilon+\alpha}\frac{ds}{s}+\int_t^\infty\Big(\frac{t}{s}\Big)^{\epsilon-\alpha}\frac{ds}{s}\right)\frac{t^{2\epsilon}}{(t+|x-z|)^{n+2\epsilon}}\\
&\qquad\le C\sup_{t>\varepsilon}\frac{t^{2\epsilon}}{(t+|x-z|)^{n+2\epsilon}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\sup_{(y,t):\varepsilon\le|x-y|<t}\int_0^\infty\Big(\frac{t}{s}\Big)^{-\alpha}|K_{t,s}(y,z)|\frac{ds}{s}\right)g(x)\,dx\\
&\quad\le C\int_{\mathbb{R}^n}\left(\sup_{t>\varepsilon}\frac{t^{2\epsilon}}{(t+|x-z|)^{n+2\epsilon}}\right)g(x)\,dx\\
&\quad\le C\left(\int_{|x-z|\le 2\varepsilon}\frac{|x-z|^{2\epsilon}}{(\varepsilon+|x-z|)^{n+2\epsilon}}g(x)\,dx+\sum_{k=2}^\infty\int_{2^k\varepsilon<|x-z|\le 2^{k+1}\varepsilon}\frac{1}{|x-z|^n}g(x)\,dx\right)\\
&\quad\le CMg(z),
\end{aligned}$$

where the constant C is independent of ε. By using the last inequality, the left-hand side of (3.1) is less than

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left(\sup_{(y,t):\varepsilon\le|x-y|<t}t^{-\alpha}\int_{\mathbb{R}^{n+1}_+}|K_{t,s}(y,z)Q_sf(z)|\frac{dz\,ds}{s}\right)g(x)\,dx\\
&\qquad\qquad\le C\int_{\mathbb{R}^n}\big(\sup_{s>0}s^{-\alpha}|Q_sf(z)|\big)Mg(z)\,dz\\
&\qquad\qquad\le C\Big\|\sup_{s>0}s^{-\alpha}|Q_sf(z)|\Big\|_{L^p(\omega)}\|Mg\|_{L^{p'}(\omega^{1-p'})},
\end{aligned}$$

and so the (3.1) follows from the weighted inequality for the maximal function.

To prove Theorem 1.3 for $1<p,q<\infty$, we use the reproducing property of Q_t and rewrite the inequality (1.17) as

$$\begin{aligned}
&\left(\int_{\mathbb{R}^n}\left(\iint_{\Gamma(x)}\left|\int_0^\infty\Big(\frac{t}{s}\Big)^{-\alpha}S_tQ_s(s^{-\alpha}Q_sf)(z)\frac{ds}{s}\right|^q\frac{dz\,dt}{t^{n+1}}\right)^{p/q}\omega(x)\,dx\right)^{1/p}\\
&\qquad\qquad\le C\left(\int_{\mathbb{R}^n}\left(\int_0^\infty\big(s^{-\alpha}|Q_sf(x)|\big)^q\frac{ds}{s}\right)^{p/q}\omega(x)\,dx\right)^{1/p}.
\end{aligned}$$

In order to show this, we consider the Banach spaces E of measurable functions on $(0,\infty)$ and F of measurable functions on $\mathbb{R}^{n+1}_+$ with norms

$$\|f\|_E=\left(\int_0^\infty|f(t)|^q\frac{dt}{t}\right)^{1/q}, \tag{3.2}$$

and

$$\|f\|_F = \Big(\int_0^\infty \int_{|w|<1} |f(w,t)|^q \, dw \frac{dt}{t}\Big)^{1/q}, \tag{3.3}$$

respectively.

Let K be the $\mathcal{L}(E,F)$-valued kernel given by

$$[K(x,y)f]_{w,t} = \int_0^\infty \Big(\frac{t}{s}\Big)^{-\alpha} K_{t,s}(x-tw,y)f(s)\frac{ds}{s}, \tag{3.4}$$

where $x, y \in \mathbb{R}^n$ are parameters, and $K_{t,s}$ is defined by (2.2). We then approach the problem by showing that the F-valued integral operator T, defined by

$$Tf(x) = \int_{\mathbb{R}^n} K(x,y)f(y)\, dy, \tag{3.5}$$

is a bounded mapping form $L^p_E(\omega)$ into $L^p_F(\omega)$ for $1 < p < \infty$, if $\omega \in \mathcal{A}_p$, that is,

$$\Big(\int_{\mathbb{R}^n} \|Tf(x)\|_F^p \omega(x)\, dx\Big)^{1/p} \le C\Big(\int_{\mathbb{R}^n} \|f(x)\|_E^p \omega(x)\, dx\Big)^{1/p}. \tag{3.6}$$

We note, by changing variable $z = x - tw$, that

$$\begin{aligned}\|Tf(x)\|_F &= \Big(\int_0^\infty \int_{|w|<1} \Big| \iint_{\mathbb{R}^{n+1}_+} \Big(\frac{t}{s}\Big)^{-\alpha} K_{t,s}(x-tw,y)f(y,s)\frac{dy\,ds}{s}\Big|^q \frac{dw\,dt}{t}\Big)^{1/q}\\ &= \Big(\iint_{\Gamma(x)} \Big|\int_0^\infty \Big(\frac{t}{s}\Big)^{-\alpha} S_tQ_sf_s(z)\frac{ds}{s}\Big|^q \frac{dz\,dt}{t^{n+1}}\Big)^{1/q},\end{aligned}$$

where $S_tQ_sf_s(z) = \int_{\mathbb{R}^n} K_{t,s}(z,y)f(y,s)\, dy$.

By using the extrapolation theorem on $\mathcal{A}_p$-weights, one can show the following theorem for vector-valued operators (see Theorem 3.16 on p. 496 of [GR] for a statement for convolution kernels).

THEOREM A. *Let E and F be Banach spaces. Suppose that T is defined by (3.5) with a $\mathcal{L}(E,F)$-valued kernel $K(x,y)$ satisfying the following estimates:*

$$\|K(x,y) - K(x,y')\|_{\mathcal{L}(E,F)} \le C\frac{|y-y'|^\delta}{|x-y|^{n+\delta}}, \quad \text{for } 4|y-y'| \le |x-y|, \tag{3.7}$$

for some $0 < \delta \le 1$. Assume that T is a bounded linear operator from $L^q_E(\omega)$ into $L^q_F(\omega)$, for some fixed q, $1 < q \le \infty$, and for all weights $\omega \in \mathcal{A}_1$, then T is bounded from $L^p_E(\omega)$ into $L^p_F(\omega)$ for all $\omega \in \mathcal{A}_p$ for $1 < p < \infty$.

Now, we note that if $p = q$ and $\omega \in \mathcal{A}_1$, it follows immediately from Fubini's theorem and the definition of $\mathcal{A}_1$ weights that

$$\begin{aligned}\|A^{\alpha,q}(Sf)\|^q_{L^q(\omega)} &= \int_{\mathbb{R}^n}\int_0^\infty \big(t^{-\alpha}|S_tf(y)|\big)^q \Big(\frac{1}{t^n}\int_{|x-y|<t} \omega(x)\, dx\Big)\frac{dt}{t}\, dy\\ &\le C\int_{\mathbb{R}^n}\int_0^\infty \big(t^{-\alpha}|S_tf(y)|\big)^q \frac{dt}{t}\omega(y)\, dy.\end{aligned}$$

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This shows that the vector-valued operator T is bounded for $p = q$ and for all weights $\omega \in \mathcal{A}_1$. Thus, by using the last theorem, we need only to verify the kernel estimate (3.7).

We notice that, by Hölder's inequality, we have

$$\begin{aligned}
&\|K(x,y) - K(x,y')\|_{\mathcal{L}(E,F)} \\
&\quad = \sup_{\|f\|_E \le 1} \|K(x,y)f - K(x,y')f\|_F \\
&\quad = \sup_{\|f\|_E \le 1} \left(\iint_{\Gamma(x)} \left| \int_0^\infty \left(\frac{t}{s}\right)^{-\alpha} \big(K_{t,s}(z,y) - K_{t,s}(z,y')\big) f(s) \frac{ds}{s} \right|^q \frac{dz\,dt}{t^{n+1}} \right)^{1/q} \\
&\quad \le \left(\iint_{\Gamma(x)} \left(\int_0^\infty \left(\frac{t}{s}\right)^{-\alpha q'} |K_{t,s}(z,y) - K_{t,s}(z,y')|^{q'} \frac{ds}{s} \right)^{q/q'} \frac{dz\,dt}{t^{n+1}} \right)^{1/q}.
\end{aligned}$$

To prove (3.7), we use Lemma 2.4. Let $x, x', y \in \mathbb{R}^n$ be such that $2|y - y'| \le |x - y|$, we then have

$$\begin{aligned}
&\int_0^\infty \left(\frac{t}{s}\right)^{-\alpha q'} |K_{t,s}(z,y) - K_{t,s}(z,y')|^{q'} \frac{ds}{s} \\
&\qquad \le C \int_0^{\frac{1}{12}(t+|z-y|)} \left(\left(\frac{s}{t}\right)^{\epsilon+\alpha-\delta} \frac{t^{2\epsilon-[\epsilon]-\delta}|y-y'|^\delta}{(t+|z-y|)^{n+2\epsilon-[\epsilon]}} \right)^{q'} \frac{ds}{s} \\
&\qquad\quad + C \int_{\frac{1}{12}(t+|z-y|)}^\infty \left(\left(\frac{t}{s}\right)^{\epsilon-\alpha-\delta} \frac{s^{2\epsilon-[\epsilon]-\delta}|y-y'|^\delta}{(s+|z-y|)^{n+2\epsilon-[\epsilon]}} \right)^{q'} \frac{ds}{s} \\
&\qquad \le C \left(\frac{t^{\epsilon-\delta}|y-y'|^\delta}{(t+|z-y|)^{n+\epsilon}} \right)^{q'}.
\end{aligned}$$

So, we obtain

$$\begin{aligned}
\|K(x,y) - K(x,y')\|^q_{\mathcal{L}(E,F)} &\le C \iint_{\Gamma(x)} \left(\frac{t^{\epsilon-\delta}|y-y'|^\delta}{(t+|z-y|)^{n+\epsilon}} \right)^q \frac{dz\,dt}{t^{n+1}} \\
&= C \left(\int_0^{|x-y|} + \int_{|x-y|}^\infty \right) \int_{|x-z|<t} \left(\frac{t^{\epsilon-\delta}|y-y'|}{(t+|z-y|)^{n+\epsilon}} \right)^q dz \frac{dt}{t^{n+1}} \\
&\le C|y-y'|^{\delta q} \left(\frac{1}{|x-y|^{(n+\epsilon)q}} \int_0^{|x-y|} t^{(\epsilon-\delta)q} \frac{dt}{t} + \int_{|x-y|}^\infty \frac{1}{t^{(n+\delta)q}} \frac{dt}{t} \right) \\
&\le C \left(\frac{|y-y'|^\delta}{|x-y|^{n+\delta}} \right)^q
\end{aligned}$$

as it required, where we have used the fact that $t + |z - y| \ge |x - y|$ if $|x - z| \le t$.

4. **A characterization of Triebel-Lizorkin spaces by tent spaces.** This section contains the generalization of some results of Littlewood-Paley-Stein theory for Triebel-Lizorkin spaces. The idea, of course, goes back to the fundamental work of Fefferman and Stein [FS] on H^p spaces.

We first note that, if ϕ is the Schwartz function in Calderón's reproducing formula, then there is a constant $c > 0$ such that $\sup_{t>0} |\hat{\phi}(t\xi)| \geq c$ for all $\xi \neq 0$. In fact, since $\int_0^\infty |\hat{\phi}(t)|^2 \, dt/t = 1$, there is a number $L > 1$ so that $\int_{1/L}^L |\hat{\phi}(t)|^2 \, dt/t \geq 1/2$, and hence

$$\sup_{t>0} |\hat{\phi}(t\xi)| \geq \sup_{1/L<t<L} |\hat{\phi}(t\xi)| \geq \Big(\frac{1}{2\log L}\int_{1/L}^{L} |\hat{\phi}(t\xi)|^2 \frac{dt}{t}\Big)^{1/2} \geq \frac{1}{2\sqrt{\log L}} > 0,$$

since $\hat{\phi}$ is radial and dt/t is the Haar measure for the multiplicative group of positive real numbers. It then follows from Theorem 5(a) of Chapter V in [ST] that we have the following estimate for extensions of distributions on $\mathbb{R}^n$ to the upper half-space $\mathbb{R}^{n+1}_+$.

LEMMA B. *Let ϕ be the Schwartz function in Calderón's reproducing formula, and $Q_t f = \phi_t * f$. Suppose that $\psi \in \mathcal{S}$ is such that $\hat{\psi} = \hat{\tau}\hat{\phi}$ near origin for some $\hat{\tau} \in C^\infty(\mathbb{R}^n)$. Then for all $0 < q < \infty$ and every integer $N > 0$, there is a constant C depending only on ϕ, ψ, q and N such that for all $f \in \mathcal{S}'$ and $(x,t) \in \mathbb{R}^{n+1}_+$, we have*

$$\begin{aligned} |(Q_t^\psi f)(x)|^q \leq C \int_0^t \int_{\mathbb{R}^n} \frac{|Q_s f(y)|^q}{(1+|x-y|/s)^{Nq}} \Big(\frac{s}{t}\Big)^{Nq} s^{-n} \, dy \frac{ds}{s} \\ + C \int_{\mathbb{R}^n} \frac{|Q_t f(y)|^q}{(1+|x-y|/t)^{Nq}} t^{-n} \, dy. \end{aligned} \tag{4.1}$$

The next proposition is a continuous version of Peetre's characterization of Triebel-Lizorkin spaces (see [P]). In the rest of section we shall fixed ℓ to be the positive integer in the definition of $\dot{F}_p^{\alpha,q}(\omega)$.

PROPOSITION 4.1. *Let $\alpha \in \mathbb{R}$ with $|\alpha| < \ell + 1$, $0 < p, q < \infty$, and $0 < r < \min\{p, q\}$. Suppose that $\omega \in \mathcal{A}_{p/r}$, then*

$$\|M_r^*(Qf)\|_{L_p^{\alpha,q}(\omega)} \leq C\|f\|_{\dot{F}_p^{\alpha,q}(\omega)}, \tag{4.2}$$

for all $f \in \dot{F}_p^{\alpha,q}(\omega)$, where $M_r^*(Q_t f)$ is defined by

$$M_r^*(Q_t f)(x) = \sup_{y\in\mathbb{R}^n} \frac{|Q_t^\varphi f(y)|}{(1+|x-y|/t)^{n/r}}, \tag{4.3}$$

and the norm $\|\cdot\|_{L_p^{\alpha,q}(\omega)}$ is defined by (2.9).

PROOF. Let $f \in \mathcal{S}'$. We have Peetre's mean value theorem: there exists a constant C independent of f and $x \in \mathbb{R}^n$ so that

$$M_r^*(Q_t f)(x) \leq C\Big(\delta^{-n/r} M\big((Q_t f)^r\big)(x)^{1/r} + \delta M_r^*\big(t\nabla(Q_t f)\big)(x)\Big),$$

for all $x \in \mathbb{R}^n$, $0 < t < \infty$ and $0 < \delta < 1$, where M is the Hardy-Littlewood maximal operator.

We claim that the $L_p^{\alpha,q}(\omega)$-norm of $M_r^*\big(t\nabla(Q_t f)(x)\big)$ is bounded by $C\|M_r^*(Qf)\|_{L_p^{\alpha,q}(\omega)}$, for some constant C independent of f.

Taking this claim momentarily for granted, we then have

$$\|M_r^*(Qf)\|_{L_p^{\alpha,q}(\omega)} \leq C\Big(\delta^{-n/r}\Big\|\Big(\int_0^\infty M\big((t^{-\alpha}Qf)^r\big)^{q/r}\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\omega)} + \delta\|M_r^*(Qf)\|_{L_p^{\alpha,q}(\omega)}\Big).$$

By a standard approximation argument, we could assume that $\|M_r^*(Qf)\|_{L_p^{\alpha,q}(\omega)}$ is finite. Then, by taking δ sufficiently small, inequality (4.2) follows from the weighted Fefferman-Stein vector-valued inequality for the maximal function, provided $\omega \in \mathcal{A}_{p/r}$.

We now prove the claim made above. Since locally we can write $\widehat{\partial\phi/\partial x_j} = \hat{\tau}\hat{\phi}$ for some $\hat{\tau} \in C(\mathbb{R}^n)$, by Lemma B with $N > |\alpha| + n/r$, we have $|t\nabla(Q_tf)(x)|^q \leq I_t(x) + II_t(x)$, where $I_t(x)$ and $II_t(x)$ are the first and second summand of the right hand side of (4.1), respectively. Thus, $M_r^*\big(t\nabla(Q_tf)\big)(x)^q \leq M_{r/q}^*(I_t)(x) + M_{r/q}^*(II_t)(x)$. Since $N > n/r$,

$$\int_{\mathbb{R}^n} \frac{t^{-n}}{(1+|x-y|/t)^{(N-n/r)q}}\,dy \leq C, \quad \text{for all } (x,t) \in \mathbb{R}_+^{n+1}. \tag{4.4}$$

It then follows immediately that $M_{r/q}^*(II_t)(x) \leq CM_r^*(Q_tf)(x)^q$ with the constant C independent of f and $(x,t) \in \mathbb{R}_+^{n+1}$. Thus, to prove the claim we only need show that

$$\int_0^\infty t^{-\alpha q}M_{r/q}^*(I_t)(x)\frac{dt}{t} \leq C\int_0^\infty \big(t^{-\alpha}M_r^*(Q_tf)(x)\big)^q\frac{dt}{t}, \tag{4.5}$$

for some constant C independent of f and x.

To this end, we first note that, if $x, y, z \in \mathbb{R}^n$ and $s > 0$, then $1 + |x-z|/s \leq 1 + |x-y|/s + |y-z|/s \leq (1+|x-y|/s)(1+|y-z|/s)$, and hence, for $t \geq s$,

$$\frac{1+|x-z|/s}{(1+|x-y|/t)(1+|y-z|/s)} \leq \frac{1+|x-y|/s}{1+|x-y|/t} \leq \Big(\frac{s}{t}\Big)^{-1}.$$

Therefore,

$$\begin{aligned}
&t^{-\alpha q}M_{r/q}^*(I_t)(x)\\
&\quad = C\sup_{y\in\mathbb{R}^n}\frac{t^{-\alpha q}I_t(x)}{(1+|x-y|/t)^{nq/r}}\\
&\quad \leq C\int_0^t \sup_{y\in\mathbb{R}^n}\Big\{\int_{\mathbb{R}^n}\Big(\frac{s^{-\alpha}|Q_sf(z)|}{(1+|x-y|/t)^{n/r}(1+|y-z|/s)^N}\Big)^q dz\Big\}\Big(\frac{s}{t}\Big)^{(N-|\alpha|)q}s^{-n}\frac{ds}{s}\\
&\quad \leq C\int_0^t\big(s^{-\alpha}M_r^*(Q_sf)(x)\big)^q\sup_{y\in\mathbb{R}^n}\Big\{\int_{\mathbb{R}^n}\frac{s^{-n}}{(1+|y-z|/s)^{(N-n/r)q}}\,dz\Big\}\Big(\frac{s}{t}\Big)^{(N-|\alpha|-n/r)q}\frac{ds}{s}\\
&\quad \leq C\int_0^t\big(s^{-\alpha}M_r^*(Q_sf)(x)\big)^q\Big(\frac{s}{t}\Big)^{(N-|\alpha|-n/r)q}\frac{ds}{s},
\end{aligned}$$

the last inequality follows from (4.4). Therefore,

$$\begin{aligned}
\int_0^\infty t^{-\alpha q}M_{r/q}^*(I_t)(x)\frac{dt}{t} &\leq C\int_0^\infty\big(s^{-\alpha}M_r^*(Q_sf)(x)\big)^q\Big(\int_s^\infty\Big(\frac{s}{t}\Big)^{(N-|\alpha|-n/r)q}\frac{dt}{t}\Big)\frac{ds}{s}\\
&\leq C\int_0^\infty\big(s^{-\alpha}M_r^*(Q_sf)(x)\big)^q\frac{ds}{s}.
\end{aligned}$$

This concludes (4.5), and hence the claim. ∎

PROPOSITION 4.2. *Let $\alpha \in \mathbb{R}$ with $|\alpha| < \ell + 1$, $0 < p < \infty$, $0 < q \leq \infty$, and $0 < r < \min\{p, q\}$. Suppose that $\omega \in \mathcal{A}_{p/r}$, then*

$$\|Qf\|_{T_p^{\alpha,q}(\omega)} \approx \|f\|_{\dot{F}_p^{\alpha,q}(\omega)}, \tag{4.6}$$

for all $f \in \dot{F}_p^{\alpha,q}(\omega)$.

PROOF. We first assume that $0 < q < \infty$. Then, for each $x \in \mathbb{R}^n$, we have $|x - y| < t$ for $y \in \Gamma(x)$, and hence

$$\begin{aligned} A^{\alpha,q}(Qf)(x)^q &\leq 2^q \iint_{\Gamma(x)} \left(\frac{t^{-\alpha}|Q_t f(y)|}{(1 + |x - y|/t)^{n/r}} \right)^q \frac{dy\,dt}{t^{n+1}} \\ &\leq 2^q \int_0^\infty \left(t^{-\alpha} M_r^*(Q_t f)(x) \right)^q \frac{dt}{t}. \end{aligned}$$

Therefore, $\|Qf\|_{T_p^{\alpha,q}(\omega)} \leq C\|f\|_{\dot{F}_p^{\alpha,q}(\omega)}$, by Proposition 4.1. Conversely, by virtue of Lemma B, we have the estimate

$$\left(\int_0^\infty \left(t^{-\alpha}|Q_t f(x)| \right)^q \frac{dt}{t} \right)^{1/q} \leq C \left(\iint_{\mathbb{R}^{n+1}_+} \left(\frac{t^{-\alpha}|Q_t f(y)|}{(1 + |x - y|/t)^N} \right)^q \frac{dy\,dt}{t^{n+1}} \right)^{1/q},$$

which is the Littlewood-Paley g_λ^*-function. A standard argument in Littlewood-Paley theory shows that the $L^p(\omega)$ norm of the g_λ^*-function is bounded by the $L^p(\omega)$ norm of $A^{\alpha,q}(Qf)$ (see, *e.g.*, p. 315 of [Tor]), and hence $\|f\|_{\dot{F}_p^{\alpha,q}(\omega)} \leq C\|Qf\|_{T_p^{\alpha,q}(\omega)}$.

Now, if $q = \infty$, we certainly have

$$\sup_{t>0} t^{-\alpha}|Q_t f(x)| \leq A^{\alpha,\infty}(Qf)(x),$$

for all $x \in \mathbb{R}^n$, and hence $\|f\|_{\dot{F}_p^{\alpha,\infty}(\omega)} \leq \|Qf\|_{T_\infty^{\alpha,q}(\omega)}$. To show the converse inequality, we follow the idea in [FS]. For each number $\varepsilon > 0$ and integer k, we define

$$M_{r,\varepsilon}^{\alpha,k}(Qf)(x) = \sup_{2^{-k}|x-y|<t<\varepsilon^{-1}} \left(\frac{t}{t + \varepsilon} \right)^{n/r} \frac{t^{-\alpha}|Q_t f(y)|}{(1 + \varepsilon|x - y|)^{n/r}}. \tag{4.7}$$

We shall write $M_{r,\varepsilon}^{\alpha}(Qf) = M_{r,\varepsilon}^{\alpha,0}(Qf)$. We notice that $M_{r,\varepsilon}^{\alpha}(Qf) \longrightarrow A^{\alpha,\infty}(Qf)$ as $\varepsilon \longrightarrow 0$. We also note that, by Lemma 13 in Chapter IV of [ST], we have

$$|\{x \in \mathbb{R}^n : M_{r,\varepsilon}^{\alpha,k}(Qf)(x) > \lambda\}|_\omega \leq C2^{npk/r}|\{x \in \mathbb{R}^n : M_{r,\varepsilon}^{\alpha}(Qf)(x) > \lambda\}|_\omega,$$

for all $\lambda > 0$, with a constant C independent of ε and k, since $|2^k B|_\omega \leq C2^{npk/r}|B|_\omega$ by the strong doubling property (1.8) of $\omega \in \mathcal{A}_{p/r}$. Therefore, we obtain

$$\int_{\mathbb{R}^n} M_{r,\varepsilon}^{\alpha,k}(Qf)(x)^p \omega(x)\,dx \leq C2^{npk/r} \int_{\mathbb{R}^n} M_{r,\varepsilon}^{\alpha}(Qf)(x)^p \omega(x)\,dx. \tag{4.8}$$

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We also let

$$m^{\alpha}_{r,\varepsilon}(Qf)(x) = \sup_{|x-y|<t<\varepsilon^{-1}} \Big(\frac{t}{t+\varepsilon}\Big)^{n/r} \frac{t^{-\alpha}|t\nabla(Q_t f)(y)|}{(1+\varepsilon|x-y|)^{n/r}}. \tag{4.9}$$

We claim that

$$\|m^{\alpha}_{r,\varepsilon}(Qf)\|_{L^p(\omega)} \le C\|M^{\alpha}_{r,\varepsilon}(Q_t f)\|_{L^p(\omega)}, \tag{4.10}$$

for a constant C independent of f and ε. To see this, we first note that, for $0 < s \le t < \varepsilon^{-1}$, we have

$$\frac{1+\varepsilon|y-z|}{1+\varepsilon|x-y|} \le \frac{1+\varepsilon|x-y|+\varepsilon|x-z|}{1+\varepsilon|x-y|} \le 1+\varepsilon|x-z| \le 1+\frac{|x-z|}{t} \le 1+\frac{|x-z|}{s}.$$

Therefore, by using Lemma B again, we have

$$\begin{aligned}
&\Big(\Big(\frac{t}{t+\varepsilon}\Big)^{n/r} \frac{t^{-\alpha}|t\nabla(Q_t f)(x)|}{(1+\varepsilon|x-y|)^{n/r}}\Big)^p \\
&\quad\le C\int_0^t \int_{\mathbb{R}^n} \Big(\Big(\frac{t}{t+\varepsilon}\Big)^{n/r} \frac{t^{-\alpha}|Q_s f(z)|}{(1+\varepsilon|x-z|)^{n/r}} \Big(\frac{s}{t}\Big)^N\Big)^p \frac{s^{-n}}{(1+|x-z|/s)^{(N-n/r)p}}\, dz\frac{ds}{s} \\
&\qquad + C\int_{\mathbb{R}^n} \Big(\Big(\frac{t}{t+\varepsilon}\Big)^{n/r} \frac{t^{-\alpha}|Q_t f(z)|}{(1+\varepsilon|x-z|)^{n/r}}\Big)^p \frac{t^{-n}}{(1+|x-z|/t)^{(N-n/r)p}}\, dz \\
&\quad\le C\sum_{k=0}^{\infty} M^{\alpha,k}_{r,\varepsilon}(Qf)(x)^p \int_0^t \Big(\frac{s}{t}\Big)^{(N-n/r-|\alpha|)p} \int_{D_k(s)} \frac{s^{-n}}{(1+|x-z|/s)^{(N-n/r)p}}\, dz\frac{ds}{s} \\
&\qquad + C\sum_{k=0}^{\infty} M^{\alpha,k}_{r,\varepsilon}(Qf)(x)^p \int_{D_k(t)} \frac{t^{-n}}{(1+|x-z|/t)^{(N-n/r)p}}\, dz \\
&\quad\le C\sum_{k=0}^{\infty} 2^{-(N-n/r-n/p)pk} M^{\alpha,k}_{r,\varepsilon}(Qf)(x)^p,
\end{aligned}$$

with a constant C independent of x and ε, where $D_0(t) = \{y \in \mathbb{R}^n : |x-y| < t\}$ and $D_k(t) = \{y \in \mathbb{R}^n : 2^{k-1}t \le |x-y| < 2^k t\}$ if $k \ge 1$. Thus, by taking the supremum over $|x-y| < t < \varepsilon^{-1}$ on the left-hand side of last inequality and then taking the integral over $\mathbb{R}^n$ with respect to the measure $\omega(x)\,dx$, we have

$$\begin{aligned}
\int_{\mathbb{R}^n} m^{\alpha}_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx &\le C\sum_{k=0}^{\infty} 2^{-(N-n/r-n/p)pk} \int_{\mathbb{R}^n} M^{\alpha,k}_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx \\
&\le C\Big(\sum_{k=0}^{\infty} 2^{-(N-2n/r-n/p)pk}\Big) \int_{\mathbb{R}^n} M^{\alpha}_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx,
\end{aligned}$$

where we have used (4.8). Thus, (4.10) follows from this, if we choose $N > 2np/r + n$.

We now consider the maximal function

$$M^{\alpha}_{r}(Qf)(x) = \sup_{B\ni x}\Big(\frac{1}{|B|}\int_B \Big(\sup_{t>0} t^{-\alpha}|Q_t f(y)|\Big)^r dy\Big)^{1/r}, \tag{4.11}$$

and the set $G_{r,\varepsilon} = \{x \in \mathbb{R}^n : cm^\alpha_{r,\varepsilon}(Qf)(x) \le M^\alpha_{r,\varepsilon}(Qf)(x)\}$, for some constant $0 < c < 1$ to be chosen. The same proof of Theorem 11 in [FS] (p. 186) yields that $M^\alpha_{r,\varepsilon}(Qf)(x) \le CM^\alpha_r(Qf)(x)$ for all $x \in G_{r,\varepsilon}$. Also, (4.10) implies that

$$\int_{\mathbb{R}^n\setminus G_{r,\varepsilon}} M^\alpha_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx \le c^p \int_{\mathbb{R}^n\setminus G_{r,\varepsilon}} m^\alpha_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx$$
$$\le \frac{1}{2}\int_{\mathbb{R}^n} M^\alpha_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx,$$

if we choose that c sufficiently small. Thus,

$$\int_{\mathbb{R}^n} M^\alpha_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx \le 2\int_{G_{r,\varepsilon}} M^\alpha_{r,\varepsilon}(Qf)(x)^p\,\omega(x)\,dx$$
$$\le C\int_{\mathbb{R}^n} M^\alpha_r(Qf)(x)^p\,\omega(x)\,dx$$
$$\le C\int_{\mathbb{R}^n}\Big(\sup_{t>0} t^{-\alpha}|Q_tf(y)|\Big)^p\omega(x)\,dx,$$

with a constant C independent of ε, the last inequality follows from the weighted inequality for the maximal function. By taking the limit $\varepsilon \longrightarrow 0$, we get $\|Qf\|_{T^{\alpha,\infty}_p(\omega)} \le C\|f\|_{\dot F^{\alpha,\infty}_p(\omega)}$. This concludes the proof of the proposition. ∎

Let $0 < p \le 1$ and $1 < q \le \infty$. We recall that a $T^{\alpha,q}_p(\omega)$-atom is a function $a(x,t)$, supported in $\hat B$ for some ball $B \subset \mathbb{R}^n$, and satisfying

$$\left(\frac{1}{|B|}\iint_{\hat B}\big(t^{-\alpha}|a(x,t)|\big)^q\frac{dx\,dt}{t}\right)^{1/q} \le \frac{1}{|B|^{1/p}_\omega},\quad 1<q<\infty, \tag{4.12}$$

or

$$\sup_{(x,t)\in\mathbb{R}^{n+1}_+} t^{-\alpha}|a(x,t)| \le \frac{1}{|B|^{1/p}_\omega},\quad q=\infty, \tag{4.13}$$

We also define a function, $b(x)$, to be an $\dot F^{\alpha,q}_p(\omega)$-atom if b is supported in some ball $B \subset \mathbb{R}^n$, satisfies the moment conditions $\int x^\gamma b(x)\,dx = 0$ for all $|\gamma| \le \ell$, and the size condition

$$\|b\|_{\dot F^{\alpha,q}_q(\omega)} \le \frac{1}{|B|^{1/p}_\omega}. \tag{4.14}$$

PROPOSITION 4.3. *Let $\alpha \in \mathbb{R}$ with $|\alpha| < \ell+1$, $0<p\le 1$, $1<q\le\infty$, and $0<r<p$. Suppose that $\omega \in \mathcal{A}_{p/r}$, then every element $f \in \dot F^{\alpha,q}_p(\omega)$ can be written as $f = \sum\lambda_j b_j$ in the distribution sense, where b_j are $\dot F^{\alpha,q}_p(\omega)$-atoms, $\lambda_j \in \mathbb{C}$, and $\sum\lambda_j^p \le C\|f\|^p_{\dot F^{\alpha,q}_p(\omega)}$.*

PROOF. The same proof as in [CMS] gives the atomic decomposition of $T^{\alpha,q}_p(\omega)$ (provided ω is doubling), that is, every element $g \in T^{\alpha,q}_p(\omega)$ can be written as $g = \sum\lambda_j a_j$, where a_j are $T^{\alpha,q}_p(\omega)$-atoms, and $\sum\lambda_j^p \le C\|g\|^p_{T^{\alpha,q}_p(\omega)}$. For instance, for the space $T^{0,2}_1(\omega)$, by using the same notation as in the proof of Theorem 1 of [CMS], and defining $\mu =$

$\int_{\Delta_j^k} |f|^2\, dy\, dt/t$, $a_j^k = f\chi_{\Delta_j^k}|B|^{1/2}\mu^{-1/2}|B|_\omega^{-1}$, and $\lambda_j^k = |B|^{-1/2}\mu^{1/2}|B|_\omega$, one can verify that each statement of the proof there is valid if L^1 is replaced by $L^1(\omega)$.

Now, if a is $T_p^{\alpha,q}(\omega)$-atom, we define

$$b(x) = C_0^{-1} \int_0^\infty Q_t a(x,t)\frac{dt}{t} \tag{4.15}$$

for some suitable constant C_0 to be chosen, which is independent of a. We claim that a is an $\dot{F}_p^{\alpha,q}(\omega)$-atom. Indeed, if a is supported in $\hat{B}$ then b is supported in $2B$, since ϕ is supported in the unit ball. The moment conditions readily follow from the moment conditions of ϕ. Finally, by using the same proof as in Section 1, we have

$$C_0\|b\|_{\dot{F}_q^{\alpha,q}(\omega)} \leq C\Big(\frac{1}{|B|}\iint_{\hat{B}}\big(t^{-\alpha}|a(x,t)|\big)^q\frac{dx\,dt}{t}\Big)^{1/q},$$

with the constant C independent of a. We then choose $C_0 = C$.

Now, if $f \in \dot{F}_p^{\alpha,q}(\omega)$, it follows from Proposition 4.2 that $Qf \in T_p^{\alpha,q}(\omega)$, and then $Qf = \sum \lambda_j a_j$, with a_j being $T_p^{\alpha,q}(\omega)$-atoms. Thus, the reproducing property of Q implies that

$$f(x) = \int_0^\infty Q_t(Q_t f)(x)\frac{dt}{t} = C_0\sum \lambda_j b_j(x)$$

in the distribution sense. Furthermore,

$$\|f\|_{\dot{F}_p^{\alpha,q}(\omega)} \leq C\|Qf\|_{T_p^{\alpha,q}(\omega)} \leq C\Big(\sum \lambda_j^p\Big)^{1/p}.$$

This proves the proposition. ■

We remark that $\|f\|_{\dot{F}_\infty^{-\alpha,q'}(\omega)}$ is equivalent to $\|Qf\|_{T_\infty^{-\alpha,q'}(\omega)}$, since the sets $\hat{B}$ and $B\times\big(0,r(B)\big)$ are comparable. Therefore, by Proposition 4.2 and (1.16), we are able to see that the dual space of $\dot{F}_1^{\alpha,q}(\omega)$ is $\dot{F}_\infty^{-\alpha,q'}(\omega)$. Moreover, one can easily prove, by using inequality (1.16) and the Hahn-Banach theorem (see Section 5 of [FJ]), that

$$\|f\|_{\dot{F}_1^{\alpha,q}(\omega)} = \sup\Big\{\int_{\mathbb{R}^n} f(x)g(x)\,dx : g\in \dot{F}_\infty^{-\alpha,q'}(\omega) \text{ with } \|g\|_{\dot{F}_\infty^{-\alpha,q'}(\omega)}\leq 1\Big\}. \tag{4.16}$$

5. **Proof of Theorem 1.2 and Theorem 1.3 (Case of $p \leq 1$ or $p = \infty$).** We are now ready to finish the proof of Theorems 1.2 and 1.3. We note that the endpoint cases for Theorem 1.2, *i.e.*, $p = q = 1$ and $p = q = \infty$, are basically known, since $\dot{F}_p^{\alpha,p}(\omega) = \dot{B}_p^{\alpha,p}(\omega)$. Also, when $p = q = 1$, Theorem 1.3 was proved in Section 3. We now study the cases of $p \leq 1$ or $p = \infty$ and $1 < q \neq p$.

Let S be an ϵ-family of operators with $S(x^\gamma) = 0$ for all $|\gamma| \leq [\epsilon]$. By Lemma 4.2, to prove (1.11) and (1.12) we need to show that

$$\Big\|\Big(\int_0^\infty (t^{-\alpha}|S_t f|)^q\frac{dt}{t}\Big)^{1/q}\Big\|_{L^p(\omega)} \leq C\|Qf\|_{T_p^{\alpha,q}(\omega)}, \tag{5.1}$$

for $f \in \dot{F}_p^{\alpha,q}(\omega)$, $1 < q \le \infty$, $|\alpha| < \epsilon$ and $\omega \in \mathcal{A}_1 \cap \mathcal{RH}_{(q/p)'}$, and that

$$\left\| \sup_{B \ni x} \left(\frac{1}{|B|} \int_B \int_0^{r(B)} (t^{-\alpha} |S_t f|)^q \frac{dt}{t}\, dx \right)^{1/q} \right\|_{L^\infty(\omega)} \le C \|Qf\|_{T_\infty^{\alpha,q}(\omega)}, \tag{5.2}$$

for $f \in \dot{F}_\infty^{\alpha,q}(\omega)$, $1 \le q < \infty$, $|\alpha| < \epsilon$, and any weight function ω. Likewise, to prove (1.17), we need to show that

$$\|Sf\|_{T_p^{\alpha,q}(\omega)} \le C \|Qf\|_{T_p^{\alpha,q}(\omega)}, \tag{5.3}$$

for $f \in \dot{F}_p^{\alpha,q}(\omega)$, $1 < q \le \infty$, $|\alpha| < \epsilon$ and $\omega \in \mathcal{A}_1 \cap \mathcal{RH}_{(q/p)'}$.

PROOF OF (5.1). Let $f \in \dot{F}_p^{\alpha,q}(\omega)$, then, by Proposition 4.2, $Qf \in T_p^{\alpha,q}(\omega)$. It follows there is a $T_p^{\alpha,q}(\omega)$-atomic decomposition $Q_t f(x) = \sum \lambda_j a_j(x,t)$, By the reproducing property, it suffices to show that there is a constant C so that

$$\int_{\mathbb{R}^n} \left(\int_0^\infty \left(t^{-\alpha} \left| \int_0^\infty S_t Q_s a(x,s) \frac{ds}{s} \right| \right)^q \frac{dt}{t} \right)^{p/q} \omega(x)\, dx \le C, \quad 1 < q < \infty \tag{5.4}$$

for all $T_p^{\alpha,q}(\omega)$-atoms a (with a necessary change if $q = \infty$).

To see this, suppose $a(x,t)$ is such an atom, associated with the ball B in $\mathbb{R}^n$ such that $\operatorname{spt}(a) \subset \hat{B}$. If $1 < q < \infty$, by applying Proposition 2.3 to function $a(x,t)$ with $p = q$ and $\omega \equiv 1$, we have

$$\begin{aligned}
\left(\int_{\mathbb{R}^n} \int_0^\infty \left(t^{-\alpha} \int_0^\infty |S_t Q_s a(x,s)| \frac{ds}{s} \right)^q \frac{dt}{t}\, dx \right)^{p/q} & le C \left(\int_{\mathbb{R}^n} \int_0^\infty \left(t^{-\alpha} |a(x,t)| \right)^q \frac{dt}{t}\, dx \right)^{p/q} \\
&= C \left(\iint_{\hat{B}} \left(t^{-\alpha} |a(x,t)| \right)^q \frac{dx\, dt}{t} \right)^{p/q} \\
&\le C \frac{|B|^{p/q}}{|B|_\omega},
\end{aligned}$$

where we have used $\operatorname{spt}(a) \subset \hat{B}$ and (4.12). By Hölder's inequality with the exponent q/p and the inequality above, we get

$$\begin{aligned}
&\int_{2B} \left(\int_0^\infty \left(t^{-\alpha} \left| \int_0^\infty S_t Q_s a(x,s) \frac{ds}{s} \right| \right)^q \frac{dt}{t} \right)^{p/q} \omega(x)\, dx \\
&\le \left(\int_{2B} \int_0^\infty \left(t^{-\alpha} \left| \int_0^\infty S_t Q_s a(x,s) \frac{ds}{s} \right| \right)^q \frac{dt}{t}\, dx \right)^{p/q} \left(\int_{2B} \omega(x)^{(q/p)'}\, dx \right)^{1/(q/p)'} \\
&\le \frac{|2B|_\omega}{|2B|^{p/q}} \left(\int_{\mathbb{R}^n} \int_0^\infty \left(t^{-\alpha} \int_0^\infty |S_t Q_s a(x,s)| \frac{ds}{s} \right)^q \frac{dt}{t}\, dx \right)^{p/q} \\
&\le C \left(\frac{|B|}{|2B|} \right)^{p/q} \frac{|2B|_\omega}{|B|_\omega} \le C,
\end{aligned}$$

where we have used the assumption that $\omega \in \mathcal{RH}_{(q/p)'}$ and the fact that $\mathcal{A}_1$ weights are doubling.

For $q=\infty$, we use (4.13) and obtain

$$\int_{2B}\Big(\sup_{t>0} t^{-\alpha}\Big|\int_0^\infty S_tQ_sa(x,s)\frac{ds}{s}\Big|\Big)^p\omega(x)\,dx$$
$$\le \int_{2B}\Big(\sup_{t>0} t^{-\alpha}\int_0^\infty\int_{\mathbb{R}^n}|K_{t,s}(x,y)|\,|a(y,s)|\,dy\frac{ds}{s}\Big)^p\omega(x)\,dx$$
$$\le \frac{1}{|B|_\omega}\int_{2B}\Big(\sup_{t>0}\int_0^\infty\Big(\frac{t}{s}\Big)^{-\alpha}\int_{\mathbb{R}^n}|K_{t,s}(x,y)|\,dy\frac{ds}{s}\Big)^p\omega(x)\,dx$$
$$\le C\frac{|2B|_\omega}{|B|_\omega}\le C,$$

since, by using Lemma 2.1 with $f\equiv 1$ there, we have the estimate

$$\int_0^\infty\Big(\frac{t}{s}\Big)^{-\alpha}\int_{\mathbb{R}^n}|K_{t,s}(x,y)|\,dy\frac{ds}{s}\le C\Big(\int_0^t\Big(\frac{s}{t}\Big)^{\epsilon+\alpha}\frac{ds}{s}+\int_t^\infty\Big(\frac{t}{s}\Big)^{\epsilon-\alpha}\frac{ds}{s}\Big)\le C.$$

To estimate the integral over $\mathbb{R}^n\setminus 2B$, we first notice that, by Lemma 2.1, we have

$$\Big(\int_0^\infty\Big(\frac{t}{s}\Big)^{-\alpha q}|K_{t,s}(x,y)|^q\frac{dt}{t}\Big)^{p/q}\le C\frac{s^\delta}{(s+|x-y|)^{n+\delta}},\quad 1<q<\infty, \tag{5.5}$$

and

$$\sup_{t>0}\Big(\frac{t}{s}\Big)^{-\alpha}|K_{t,s}(x,y)|\le C\frac{s^\delta}{(s+|x-y|)^{n+\delta}},\quad q=\infty, \tag{5.6}$$

for any δ with $0<\delta<\min\{\epsilon,\epsilon+\alpha\}$. In fact, (5.5) is easy to see by breaking the integral into two pieces at s and using the estimates (2.4) and (2.5). A similar approach yields (5.6). Moreover,

$$\int_{\mathbb{R}^n\setminus 2B}\Big(\iint_{\hat B}\Big(\frac{s^\delta}{(s+|x-y|)^{n+\delta}}\Big)^{q'}\frac{dy\,ds}{s}\Big)^{p/q'}\omega(x)\,dx$$
$$\le\sum_{k=1}^\infty\int_{2^{k+1}B\setminus 2^kB}\Big(\iint_{\hat B}\Big(\frac{s^\delta}{(s+|x-y|)^{n+\delta}}\Big)^{q'}\frac{dy\,ds}{s}\Big)^{p/q'}\omega(x)\,dx$$
$$\le\sum_{k=1}^\infty\int_{2^{k+1}B}\frac{|B|^{p/q'}}{\big(2^kr(B)\big)^{(n+\delta)p}}\Big(\int_0^{r(B)}s^{\delta q'}\frac{ds}{s}\Big)^{p/q'}\omega(x)\,dx$$
$$\le\sum_{k=1}^\infty\Big(\frac{|B|^{1/q'}r(B)^\delta}{\big(2^kr(B)\big)^{n+\delta}}\Big)^p|2^{k+1}B|_\omega$$
$$\le C\frac{|B|_\omega}{|B|^{p/q}}\sum_{k=1}^\infty\frac{2^{(k+1)n}}{2^{k(n+\delta)p}}\le C\frac{|B|_\omega}{|B|^{p/q}},$$

where we have used the strong doubling property of $\mathcal{A}_1$ weights, *i.e.*

$$|2^{k+1}B|_\omega\le C2^{(k+1)n}|B|_\omega,$$

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and $(n+\delta)p > n$ which follows from the assumption that

$$p > \max\left\{\frac{n}{n+\epsilon}, \frac{n}{n+\epsilon+\alpha}\right\},$$

and the choice of δ so that it is less than but close enough to $\min\{\epsilon, \epsilon+\alpha\}$.

If $1 < q < \infty$, we use Minkowski's inequality, the last two estimates, and the fact of $\operatorname{spt}(a) \subset \hat{B}$, and then we have

$$\begin{aligned}
&\int_{\mathbb{R}^n\setminus 2B}\left(\int_0^\infty\left(t^{-\alpha}\left|\int_0^\infty S_tQ_sa(x,s)\frac{ds}{s}\right|\right)^q\frac{dt}{t}\right)^{p/q}\omega(x)\,dx\\
&\quad\le \int_{\mathbb{R}^n\setminus 2B}\left(\iint_{\mathbb{R}^{n+1}_+}\left(\int_0^\infty\left(\frac{t}{s}\right)^{-\alpha q}|K_{t,s}(x,y)|^q\frac{dt}{t}\right)^{1/q}s^{-\alpha}|a(y,s)|\frac{dy\,ds}{s}\right)^p\omega(x)\,dx\\
&\quad\le C\int_{\mathbb{R}^n\setminus 2B}\left(\iint_{\hat{B}}\left(\frac{s^\delta}{(s+|x-y|)^{n+\delta}}\right)^{q'}\frac{dy\,ds}{s}\right)^{p/q'}\omega(x)\,dx\\
&\qquad\times\left(\iint_{\hat{B}}\left(s^{-\alpha}|a(y,s)|\right)^q\frac{dy\,ds}{s}\right)^{p/q}\\
&\quad\le C\frac{|B|_\omega}{|B|^{p/q}}\left(\iint_{\hat{B}}\left(s^{-\alpha}|a(y,s)|\right)^q\frac{dy\,ds}{s}\right)^{p/q}\le C.
\end{aligned}$$

A similar estimate holds for $q=\infty$, if we pull out the sup-norm of $s^{-\alpha}|a(y,s)|$ instead of using Minkowski's inequality. Thus, the proof of (5.4) is completed. ∎

PROOF OF (5.2). Let $x_0 \in \mathbb{R}^n$ be given, and B be a ball in $\mathbb{R}^n$ which contains x_0. We first apply Proposition 2.3 to the function $g(y,s) = Q_sf(y)\chi_{\widehat{4B}}(y,s)$, and we have

$$\begin{aligned}
\left(\int_B\int_0^{r(B)}\left(t^{-\alpha}\int_0^\infty|S_tQ_sg(x,s)|\frac{ds}{s}\right)^q\frac{dt}{t}\,dx\right)^{1/q} &\le C\left(\int_{\mathbb{R}^n}\int_0^\infty\left(t^{-\alpha}|g(x,t)|\right)^q\frac{dt}{t}\,dx\right)^{1/q}\\
&= C\left(\iint_{\widehat{4B}}\left(t^{-\alpha}|Q_tf(x)|\right)^q\frac{dx\,dt}{t}\right)^{1/q}\\
&\le C|B|^{1/q}C^{\alpha,q}f(x_0).
\end{aligned}$$

We then write $\mathbb{R}^{n+1}_+ \setminus \widehat{4B} = \bigcup_{k=2}^\infty \widehat{2^{k+1}B}\setminus\widehat{2^kB}$. We claim that, for a fixed δ with $0 < \delta < \epsilon - |\alpha|$ and each k, we have

$$(5.7)\qquad \left(\iint_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}}\left(\frac{t}{s}\right)^{-\alpha q'}|K_{t,s}(x,y)|^{q'}\frac{dy\,ds}{s}\right)^{1/q'} \le \frac{C}{|2^{k+1}B|^{1/q}}\left(\frac{t}{2^kr(B)}\right)^\delta.$$

Indeed, it is obvious by inspection that $s+|x-y| \ge C^{-1}2^kr(B)$, for all $x\in B$ and

$(y,s)\notin \widehat{2^kB}$, and therefore, by Lemma 2.1, we get

$$\begin{aligned}\iint_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}}&\Big(\frac{t}{s}\Big)^{-\alpha q'}|K_{t,s}(x,y)|^{q'}\frac{dy\,ds}{s}\\ &=\iint_{\substack{\widehat{2^{k+1}B}\setminus\widehat{2^kB}\\ s\le t}}\cdots+\iint_{\substack{\widehat{2^{k+1}B}\setminus\widehat{2^kB}\\ t\le s}}\cdots\\ &\le C\frac{|2^{k+1}B|}{\big(2^kr(B)\big)^{(n+\delta)q'}}\Big(\int_0^t t^{\delta q'}\Big(\frac{s}{t}\Big)^{(\epsilon+\alpha)q'}\frac{ds}{s}+\int_t^\infty s^{\delta q'}\Big(\frac{t}{s}\Big)^{(\epsilon-\alpha)q'}\frac{ds}{s}\Big)\\ &\le C\Big(\frac{t^\delta}{\big(2^kr(B)\big)^\delta|2^{k+1}B|}\Big)^{q'}|2^{k+1}B|.\end{aligned}$$

By using this after applying Hölder inequality, we obtain

$$\begin{aligned}\Big(\int_B\int_0^{r(B)}&\Big(t^{-\alpha}\iint_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}}|K_{t,s}(x,y)|\,|Q_sf(y)|\frac{dy\,ds}{s}\Big)^q\frac{dt}{t}\,dx\Big)^{1/q}\\ &\le\left(\int_B\int_0^{r(B)}\Big(\iint_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}}\Big(\frac{t}{s}\Big)^{-\alpha q'}|K_{t,s}(x,y)|^{q'}\frac{dy\,ds}{s}\Big)^{q/q'}\frac{dt}{t}\,dx\right)^{1/q}\\ &\quad\times\Big(\iint_{\widehat{2^{k+1}B}}\big(s^{-\alpha}|Q_sf(y)|\big)^q\frac{dy\,ds}{s}\Big)^{1/q}\\ &\le\frac{C}{2^{\delta k}r(B)^\delta}\Big(\int_0^{r(B)}t^{\delta q}\frac{dt}{t}\Big)^{1/q}\Big(\frac{|B|}{|2^{k+1}B|}\iint_{\widehat{2^{k+1}B}}\big(s^{-\alpha}|Q_sf(y)|\big)^q\frac{dy\,ds}{s}\Big)^{1/q}\\ &\le\frac{C}{2^{\delta k}}\Big(\frac{|B|}{|2^{k+1}B|}\iint_{\widehat{2^{k+1}B}}\big(s^{-\alpha}|Q_sf(y)|\big)^q\frac{dy\,ds}{s}\Big)^{1/q}.\end{aligned}$$

Finally, by Minkowski's inequality, we get

$$\begin{aligned}\Big(\frac{1}{|B|}\int_B\int_0^{r(B)}&\Big(t^{-\alpha}\iint_{\mathbb{R}^{n+1}_+\setminus\widehat{4B}}|K_{t,s}(x,y)|\,|Q_sf(y)|\frac{dy\,ds}{s}\Big)^q\frac{dt}{t}\,dx\Big)^{1/q}\\ &\le\sum_{k=2}^\infty\Big(\frac{1}{|B|}\int_B\int_0^{r(B)}\Big(t^{-\alpha}\iint_{\widehat{2^{k+1}B}\setminus\widehat{2^kB}}|K_{t,s}(x,y)|\,|Q_sf(y)|\frac{dy\,ds}{s}\Big)^q\frac{dt}{t}\,dx\Big)^{1/q}\\ &\le C\sum_{k=2}^\infty\frac{1}{2^{\delta k}}\Big(\frac{1}{|2^{k+1}B|}\iint_{\widehat{2^{k+1}B}}\big(s^{-\alpha}|Q_sf(y)|\big)^q\frac{dy\,ds}{s}\Big)^{1/q}\\ &\le C\Big(\sum_{k=2}^\infty\frac{1}{2^{\delta k}}\Big)C^{\alpha,q}f(x_0).\end{aligned}$$

As a result, we have shown that, for each $x_0\in\mathbb{R}^n$, there is a constant C, independent of x_0, so that

$$\Big(\frac{1}{|B|}\int_B\int_0^{r(B)}\big(t^{-\alpha}|S_tf(x)|\big)^q\frac{ds}{s}\,dx\Big)^{1/q}\le CC^{\alpha,q}(Qf)(x_0),$$

for all ball B containing x_0. This yields (5.2). ■

PROOF OF (5.3). The proof is very similar to the proof of (5.1). We only need to show that

$$\Big\|\int_0^\infty S_{(\cdot)}Q_s a(\cdot,s)\frac{ds}{s}\Big\|_{T_p^{\alpha,q}(\omega)} \le C, \quad 1<q\le\infty \tag{5.8}$$

for all $T_p^{\alpha,q}(\omega)$-atoms a.

Suppose $a(x,t)$ is a given $T_p^{\alpha,q}(\omega)$-atom, associated with the ball B in $\mathbb{R}^n$ such that $\operatorname{spt}(a)\subset\hat{B}$. If $1<q<\infty$, the same proof as in Section 3 by using Theorem A, we have

$$\begin{aligned}\Big\|\int_0^\infty S_{(\cdot)}Q_s a(\cdot,s)\frac{ds}{s}\Big\|_{T_q^{\alpha,q}(\omega)} &\le C\Big(\int_{\mathbb{R}^n}\int_0^\infty \big(t^{-\alpha}|a(x,t)|\big)^q\frac{dt}{t}\,dx\Big)^{p/q}\\ &\le C\Big(\iint_{\hat{B}}\big(t^{-\alpha}|a(x,t)|\big)^q\frac{dx\,dt}{t}\Big)^{p/q}\\ &\le C\frac{|B|^{p/q}}{|B|_\omega},\end{aligned}$$

since $\operatorname{spt}(a)\subset\hat{B}$. As before, Hölder's inequality and the inequality above imply that

$$\begin{aligned}&\int_{2B}\Big(\int_{\Gamma(x)}\Big(t^{-\alpha}\Big|\int_0^\infty S_tQ_s a(y,s)\frac{ds}{s}\Big|\Big)^q\frac{dy\,dt}{t^{n+1}}\Big)^{p/q}\omega(x)\,dx\\ &\le \Big\|\int_0^\infty S_{(\cdot)}Q_s a(\cdot,s)\frac{ds}{s}\Big\|_{T_q^{\alpha,q}(\omega)}\Big(\int_{2B}\omega(x)^{(q/p)'}\,dx\Big)^{1/(q/p)'}\\ &\le C\Big(\frac{|B|}{|2B|}\Big)^{p/q}\frac{|2B|_\omega}{|B|_\omega}\le C.\end{aligned}$$

A similar approach, as in the proof of (5.1), yields the same estimate for $q=\infty$.

Finally, by using Lemma 2.1 and the definition of $\Gamma(x)$, we have the kernel estimates:

$$\Big(\int_{\Gamma(x)}\Big(\frac{t}{s}\Big)^{-\alpha q}|K_{t,s}(y,z)|^q\frac{dy\,dt}{t^{n+1}}\Big)^{p/q}\le C\frac{s^\delta}{(s+|x-z|)^{n+\delta}},\quad 1<q<\infty, \tag{5.9}$$

and

$$\sup_{(y,t)\in\Gamma(x)}\Big(\frac{t}{s}\Big)^{-\alpha}|K_{t,s}(y,z)|\le C\frac{s^\delta}{(s+|x-z|)^{n+\delta}},\quad q=\infty, \tag{5.10}$$

for any $0<\delta<\min\{\epsilon,\epsilon+\alpha\}$. Again, a similar argument as before involving Minkowski's inequality gives us (5.8). ∎

Finally, the proof of (1.18) is similar as the one for (5.1). By using the atomic decomposition of $g\in T_p^{\alpha,q}(\omega)$, we only need to show that

$$\int_{\mathbb{R}^n}\Big(\int_0^\infty\Big(s^{-\alpha}\Big|\int_0^\infty Q_sS_t^* a(x,t)\frac{dt}{t}\Big|\Big)^q\frac{ds}{s}\Big)^{p/q}\omega(x)\,dx\le C,\quad 1<q<\infty \tag{5.11}$$

for all $T_p^{\alpha,q}(\omega)$-atoms a (with a necessary change if $q=\infty$). Since the kernel of $Q_sS_t^*$ satisfies the same estimates (2.4) and (2.5) as the kernel of S_tQ_s, the proof of (5.11) is the same as the one for (5.4).

6. **Some applications.** In this section, we give two applications of the results (and their proofs) stated in Section 1. First, we show the lift property of Triebel-Lizorkin (see [Tr]) in our settings, which asserts that the fractional operator I_β is an isomorphic mapping from $\dot{F}_p^{\alpha,q}(\omega)$ onto $\dot{F}_p^{\alpha+\beta,q}(\omega)$. The operator I_β is defined by means of its Fourier transform, $(\widehat{I_\beta f})(\xi) = |\xi|^{-\beta}\hat{f}(\xi)$ for $f \in \mathcal{S}'/\mathcal{P}$. It is well-known that I_β is the fractional integral (or the Riesz potential) of order β when $\beta > 0$, and is the fractional derivative of order β when $\beta < 0$. By checking Fourier transforms, we have the following representation formula of I_β: there exists radial function $\eta \in \mathcal{S}$ so that $\hat{\eta} \in C^\infty$ with compact support and vanishing near the origin, and that

$$I_\beta f(x) = c\int_0^\infty t^\beta Q_t^\eta f(x)\frac{dt}{t}, \quad \text{for } f \in \mathcal{S}'/\mathcal{P}, \tag{6.1}$$

holds in the distribution sense.

THEOREM 6.1. *Suppose Hypothesis 1.1, and for $\beta \in \mathbb{R}$ with $|\beta| < \epsilon$, define*

$$T_\beta f(x) = \int_0^\infty t^\beta S_t^* f(x)\frac{dt}{t}, \quad \textit{for } f \in \mathcal{S}'/\mathcal{P}. \tag{6.2}$$

Then the operator T_β is bounded from $\dot{F}_p^{\alpha,q}(\omega)$ to $\dot{F}_p^{\alpha+\beta,q}(\omega)$ for $|\alpha| < \epsilon - |\beta|$. In particular, the fractional operator I_β is bounded from $\dot{F}_p^{\alpha,q}(\omega)$ to $\dot{F}_p^{\alpha+\beta,q}(\omega)$ for α and $\beta \in \mathbb{R}$.

PROOF. The second statement follows from the first and the representation formula (6.1) of I_β. To proof the first statement, we need to show that $\|T_\beta f\|_{\dot{F}_p^{\alpha+\beta,q}(\omega)} \le C\|f\|_{\dot{F}_p^{\alpha,q}(\omega)}$. By Theorem 1.2, it is enough to prove that $\{s^{-\beta}Q_sT_\beta\}$ is a δ-family of operators with $|\alpha| < \delta < \epsilon - |\beta|$. Let $K_t(x,y)$ be the kernel of S_t, then the kernel of $s^{-\beta}Q_sT_\beta$ is given by

$$\tilde{K}_t(x,y) = \int_0^\infty \left(\frac{t}{s}\right)^\beta K_{t,s}^*(x,y)\frac{dt}{t},$$

where $K_{t,s}^*(x,y)$ is given by (2.3). It is not hard to check that estimates similar to (2.4) and (2.5) in Lemma 2.1 and (2.10) and (2.11) in Lemma 2.4 are valid for the kernel $\tilde{K}_{t,s}(x,y)$ defined above. It then follows that kernel $\tilde{K}_s(x,y)$ satisfies the kernel estimates (1.1) and (1.2) with the smoothness parameter $\epsilon = \delta$ there. We leave the details to the reader. ∎

We next consider a class of generalized Calderón-Zygmund operators. Recently, there has been a great deal of work on the boundedness of Calderón-Zygmund operators in Besov and Triebel-Lizorkin spaces. In particular, Han and Sawyer [HS], Han, Jawerth, Taibleson and Weiss [HJTW], Han and Hofmann [HH], and Torres [T] were able to show that Calderón-Zygmund operators are bounded on certain Triebel-Lizorkin spaces under various weakened or generalized smoothness conditions. In this section, we prove a sharp version of their results.

Let T be a continuous linear operator from the Schwartz space of compactly supported test functions $\mathcal{D}$ to its dual space $\mathcal{D}'$ associated to a kernel $K(x,y)$ in the sense that

$$(Tf)(g) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} g(x)K(x,y)f(y)\,dy\,dx, \tag{6.3}$$

where $f, g \in \mathcal{D}$ have disjoint supports. We consider the following generalized Calderón-Zygmund conditions for the kernel $K(x,y)$ (see [T] and [HS]): For some $\beta \in \mathbb{R}$ and $\epsilon > 0$,

$$|K(x,y)| \leq C\frac{1}{|x-y|^{n+\beta}}, \quad \text{for } x \neq y, \tag{6.4}$$

and

$$|K(x,y) - P^{[\epsilon]}_{x-z}K(z,y)| \leq C\frac{|x-z|^{\epsilon}}{|x-y|^{n+\beta+\epsilon}}, \quad \text{for } 2|x-z| \leq |x-y|. \tag{6.5}$$

For the simplicity, we shall assume that ϵ is not an integer.

We also recall (see [T]) that a linear and continuous operator $T\colon \mathcal{D} \longrightarrow \mathcal{D}'$ is said to satisfy the weak boundedness property (WBP) of order β if

$$|(T\varphi)(\psi)| \leq Cr^{n-\beta}(\|\varphi\|_{L^\infty} + r\|\nabla\varphi\|_{L^\infty})(\|\psi\|_{L^\infty} + r\|\nabla\psi\|_{L^\infty}), \tag{6.6}$$

for all φ and $\psi \in \mathcal{D}$ with supports in a ball of radius r.

The following theorem is not completely new, of course. But, we believe that some combinations of parameters covered in the theorem are missing in the literature.

THEOREM 6.2. *Let $\epsilon > 0$ (which is assumed not to be an integer) and $-\min\{\epsilon, n\} < \beta < 1 + [\epsilon] - \epsilon$. Suppose that $\max\{\beta, 0\} < \alpha < \beta + \epsilon$,*

$$p > \max\left\{\frac{n}{n+\epsilon+\beta}, \frac{n}{n+\alpha-\beta}, \frac{n}{n+\alpha}\right\},$$

and $1 \leq q \leq \infty$, and that ω satisfies condition (3) of Hypothesis 1.1. Then the operator T, which is associated with a kernel function $K(x,y)$ satisfying conditions (6.4), (6.5) and (6.6), is bounded from $\dot{F}^{\alpha,q}_p(\omega)$ to $\dot{F}^{\alpha-\beta,q}_p(\omega)$ provided $T(x^\gamma) = 0$ for all $|\gamma| \leq [\epsilon]$.

Again, the proof is similar to the one for Theorem 1.2, except we shall use the following kernel estimates, which is similar to a lemma given in [HS].

LEMMA 6.3. *Suppose that the kernel $K(x,y)$ satisfies the hypothesis of Theorem 6.2. Let $K_{t,s}(x,y)$ be the kernels associated with the operators $t^\beta Q_t T Q_s$, then*

$$|K_{t,s}(x,y)| \leq C\left(\frac{s}{t}\right)^{-\beta}\left(1 + \log\frac{t}{s}\right)\frac{t^{\beta+\epsilon}}{(t+|x-y|)^{n+\beta+\epsilon}} \quad \text{for } 0 < s \leq t, \tag{6.7}$$

if $\beta \geq 0$,

$$|K_{t,s}(x,y)| \leq C\frac{t^{\beta+\epsilon}}{(t+|x-y|)^{n+\beta+\epsilon}} \quad \text{for } 0 < s \leq t. \tag{6.8}$$

if $\beta < 0$, and

$$|K_{t,s}(x,y)| \leq C\left(\frac{t}{s}\right)^{\beta+\epsilon}\frac{s^{\beta+\epsilon}}{(s+|x-y|)^{n+\beta+\epsilon}} \quad \text{for } 0 < t \leq s. \tag{6.9}$$

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