2

One-Parameter Families

In Kollár (2013b) we studied in detail canonical and semi-log-canonical varieties, especially their singularities; a summary of the main results is given in Section 11.1. These are the objects that correspond to the points in a moduli functor/stack of canonical and semi-log-canonical varieties. We start the study of the general moduli problem with one-parameter families.

In traditional moduli theory – for instance, for curves, smooth varieties or sheaves – the description of all families over one-dimensional regular schemes pretty much completes the story: the definitions and theorems have obvious generalizations to families over an arbitrary base. The best examples are the valuative criteria of separatedness and properness; we discussed these in (1.20). In our case, however, much remains to be done in order to work over arbitrary base schemes.

Two notions of locally stable or semi-log-canonical families are introduced in Section 2.1; their equivalence is proved in characteristic 0. For surfaces, one can give a rather complete étale-local description of all locally stable families; this is worked out in Section 2.2.

A series of higher dimensional examples is presented in Section 2.3. These show that stable degenerations of smooth projective varieties can get rather complicated.

Next we turn to global questions and define our main objects, stable families, in Section 2.4. The main result says that stable families satisfy the valuative criteria of separatedness and properness.

Cohomological properties of stable families are studied in Section 2.5. In particular, we show that in a proper, locally stable family $f: X \to C$, the basic numerical invariants $h^i(X_c, \mathscr{O}_{X_c})$ and $h^i(X_c, \omega_{X_c})$ are independent of $c \in C$. We also show that X_c being Cohen–Macaulay (10.4) is also independent of $c \in C$.

In the next two sections, we turn to a key problem of the theory: understanding the difference between the divisor-theoretic and the scheme-theoretic

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restriction of divisors, equivalently, the role of embedded points. The general theory is outlined in Section 2.6. Then in Section 2.7 we show that if all the coefficients of the boundary divisor are $> \frac{1}{2}$, then we need not worry about embedded points in moduli questions.

Checking local stability is easier in codimension ≥ 3, we discuss this and its relation to Grothendieck–Lefschetz-type theorems in Section 2.8.

From now on we use many definitions and results about log canonical and semi-log canonical pairs as in Kollár (2013b). The most important ones are summarized in Section 11.1.

Assumptions The basic definitions in Section 2.1 are formulated for schemes. In the rest of Sections 2.1–2.5 and 2.7, we work in characteristic 0, unless a more general set-up is specified.

In Section 2.6 we work with arbitrary Noetherian schemes.

2.1 Locally Stable Families

Following the pattern established in Section 1.4, we expect that the definition of a stable family $f\colon (X, \Delta) \to S$ consists of some local conditions describing the singularities of f, and a global condition, that $K_{X/S} + \Delta$ be f-ample. We are now ready to formulate the correct local condition, at least for one-parameter families.

Note on $\mathbb{R}$-divisors From now on, we state definitions and results for $\mathbb{R}$-divisors, which seems the natural level of generality; see Section 11.4 for a detailed treatment. However, there will be no major differences in the proofs between $\mathbb{Q}$- and $\mathbb{R}$-divisors until Chapter 6.

We already defined stable varieties in (1.41). The basic objects of our moduli theory are their generalizations.

Definition 2.1 (Stable and locally stable pairs) A *locally stable pair* (X, Δ) over a field k consists of a pure dimensional, geometrically reduced k-scheme X and an effective $\mathbb{R}$-divisor Δ such that (X, Δ) has semi-log-canonical (abbreviated as slc) singularities (11.37).

(X, Δ) is a *stable pair* if, in addition, X is proper and $K_X + \Delta$ is an ample $\mathbb{R}$-Cartier divisor (11.51). Thus a locally stable pair is the same as an slc pair; we usually use the former terminology for fibers of families.

If $\Delta = 0$, we have a *stable variety* as in (1.41).

Definition 2.2 Let C be a regular one-dimensional scheme. A *family of varieties* over C is a flat morphism of finite type $f\colon X \to C$, whose fibers are pure dimensional and geometrically reduced. We also call this a *one-parameter family*. For $c \in C$, let $X_c := f^{-1}(c)$ denote the fiber of f over c.

A *family of pairs* over C is a family of varieties $f\colon X \to C$ plus an effective Mumford $\mathbb{R}$-divisor Δ (p.xv) on X, That is, for every $c \in C$, the support of Δ does not contain any irreducible component of X_c and none of the irreducible components of $X_c \ \cap\ \operatorname{Supp}\Delta$ is contained in $\operatorname{Sing} X_c$; see (4.16.4) and Section 4.8 for details. This condition holds if the fibers are slc pairs. It turns out to be technically crucial, so it is much easier to assume it from the beginning.

The assumptions imply that X is regular at the generic points of $X_c \cap \operatorname{Supp}\Delta$. We can thus define Δ_c as the closure of the restriction of Δ to $X_c \setminus \operatorname{Sing} X_c$.

Warning For non-Cartier divisors, the divisor-theoretic restriction is a divisor, but the scheme-theoretic restriction $\Delta \cap X_c$ may have extra embedded points. This becomes quite important starting from Section 2.6.

Our main interest is in families with demi-normal (11.36) fibers, but we also want to understand to what extent this follows from other assumptions. However, we do not wish to get bogged down in technicalities, so we almost always assume the following conditions, both of which hold if the fibers are demi-normal.

(2.2.1) X satisfies Serre's condition S_2. Since the fibers are assumed reduced, X is S_2 iff the generic fiber X_g is S_2.

(2.2.2) The canonical sheaf ω_{X_c} of the fiber X_c is locally free at codimension 1 points for every $c \in C$. Equivalently, the relative canonical sheaf $\omega_{X/C}$ (2.5) is locally free at codimension 1 points of X_c. Thus the relative canonical class exists; we denote it by $K_{X/C}$ (2.5).

We can now define local stability for one-parameter families in characteristic 0. (We define stable families in (2.46).)

Definition–Theorem 2.3 Let C be a one-dimensional, regular scheme over a field of characteristic 0 and $f\colon (X, \Delta) \to C$ a family of pairs satisfying (2.2.1–2). We say that f is *locally stable* or *semi-log-canonical* at a point $p \in X_c$, if the following equivalent conditions hold:

(2.3.1) $K_{X/C} + \Delta$ is $\mathbb{R}$-Cartier at p and (X_c, Δ_c) is slc at p.

(2.3.2) $K_{X/C} + \Delta$ is $\mathbb{R}$-Cartier at p and $(\bar{X}_c, \operatorname{Diff}_{\bar{X}_c}(\Delta))$ (11.14) is log canonical at $\pi^{-1}(p)$, where $\pi\colon \bar{X}_c \to X_c$ denotes the normalization.

(2.3.3) $(X, X_c + \Delta)$ is slc at p.

(2.3.4) There is an open neighborhood $p \in X^\circ \subset X$ such that $(X, X_{f(q)} + \Delta)$ is slc at q for every $q \in X^\circ$.

Proof If (2) holds, then inversion of adjunction (11.17) shows that $(X, X_c + \Delta)$ is slc at p. The converse also holds since (11.17) works both ways. Thus (2) ⇔ (3) and Kollár (2013b, 4.10) shows that (3) ⇔ (4).

Since X_c is a Cartier divisor in X, the restriction Δ_c equals the different $\mathrm{Diff}_{X_c}(\Delta)$ by (11.15). Furthermore, by (11.14.5)

$$K_{\bar{X}_c} + \mathrm{Diff}_{\bar{X}_c}(\Delta) = \pi^*(K_{X_c} + \mathrm{Diff}_{X_c}(\Delta)).$$

Thus (11.37) shows that (1) ⇒ (2). Note that (11.37) is an equivalence, but in order to apply it we need to know that X_c is demi-normal.

By assumption, X_c is geometrically reduced. A local computation shows that X_c is either smooth or has nodes at codimension 1 points; see Kollár (2013b, 2.33). Thus it remains to prove that X_c is S_2.

This is actually quite subtle. We outline three different approaches, all of which provide valuable insight.

First, if the generic fiber is klt, then, by (2.15), (X, Δ) is klt. Thus X is CM (10.4) by (11.18), so is every fiber X_c. In general, however, (X, Δ) is not klt and X is not CM. However, CM is much more than we need.

We should look carefully at weaker versions of CM that still imply that the fibers are S_2. Since the X_c are Cartier divisors in X, it would be enough to prove that X is S_3. However, as noted in Kollár (2013b, 3.6), X is not S_3 in general. Fortunately this is not a problem for us. If $g \in C$ is the generic point, then a local ring of X_g is also a local ring of X, hence X_g is S_2 if X is S_2. Therefore, (X_g, Δ_g) is slc. If $c \in C$ is a closed point and $p \in X_c$ has codimension ≥ 2, then $p \in X$ has codimension ≥ 3, thus $\mathrm{depth}_p\, \mathscr{O}_X \geq 3$ by (11.21), hence $\mathrm{depth}_p\, \mathscr{O}_{X_c} \geq 2$. Thus again X_c is S_2.

Third, we know that X_c is a Cartier divisor on a demi-normal scheme. A local version of the Enriques–Severi–Zariski lemma (2.93) implies that if $p \in X_c$ is a point of codimension ≥ 2, then $\hat{X}_{c,p} \setminus \{p\}$ is connected, where $\hat{X}_{c,p}$ denotes the completion of X_c at p.

Furthermore, X_c is the union of log canonical centers of $(X, X_c + \Delta)$. Therefore, X_c is seminormal by (11.12.2). These two observations together imply that X_c is S_2, hence demi-normal. □

Comment 2.3.5 For proofs, the versions (2.3.3–4) are the most useful, but it is not clear how they could be generalized to families over higher dimensional bases. By contrast, the variants (2.3.1–2) are harder to use directly, but they make sense in general. This observation leads to the general definition of our moduli functor in Chapters 6–8.

2.4 (Positive characteristic) For arbitrary regular, one-dimensional schemes C, the conditions (2.3.1–4) are equivalent if the relative dimension of X/C is

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1, and are expected to be equivalent if the relative dimension of X/C is 2. However, the examples of Kollár (2022) show that they are **not** equivalent if the relative dimension of X/C is ≥ 3. We discuss this in Section 8.8.

Here we adopt (2.3.4) as the definition of local stability in positive and mixed characteristics. This is dictated by the proof of (2.51), but few of the arguments work in full generality; see (2.15), (2.50), and (2.55).

2.5 (The relative canonical or dualizing sheaf I) Let C be a regular scheme of dimension 1 and $f: X \to C$ a flat morphism of finite type. Then the relative canonical or dualizing sheaf $\omega_{X/C}$ exists; see (2.68) or (11.2) for discussions.

If C is a smooth curve over a field, then $\omega_{X/C} = \omega_X \otimes f^*\omega_C^{-1}$.

If each ω_{X_c} is locally free in codimension 1 (for example, the fibers are normal or demi-normal) then $\omega_{X/C}$ is also locally free in codimension 1 and determines the relative canonical class $K_{X/C}$.

By (11.13), for $c \in C$ there is a Poincaré residue (or adjunction) map

$$\mathcal{R} : \omega_{X/C}|_{X_c} \to \omega_{X_c}. \tag{2.5.1}$$

The map exists for any flat morphism $f: X \to C$. General duality theory implies that it is an isomorphism if the fibers are CM, see (2.68.2). It is, however, not an isomorphism in general, but we prove in (2.67) that, for locally stable morphisms, the adjunction map is an isomorphism. Thus $\omega_{X/C}$ can be thought of as a flat family of the canonical sheaves of the fibers.

The isomorphism in (2.5.1) is easy to prove if the fibers are dlt, or if $K_{X/C}$ is $\mathbb{Q}$-Cartier (2.79.2). For the general case, see Section 2.5.

It is also worth noting that the reflexive powers (3.25) of the residue map

$$\mathcal{R}^m : \omega_{X/C}^{[m]}|_{X_c} \to \omega_{X_c}^{[m]} \tag{2.5.2}$$

are isomorphisms for locally stable maps if $\Delta = 0$, but not in general; see (2.79.2) and (2.44).

In (2.3.1) we make a fiber-wise assumption, that (X_c, Δ_c) be slc, and a total space assumption, that $K_{X/C} + \Delta$ be $\mathbb{R}$-Cartier. As in Section 1.4, usually (2.3.1) cannot be reformulated as a condition about the fibers of f only.

However, if ω_{X_c} is locally free then (2.5.1) implies that $\omega_{X/C}$ is also locally free along X_c. Thus (2.67) and (2.3) imply the following.

Lemma 2.6 *Let C be a smooth curve over a field of characteristic 0 and $f: X \to C$ a flat morphism of finite type such that X_c is slc and ω_{X_c} is locally free for some $c \in C$. Then $\omega_{X/C}$ is locally free along X_c and f is locally stable near X_c.* □

Note that (2.6) is a special property of slc varieties. Analogous claims fail both for normal varieties (2.45) and for pairs (X, D). To see the latter, consider a flat family X_c of smooth quadrics in $\mathbb{P}^3$ becoming a quadric cone for $c = 0$. Let $D_c \subset X_c$ be two disjoint lines that degenerate to a pair of distinct lines on X_0. Then K_{X_c} and D_c are both Cartier divisors for every c, but on the total space X they give a divisor $K_X + D$ that is not even $\mathbb{Q}$-Cartier.

In Section 1.4, we saw families of surfaces with quotient singularities where $K_{X/C}$ is not $\mathbb{R}$-Cartier, but the situation gets better in dimension ≥ 3.

Theorem 2.7 (Kollár, 2013a, Thm.18) *Let C be a smooth curve over a field of characteristic 0 and $f\colon (X, \Delta) \to C$ a family of pairs over C satisfying (2.2.1–2). Let $c \in C$ be a closed point and $Z_c \subset X_c$ a closed subset of codimension ≥ 3. Assume that*

(2.7.1) *f is locally stable along $X_c \setminus Z_c$, and*

(2.7.2) $(\bar{X}_c, \mathrm{Diff}_{\bar{X}_c}(\Delta))$ *(11.14) is log canonical.*

Then f is locally stable along X_c.

Note that $\mathrm{Diff}_{\bar{X}_c}(\Delta)$ is the closure of $\mathrm{Diff}_{\bar{X}_c \setminus \bar{Z}_c}(\Delta)$, which is defined by (2.7.1). We prove this in Section 2.8; see (5.6) for higher dimensonal base spaces.

If X_c is canonical, then K_{X_c} is Cartier in codimension 2. We can thus use (2.6) in codimension 2 and then (2.7) to obtain the next result.

Corollary 2.8 (Families with canonical fibers) *Let C be a smooth curve over a field of characteristic 0 and $f : X \to C$ a flat morphism of finite type such that X_c has canonical singularities for some $c \in C$. Then K_X is $\mathbb{Q}$-Cartier along X_c and f is locally stable near X_c.* □

Next we study permanence properties of local stability. We start with the invariance of local stability for morphisms that are quasi-étale, that is, étale outside a subset of codimension ≥ 2.

Lemma 2.9 *Let C be a smooth curve over a field of characteristic 0 and $f\colon (X, \Delta) \to C$ a family of pairs over C satisfying (2.2.1). Let $\pi : Y \to X$ be quasi-étale, where Y is S_2. If f is locally stable then so is $f \circ \pi$. The converse also holds if π is surjective.*

Proof This follows directly from (2.3) and (11.23.3). □

Note that $\pi_c : Y_c \to X_c$ need not be quasi-étale, but codimension 1 ramification can occur only along the singular locus of X_c. A typical example is given by $\mathbb{A}^2_{xy} \xrightarrow{\pi} \mathbb{A}^2/\frac{1}{n}(1,-1) \xrightarrow{\tau} \mathbb{A}^1$, where $\pi \circ \tau(x,y) = xy$.

Next we consider base changes $C' \to C$.

Proposition 2.10 *Let C be a smooth curve over a field of characteristic 0 and $g\colon C' \to C$ a quasi-finite morphism. If $f\colon (X,\Delta) \to C$ is locally stable, then so is the pull-back*

$$g^* f\colon (X',\Delta') := (X \times_C C', \Delta \times_C C') \to C'.$$

Proof We may assume that $g\colon (c',C') \to (c,C)$ is a finite, local morphism, étale away from c'. Set $D := X_c$ and $D' := X'_{c'}$. By (11.23.5), $(X, D+\Delta)$ is lc iff $(X', D'+\Delta')$ is. The rest follows from (2.3). □

The following is useful for dimension induction.

Lemma 2.11 *Let C be a smooth curve over a field of* char *0 and $f\colon (X, D+\Delta) \to C$ a locally stable morphism, where D is a $\mathbb{Z}$-divisor with normalization $n\colon \bar{D} \to D$. Then $f \circ n\colon (\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) \to C$ is also locally stable.*

Proof For any $c \in C$, the fiber X_c is a Cartier divisor, thus

$$\operatorname{Diff}_{\bar{D}}(\Delta + X_c) = (\operatorname{Diff}_{\bar{D}} \Delta) + X_c|_{\bar{D}} = (\operatorname{Diff}_{\bar{D}} \Delta) + \bar{D}_c.$$

Together with adjunction (11.17), this shows that $f_D\colon (\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta) \to C$ is locally stable. □

Complement 2.11.1 Since $K_{\bar{D}} + \operatorname{Diff}_{\bar{D}} \Delta \sim_{\mathbb{Q}} n^*(K_X + D + \Delta)$ and $\bar{D} \to D$ is finite, if $K_X + D + \Delta$ is f-ample, then $K_{\bar{D}} + \operatorname{Diff}_{\bar{D}}$ is $f \circ n$-ample. Thus if f is stable (2.46), then so is $f \circ n$.

The following result shows that one can usually reduce questions about locally stable families to the special case when X is normal; see also (2.54).

Proposition 2.12 *Let C be a smooth curve over a field of characteristic 0 and $f\colon (X,\Delta) \to C$ a family of pairs over C. Assume that X is demi-normal and let $\pi\colon \bar{X} \to X$ denote the normalization with conductor $\bar{D} \subset \bar{X}$ (11.36).*

(2.12.1) *If $f\colon (X,\Delta) \to C$ is locally stable, then so is $f \circ \pi\colon (\bar{X}, \bar{D} + \bar{\Delta}) \to C$.*

(2.12.2) *If $K_X + \Delta$ is $\mathbb{R}$-Cartier and $f \circ \pi\colon (\bar{X}, \bar{D} + \bar{\Delta}) \to C$ is locally stable, then so is $f\colon (X,\Delta) \to C$.*

Proof Fix a closed point $c \in C$. By (11.38) or Kollár (2013b, 5.38), if $K_X + \Delta$ is $\mathbb{R}$-Cartier, then $(X, X_c + \Delta)$ is slc iff $(\bar{X}, \bar{X}_c + \bar{D} + \bar{\Delta})$ is lc. □

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The next result allows us to pass to hyperplane sections. This is quite useful in proofs that use induction on the dimension. (As with many Bertini-type theorems, the characteristic 0 assumption is essential.)

Proposition 2.13 (Bertini theorem for local stability) *Let C be a smooth curve over a field of* char *0 and $f\colon (X,\Delta)\to C$ a locally stable morphism. Fix a point $c\in C$ and let H be a general divisor in a basepoint-free linear system on X. Then there is an open $c\in C^\circ\subset C$ such that the following morphisms are also locally stable over C°:*

(2.13.1) $f\colon (X, H+\Delta)\to C$,

(2.13.2) $f|_H\colon (H,\Delta|_H)\to C$, *and*

(2.13.3) *the composite $f\circ\pi\colon (Y,\pi^{-1}(\Delta))\to C$ where $\pi\colon Y\to X$ is a μ_m-cover ramified along H; see (11.24).*

Proof As we noted in (2.12), we can assume that X is normal. Let $p\colon Y\to X$ be a log resolution (p.xvi) of (X,Δ) such that $p^{-1}(\operatorname{Supp}\Delta)+\operatorname{Ex}(p)+Y_c$ is an snc divisor. Pick H such that $p^{-1}(H)=p_*^{-1}(H)$ and

$$p^{-1}(H)+p^{-1}(\operatorname{Supp}\Delta)+\operatorname{Ex}(p)+Y_c$$

is an snc divisor. Then every exceptional divisor of p has the same discrepancy with respect to $(X, X_c+\Delta)$ and $(X, X_c+H+\Delta)$. Therefore, $(X, X_c+H+\Delta)$ is slc near X_c. Thus $f\colon (X,H+\Delta)\to C$ is locally stable over some $C^\circ\subset C$, proving (1). By adjunction, this implies that $(H, H_{c'}+\Delta|_H)$ is slc for every $c'\in C^\circ$, proving (2). By (11.23),

$$(Y, Y_{c'}+\pi^{-1}(\Delta))\quad\text{is slc}\quad\Leftrightarrow\quad (X, X_{c'}+(1-\tfrac{1}{m})H+\Delta)\quad\text{is slc.}$$

The latter holds since even $(X, X_{c'}+H+\Delta)$ is slc for every $c'\in C^\circ$. □

2.14 (Inverse Bertini theorem, weak form) Let $H\subset X$ be any Cartier divisor. If $f|_H\colon (H,\Delta|_H)\to C$ is locally stable, then $f\colon (X,H+\Delta)\to S$, and hence also $f\colon (X,\Delta)\to S$, are locally stable in a neighborhood of H by (11.17). Stronger results are in (2.7) and (5.7).

The following simple result shows that if $f\colon (X,\Delta)\to C$ is locally stable, then (X,Δ) behaves as if it were *canonical*, as far as divisors over closed fibers are concerned. In some situations, for instance in (2.50), this is a very useful observation, but at other times the technical problems caused by log canonical centers in the generic fiber are hard to overcome.

Proposition 2.15 *Let $f\colon (X,\Delta)\to C$ be a locally stable morphism. Let E be a divisor over X (p.xv) such that* $\operatorname{center}_X E\subset X_c$ *for some closed point $c\in C$.*

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Then $a(E, X, \Delta) \geq 0$. *Therefore, every log center (11.11) of* (X, Δ) *dominates* C. *In particular, if the generic fiber is klt (resp. canonical) then* (X, Δ) *is also klt (resp. canonical).*

Proof Since $(X, X_c + \Delta)$ is slc, $a(E, X, X_c + \Delta) \geq -1$. Let $\pi\colon Y \to X$ be a proper birational morphism such that E is a divisor on Y and let b_E denote the coefficient of E in $\pi^*(X_c)$. Then b_E is an integer and it is positive since $\operatorname{center}_X E \subset X_c$. Thus,

$$a(E, X, \Delta) = a(E, X, X_c + \Delta) + b_E \geq -1 + b_E \geq 0.$$

In particular, none of the log centers of (X, Δ) are contained in X_c. □

2.16 (Some results in positive characteristic) As we already noted, very few of the previous theorems are known in positive characteristic, but the following partial results are sometimes helpful.

(2.16.1) Let (X, Δ) be a pair and $g\colon Y \to X$ a smooth morphism. By Kollár (2013b, 2.14.2), if (X, Δ) is slc, lc, klt, ... then so is $(Y, g^*\Delta)$.

(2.16.2) As a special case of Kollár (2013b, 2.14.4) we see that if (X, Δ) is slc then, for every smooth curve C, the trivial family $(X, \Delta) \times C \to C$ is locally stable.

(2.16.3) The proof of (2.15) works in any characteristic. Applying this to a trivial family will have useful consequences in (8.64).

(2.16.4) Let (X_i, Δ_i) be two pairs that are slc, lc, klt, ... Then their product $(X_1 \times X_2, X_1 \times \Delta_2 + \Delta_1 \times X_2)$ is also slc, lc, klt, ... This is a generalization of (2.16.2) and can be proved by the same method as in Kollár (2013b, 2.14.2), using Kollár (2013b, 2.22).

(2.16.5) Assume that $f\colon (X, \Delta) \to C$ is locally stable and let $g\colon C' \to C$ be a tamely ramified morphism. Then $g^*f\colon (X \times_C C', \Delta \times_C C') \to C'$ is also locally stable. This follows from (11.23.3) as in (2.10); see Kollár (2013b, 2.42) for details.

(2.16.6) Neither the wildly ramified nor the inseparable case of (2.16.5) is known. By Hu and Zong (2020), the inseparable case would imply the wildly ramified one. The case when all fibers are snc divisors is treated in (2.55).

The dualizing sheaf plays a very special role in algebraic geometry, thus it is natural to focus on understanding the powers of the relative dualizing sheaf. The next result, closely related to Lee and Nakayama (2018, 7.18), says that the relative dualizing sheaf is the "best" deformation of the dualizing sheaf.

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Proposition 2.17 *Let C be a smooth curve over a field of characteristic 0 and $f: X \to C$ a flat morphism of finite type such that X_c is slc for some $c \in C$.*

Let L be a rank 1, reflexive sheaf on X such that a reflexive power $L^{[n]}$ (3.25) is locally free for some $n > 0$ and $L|_{X_c \setminus Z} \simeq \omega_{X_c \setminus Z}$ for some closed subset $Z \subset X_c$ of codimension ≥ 2.

Then there is a line bundle M such that $L \simeq \omega_{X/C} \otimes M$, near X_c.

Proof We may assume that X is local, hence $L^{[n]}$ is free. By (11.24) we can take a cyclic cover $\pi: Y \to X$, giving direct sum decompositions into μ_n-eigensheaves $\pi_* \mathscr{O}_Y = \oplus_{i=0}^{n-1} L^{[-i]}$ and

$$\pi_* \omega_{Y/C} \simeq \mathcal{H}om_X(\pi_* \mathscr{O}_Y, \omega_{X/C}) = \oplus_{i=0}^{n-1} L^{[i]} \,[\otimes]\, \omega_{X/C},$$

where $[\otimes]$ is the reflexive tensor product (3.25.1).

The resulting $g: Y \to C$ is locally stable by (2.9) and ω_{Y_c} is locally free. Therefore, $\omega_{Y/C}$ is locally free by (2.6), hence free since Y is semilocal. Thus $\pi_* \omega_{Y/C} \simeq \pi_* \mathscr{O}_Y$, so one of the summands $L^{[i]} \,[\otimes]\, \omega_{X/C}$ is free. Restriction to X_c tells us that in fact $i = n - 1$. Next note that

$$\omega_{X/C} \simeq \omega_{X/C} \,[\otimes]\, L^{[n-1]} \,[\otimes]\, L \,[\otimes]\, L^{[-n]} \simeq \mathscr{O}_X \otimes L \otimes \mathscr{O}_X \simeq L,$$

where at the end we changed to the usual tensor product, since the tensor product of a reflexive sheaf and of a line bundle is reflexive. □

2.2 Locally Stable Families of Surfaces

In this section, we develop a rather complete local picture of slc families of surfaces. That is, we start with a pointed, local slc pair $(x \in X_0, \Delta_0)$ and aim to describe all locally stable deformations over local schemes $0 \in S$

$$\begin{array}{ccc} (X_0, \Delta_0) & \hookrightarrow & (X_S, \Delta_S) \\ \downarrow & & \downarrow \\ 0 & \in & S. \end{array}$$

In the study of singularities it is natural to work étale-locally.

Definition 2.18 Following Stacks (2022, tag 02LD), an étale morphism $\pi: (s', S') \to (s, S)$ is called *elementary étale* if the induced map on the residue fields $\pi^*: k(s) \to k(s')$ is an isomorphism. (This notion is also called strictly étale or strongly étale in the literature.) The inverse limit of all elementary étale morphisms is the *Henselisation* of (s, S), denoted by $(s^{\mathrm{h}}, S^{\mathrm{h}})$.

The inverse limit of all étale morphisms is the *strict Henselisation* of (s, S), denoted by $(s^{\mathrm{sh}}, S^{\mathrm{sh}})$. See Stacks (2022, tag 0BSK) for details.

For deformation purposes, two pointed schemes $(x_1 \in X_1)$ and $(x_2 \in X_2)$ are considered the “same” if they have isomorphic Henselisations. Equivalently, there is a third pointed scheme $(x_3 \in X_3)$ and elementary étale morphisms

$$(x_1 \in X_1) \stackrel{\pi_1}{\leftarrow} (x_3 \in X_3) \stackrel{\pi_2}{\rightarrow} (x_2 \in X_2).$$

Since we have not yet defined the notion of a locally stable family in general, we concentrate on the case when S is the spectrum of a DVR.

We start by recalling the classification of lc surface singularities. This has a long history, starting with Du Val (1934). For simplicity, we work over an algebraically closed field. It turns out that lc surface singularities have a very clear description using their dual graphs and this is independent of the characteristic. (By contrast, the equations of the singularities depend on the characteristic.)

Definition 2.19 (Dual graph) Let $(0 \in S)$ be a normal surface singularity over an algebraically closed field and $f\colon S' \to S$ the minimal resolution with irreducible exceptional curves $\{C_i\}$. We associate to this a *dual graph* $\Gamma = \Gamma(0 \in S)$ whose vertices correspond to the C_i. We use the *negative* of the self-intersection number $(C_i \cdot C_i)$ to represent a vertex and connect two vertices C_i, C_j by r edges iff $(C_i \cdot C_j) = r$. In the lc cases, the C_i are almost always smooth rational curves and $(C_i \cdot C_j) \le 1$, so we get a very transparent picture.

The *intersection matrix* of the resolution is $(-(C_i \cdot C_j))$. This matrix is positive definite (essentially by the Hodge index theorem). Its determinant is denoted by $\det(\Gamma) := \det(-(C_i \cdot C_j))$.

Let B be a curve on S and B_i the local analytic branches of B that pass through $0 \in S$. The *extended dual graph* (Γ, B) has an additional vertex for each B_i, represented by $\bullet$, and it is connected to C_j by r edges if $(f_*^{-1}B_i \cdot C_j) = r$.

Definition 2.20 A connected graph is a *twig* if all vertices have ≤ 2 edges. Thus such a graph is of the form

$$c_1 \text{ ------ } c_2 \text{ ------ } \cdots \text{ ------ } c_n$$

Here $\det(\Gamma)$ is also the numerator of the continued fraction (6.70.4).

A connected graph is a *tree with one fork* if there is a vertex (the root) with three edges and all other vertices have ≤ 2 edges. Such a dual graph is of the form

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$$\begin{array}{ccccc} \Gamma_1 & — & c_0 & — & \Gamma_2 \\ & & | & & \\ & & \Gamma_3 & & \end{array}$$

where each Γ_i is a twig joined to c_0 at an end vertex.

Next we list the dual graphs of all lc pairs $(0 \in S, B)$, starting with the terminal and canonical ones. For proofs see Alexeev (1993) or Kollár (2013b, Sec.3.3).

2.21 (List of log canonical surface singularities) Here $(0 \in S)$ is a normal surface singularity over an algebraically closed field and $B \subset S$ a curve (with coefficient 1).

Case 2.21.1 (Terminal). $(0 \in S, B)$ is terminal iff $B = \emptyset$ and S is smooth.

Case 2.21.2 (Canonical). $(0 \in S, B)$ is canonical iff either B and S are both smooth at 0, or $B = \emptyset$ and Γ is one of the following. The corresponding singularities are called *Du Val* singularities or *rational double points* or *simple* surface singularities. See Durfee (1979) for more information. The following equations are correct only in characteristic 0; see Artin (1977), in general.

A_n: $x^2 + y^2 + z^{n+1} = 0$, with $n \geq 1$ curves in the dual graph:

$$2 — 2 — \cdots — 2 — 2$$

D_n: $x^2 + y^2z + z^{n-1} = 0$, with $n \geq 4$ curves in the dual graph:

$$\begin{array}{ccccccccc} & & & & & & 2 & & \\ & & & & & & | & & \\ 2 & — & 2 & — & \cdots & — & 2 & — & 2 \end{array}$$

E_n: with n curves in the dual graph:

$$\begin{array}{ccccccc} & & & & 2 & & \\ & & & & | & & \\ 2 & — & 2 & — & 2 & — & \Gamma \end{array}$$

There are 3 possibilities:

E_6: $x^2 + y^3 + z^4 = 0$ and $\Gamma = 2 - 2$,

E_7: $x^2 + y^3 + yz^3 = 0$ and $\Gamma = 2 - 2 - 2$,

E_8: $x^2 + y^3 + z^5 = 0$ and $\Gamma = 2 - 2 - 2 - 2$.

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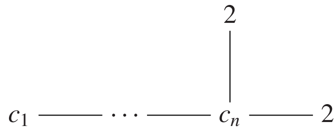
Case 2.21.3 (Purely log terminal) The names here reflect that, at least in characteristic 0, these singularities are obtained as the quotient of $\mathbb{C}^2$ by the indicated type of group. See Brieskorn (1967/1968) and (6.65).
Subcase 2.21.3.1 (Cyclic quotient) B is smooth at 0 (or empty) and (Γ, B) is

$$\bullet \text{ --- } c_1 \text{ --- } \cdots \text{ --- } c_n \quad \text{or} \quad c_1 \text{ --- } \cdots \text{ --- } c_n$$

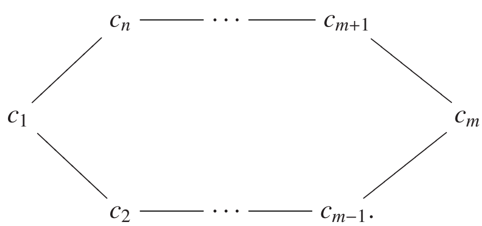
We discuss these in detail in (6.65–6.70).
Subcase 2.21.3.2 (Dihedral quotient)



Subcase 2.21.3.3 (Other quotient) The dual graph is a tree with one fork (2.20) with three possibilities for $(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3))$:

(Tetrahedral) (2,3,3)
(Octahedral) (2,3,4)
(Icosahedral) (2,3,5).

Case 2.21.4 (Log canonical with $B = 0$)
Subcase 2.21.4.1 (Simple elliptic) There is a unique exceptional curve E; it is smooth and of genus 1. If the self-intersection $r := -(E^2)$ is ≥ 3 then the singularity is isomorphic to the cone over the elliptic normal curve $E \subset \mathbb{P}^{r-1}$ of degree r.
Subcase 2.21.4.2 (Cusp) The dual graph is a circle of smooth rational curves



The cases $n = 1, 2$ are exceptional. For $n = 2$, we have two smooth rational curves meeting at two points, and for $n = 1$, the unique exceptional curve is a rational curve with a single node. We can draw the dual graphs as

$$c_1 = c_2 \quad \text{and} \quad \circlearrowleft c_1.$$

For example the dual graphs of the three singularities $(z(xy - z^2) = x^4 + y^4)$, $(z^2 = x^2(x + y^2) + y^7)$, and $(z^2 = x^2(x^2 + y^2) + y^5)$ are

$$3 = 4, \quad \circlearrowleft 1 \quad \text{and} \quad \circlearrowleft 2.$$

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Subcase 2.21.4.3 ($\mathbb{Z}/2$-quotient of a cusp)

(For $n = 1$, it is a $\mathbb{Z}/2$-quotient of a simple elliptic singularity.)
Subcase 2.21.4.4 (Simple elliptic quotient) The dual graph is a tree with one fork (2.20) with three possibilities for $(\det(\Gamma_1), \det(\Gamma_2), \det(\Gamma_3))$:

($\mathbb{Z}/3$-quotient) (3,3,3)
($\mathbb{Z}/4$-quotient) (2,4,4)
($\mathbb{Z}/6$-quotient) (2,3,6).

Case 2.21.5 (Log canonical with $B \neq 0$)
Subcase 2.21.5.1 (Cyclic) B has two smooth branches meeting transversally at 0 and (Γ, B) is

Subcase 2.21.5.2 (Dihedral)

2.22 (List of slc surface singularities) The dual graphs are very similar to the previous ones, but there are two possible changes due to the double curve of the surface S passing through the chosen point $0 \in S$.

In the normal case, the local picture represented by an edge is

$$(xy = 0) \subset \mathbb{A}^2, \quad \text{denoted by} \quad \circ - \circ \quad \text{or} \quad \bullet - \circ,$$

where $(y = 0)$ is an exceptional curve and $(x = 0)$ is either an exceptional curve or a component of B. We can now have a nonnormal variant

$$(xy = z = 0) \subset (xy = 0) \subset \mathbb{A}^3, \quad \text{denoted by} \quad \circ \overset{\mathbf{d}}{-} \circ \quad \text{or} \quad \bullet \overset{\mathbf{d}}{-} \circ,$$

where the d over the edge signifies that the two curves denoted by $\circ$ or $\bullet$ (here $(x = z = 0)$ and $(y = z = 0)$) meet at a point that is also on a double curve of the surface (here $(x = y = 0)$).

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The local picture represented by $\bullet$ — $\circ$ also has another nonnormal variant where (as long as char $\neq 2$) we create a pinch point by identifying the points $(0, y) \leftrightarrow (0, -y)$. The local equation is

$$(xy = z = 0) \subset (z^2 = xy^2) \subset \mathbb{A}^3, \quad \text{denoted by} \quad \mathbf{p} — \circ,$$

where $(y = z = 0)$ is the double curve of the surface and $(x = z = 0)$ an exceptional curve.

CASE 2.22.1 (Semi-plt)
Subcase 2.22.1.1 (Higher pinch points) These are obtained from the cyclic dual graph of (2.21.3.1) by replacing $\bullet$ — $\circ$ by $\mathbf{p}$ — $\circ$.

The simplest one is the pinch point, whose dual graph is $\mathbf{p}$ — 1. The equation of the pinch point is $(x^2 = zy^2)$; it is its own semi-resolution Kollár (2013b, sec.10.4).

As another example, start with the A_n singularity $(xy = z^{n+1})$ and pinch it along the line $(x = z = 0)$. The dual graph is

$$\mathbf{p} — 2 — \cdots — 2$$

with two occurring n-times. As a subring of $k[x, y, z]/(xy - z^{n+1})$, the coordinate ring is generated by (x, z, y^2, xy, yz), but $xy = z^{n+1}$. Thus $u_1 = x, u_2 = z, u_3 = y^2, u_4 = yz$ gives an embedding into $\mathbb{A}^4$. The image is a triple point whose equations can be written as

$$\operatorname{rank}\begin{pmatrix} u_2^n & u_4 & u_3 \\ u_1 & u_2^2 & u_4 \end{pmatrix} \leq 1.$$

Subcase 2.22.1.2 The dual graph is $\Gamma_1 \overset{\mathbf{d}}{—} \Gamma_2$, where the Γ_i are twigs such that $\det(\Gamma_1) = \det(\Gamma_2)$. Note that here we allow $\Gamma_i = \{1\}$ and $1 \overset{\mathbf{d}}{—} 1$ corresponds to $(xy = 0) \subset \mathbb{A}^3$. Similarly $2 \overset{\mathbf{d}}{—} 2$ corresponds to

$$(x_1 y - z_1^2 = x_2 = z_2 = 0) \cup (x_2 y - z_2^2 = x_1 = z_1 = 0) \subset \mathbb{A}^5.$$

Aside It is a good exercise to check that if $\det(\Gamma_1) \neq \det(\Gamma_2)$ then the canonical class of the resulting surface is not $\mathbb{Q}$-Cartier. The case $2 \overset{\mathbf{d}}{—} 1$ is easy to compute by hand. The key in general is to compute the different (11.14) on the double curve; see Kollár (2013b, 5.18) for details. This is one of the special cases of (11.38).

Case 2.22.2 (Slc and $K_S + B$ Cartier)
Subcase 2.22.2.1 (Degenerate cusp) Here $B = 0$ and these are obtained from the dual graph of a cusp (2.21.4.2) by replacing some of the edges $\circ - \circ$ with $\circ \overset{\mathbf{d}}{-} \circ$.

The cases $n = 1, 2$ are again exceptional. For $n = 2$ we can replace either of the edges $\circ - \circ$ with $\circ \overset{\mathbf{d}}{-} \circ$. For example, $(z^2 = x^2y^2)$ and $(z^2 = x^2y^2 + y^5)$ correspond to the dual graphs

$$1 \underset{\mathbf{d}}{\overset{\mathbf{d}}{=\!=}} 1 \quad \text{and} \quad 2 \overset{\mathbf{d}}{=\!=} 2.$$

For $n = 1$ the unique exceptional curve is a rational curve with a single node. We can think of the dual graph as

$$\mathbf{d} \circlearrowleft c_1.$$

For example the singularities $(z^2 = x^2(x + y^2))$ and $(z^2 = x^2(x^2 + y^2))$ give the dual graphs

$$\mathbf{d} \circlearrowleft 1 \quad \text{and} \quad \mathbf{d} \circlearrowleft 2.$$

Subcase 2.22.2.2 These are obtained from the cyclic dual graph of (2.21.5.1) by replacing some of the edges $\circ - \circ$ with $\circ \overset{\mathbf{d}}{-} \circ$.

Case 2.22.3 (Slc and $2(K_S + B)$ Cartier)
Subcase 2.22.3.1 Here $B = 0$ and these are obtained from the dual graph of a $\mathbb{Z}/2$-quotient of a cusp (2.21.4.3) by replacing some of the horizontal edges $\circ - \circ$ with $\circ \overset{\mathbf{d}}{-} \circ$.
Subcase 2.22.3.2 These are obtained from the cyclic dual graph of (2.21.5.1) by replacing at least one of $\bullet - \circ$ by $\mathbf{p} - \circ$ and replacing some of the edges $\circ - \circ$ with $\circ \overset{\mathbf{d}}{-} \circ$.
Subcase 2.22.3.3 These are obtained from the dual graph of (2.21.5.2) by replacing $\bullet - \circ$ by $\mathbf{p} - \circ$ and replacing some of the horizontal edges $\circ - \circ$ with $\circ \overset{\mathbf{d}}{-} \circ$.

This completes the list of all slc surface singularities and now we turn to describing their locally stable deformations. An slc surface can be singular along a curve and the transversal hyperplane sections are nodes. Deformations of nodes are described in (11.35).

The situation is much more complicated for surfaces, so we start with the case $\Delta_0 = 0$. It would be natural to first try to understand all flat deformations of $(x \in X_0)$ and then decide which of these are locally stable. However, in

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many interesting cases, flat deformations are rather complicated, but a good description of all locally stable deformations can be obtained by relating them to locally stable deformations of certain cyclic covers of X (11.24).

Proposition 2.23 *Let k be a field and (X, D) a local, slc scheme over k with D reduced. Assume that $\omega_X^{[m]}(mD) \simeq \mathscr{O}_X$ for some $m \geq 1$ that is not divisible by* $\operatorname{char} k$. *Let $\pi\colon (\tilde{X}, \tilde{D}) \to (X, D)$ be a corresponding μ_m-cover (11.24). Let R be a complete DVR with residue field k and set $S = \operatorname{Spec} R$.*

Taking μ_m-invariants establishes a bijection between the sets:

(2.23.1) *Flat, local, slc morphisms $\tilde{f}\colon (\tilde{X}_S, \tilde{D}_S) \to S$ such that $(\tilde{X}_0, \tilde{D}_0) \simeq (\tilde{X}, \tilde{D})$, plus a μ_m-action on $(\tilde{X}_S, \tilde{D}_S)$ extending the μ_m-action on $(\tilde{X}, \tilde{D})$.*

(2.23.2) *Flat, local, slc morphisms $f\colon (X_S, D_S) \to S$ such that $(X_0, D_0) \simeq (X, D)$.*

Note that $\omega_{\tilde{X}}(\tilde{D})$ is locally free, and, in many cases, this makes $(\tilde{X}, \tilde{D})$ much simpler than (X, D). This reduction step is especially useful when $D = 0$, in which case $\omega_{\tilde{X}}$ is locally free. As we saw in (2.6), then all flat deformations of $\tilde{X}$ are slc. For surfaces, this leads to an almost complete description of all slc deformations.

Aside 2.24 (Deformations of quotients) Let $\tilde{X}$ be a scheme and G a finite group acting on it. The proof of (2.23) shows that G-equivariant deformations of $\tilde{X}$ always induce flat deformations of $X := \tilde{X}/G$ provided the characteristic does not divide $|G|$.

The converse is, however, quite subtle, and usually deformations of X are not related to any deformation of $\tilde{X}$. As an example, consider the family $(xy - z^n - tz^m = 0)$ for $m < n$. For $t = 0$, the fiber is isomorphic to $\mathbb{C}^2/\mathbb{Z}_n$ and, for $t \neq 0$, the fiber has a singularity (analytically) isomorphic to $\mathbb{C}^2/\mathbb{Z}_m$. There is no relation between the corresponding degree n cover of the central fiber and the (local analytic) degree m cover of a general fiber. However, if G acts freely outside a subset of codimension ≥ 3 and $\tilde{X}$ is S_3, then every deformation of X arises from a deformation of $\tilde{X}$ (Kollár, 1995a, 12.7).

The following two examples show that the codimension ≥ 3 condition is not enough, not even for μ_m-covers.

(2.24.1) Let E be an elliptic curve and S a K3 surface with a fixed point free involution τ. Set $Y = E \times S$ and $X = Y/\sigma$ where σ is the involution $(-1, \tau)$. Note that $p\colon Y \to X$ is an étale double cover, $h^1(Y, \mathscr{O}_Y) = 1$ and $h^1(X, \mathscr{O}_X) = 0$. Let H_X be a smooth ample divisor on X and H_Y its pull-back to Y. Consider the cones (2.35) and general projections

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$$\begin{array}{ccc} C_a(Y,H_Y) & \xrightarrow{p_C} & C_a(X,H_X) \\ \pi_Y\big\downarrow & & \big\downarrow\pi_X \\ \mathbb{A}^1 & = & \mathbb{A}^1. \end{array}$$

Since $h^1(X,\mathscr{O}_X) = 0$, the central fiber of π_X is the cone over H_X by (2.36). By contrast, the central fiber F_0 of π_Y is not S_2 since $h^1(Y,\mathscr{O}_Y) \neq 0$, again by (2.36). Thus, although the normalization of F_0 is the cone over H_Y, it is not isomorphic to it.

(2.24.2) Let $g\colon X \to B$ be a smooth projective morphism to a smooth curve, H an ample line bundle on X and choose m large enough. Then the direct images $g_*\mathscr{O}_X(rmH)$ commute with base change for every $r \in \mathbb{N}$, hence the cones $C_a(X_b, \mathscr{O}_{X_b}(mH|_{X_b}))$ form a flat family.

The cones $C_a(X_b, \mathscr{O}_{X_b}(H|_{X_b}))$ are μ_m-covers of the cones $C_a(X_b, \mathscr{O}_{X_b}(mH|_{X_b}))$, but they form a flat family only if $g_*\mathscr{O}_X(rH)$ commutes with base change for every r. That is, we get the required examples whenever $H^0(X_b, \mathscr{O}_{X_b}(H|_{X_b}))$ jumps for special values of b. The latter is easy to arrange, even on a family of smooth curves, as long as $\deg H|_{X_b} < 2g-2$.

Proof of 2.23 Let us start with $f\colon (X_S, D_S) \to S$. Since $\omega_{X_S}^{[m]}(mD_S)$ is locally free, the restriction map

$$\omega_{X_S}^{[m]}(mD_S) \twoheadrightarrow \omega_{X_0}^{[m]}(mD_0) \simeq \mathscr{O}_{X_0}$$

is surjective. Since X_S is affine, the constant 1 section lifts to a nowhere zero section $s\colon \mathscr{O}_{X_S} \simeq \omega_{X_S}^{[m]}(mD_S)$. Let $\tilde{f}\colon (\tilde{X}_S, \tilde{D}_S) \to S$ be the corresponding μ_m-cover (11.24).

The map $\tilde{f}$ is also locally stable by (2.9). By (2.3), this implies that $\tilde{X}_0$ is S_2, hence it agrees with the μ_m-cover of (X_0, D_0).

To see the converse, let $g\colon Y \to S$ be any flat, affine morphism and G a reductive group (or group scheme) acting on Y with quotient $g/G\colon Y/G \to S$. Then $(g/G)_*\mathscr{O}_{Y/G} = (g_*\mathscr{O}_Y)^G$ is a direct summand of $g_*\mathscr{O}_Y$, hence g/G is also flat. Taking invariants commutes with base change since G is reductive. This shows that (1) $\Rightarrow$ (2). □

Assumptions For the rest of this section, we work in characteristic 0, though almost everything works in general as long as the characteristic does not divide m in (2.25), but very little has been proved otherwise.

2.25 (Classification plan) We establish an étale-local description of all slc deformations of surface singularities in four steps.

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(2.25.1) Classify all slc surface singularities $(0, \tilde{S})$ with $\omega_{\tilde{S}}$ locally free.
(2.25.2) Classify all flat deformations of these $(0, \tilde{S})$.
(2.25.3) Classify all μ_m-actions on these surfaces and decide which ones correspond to our μ_m-covers.
(2.25.4) Describe the μ_m-actions on the deformation spaces of the $(0, \tilde{S})$.

The first task was already accomplished in (2.21–2.22); we have Du Val singularities (2.21.2), simple elliptic singularities and cusps (2.21.4.1–2), and degenerate cusps (2.22.1). We can thus proceed to the next step (2.25.2).

2.26 (Deformations of slc surface singularities with K_S Cartier)
2.26.1 (Du Val singularities) It is easy to work out the miniversal deformation space from the equations and (2.27). For each of the A_n, D_n, E_n cases the dimension of the miniversal deformation space is exactly n. For instance, for A_n we get (in char 0)

$$\begin{array}{ccccc} (xy + z^{n+1} = 0) & \hookrightarrow & (xy + z^{n+1} + \sum_{i=0}^{n-1} t_i z^i = 0) & \hookrightarrow & \mathbb{A}^3_{xyz} \times \mathbb{A}^n_{\mathbf{t}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}^n_{\mathbf{t}} & = & \mathbb{A}^n_{\mathbf{t}}. \end{array}$$

2.26.2 (*Elliptic/cusp/degenerate cusp*) Let $(0 \in S)$ be one of these singularities and C_i the exceptional curves of the minimal (semi)resolution. Set $m = -(\sum C_i)^2$ and write $(0 \in S_m)$ to indicate such a singularity.

If $m = 1, 2, 3$ then $(0 \in S_m)$ is (isomorphic to) a singular point on a surface in $\mathbb{A}^3$ by Saito (1974); Laufer (1977). Their deformations are completely described by (2.27).

If $m = 4$, then $(0 \in S_4)$ is (isomorphic to) a singular point on a surface in $\mathbb{A}^4$ that is a complete intersection of two hypersurfaces. The miniversal deformation space of a complete intersection can be described in a manner similar to (2.27); see Artin (1976); Looijenga (1984); or Hartshorne (2010).

If $m = 5$ then the deformations are fully described by the method of Buchsbaum and Eisenbud (1977); see Hartshorne (2010, sec.9).

If $m \geq 3$ and $(0 \in S_m)$ is simple elliptic, then it is (isomorphic to) the singular point of a projective cone $\bar{S}_m \subset \mathbb{P}^m$ over an elliptic normal curve $E_m \subset \mathbb{P}^{m-1}$. By Pinkham (1974, sec.9), every deformation of $(0 \in S_m)$ is the restriction of a deformation of $\bar{S}_m \subset \mathbb{P}^m$. In particular, any smoothing corresponds to a smooth surface of degree m in $\mathbb{P}^m$. The latter have been fully understood classically: these are the del Pezzo surfaces embedded by $|-K|$. In particular, a simple elliptic singularity $(0 \in S_m)$ is smoothable only for $m \leq 9$ Pinkham (1974, sec.9).

The $m = 9$ case is especially interesting. Given an elliptic curve E, a degree 9 embedding $E_9 \hookrightarrow \mathbb{P}^8$ is given by global sections of a line bundle L_9 of degree 9 on E. Embeddings of E into $\mathbb{P}^2$ are given by line bundles L_3 of degree 3. If we take $(E \hookrightarrow \mathbb{P}^2)$ given by L_3, and then embed $\mathbb{P}^2$ into $\mathbb{P}^9$ by $\mathscr{O}_{\mathbb{P}^2}(3)$, then E is mapped to E_9 iff $L_3^{\otimes 3} \simeq L_9$. For a fixed L_9 this gives nine choices of L_3. Thus a given $E_9 \hookrightarrow \mathbb{P}^8$ is a hyperplane section of a $\mathbb{P}^2 \hookrightarrow \mathbb{P}^9$ in nine different ways. Correspondingly, the deformation space $(0 \in S_9)$ has nine smoothing components. (This was overlooked in Pinkham (1974, sec.9).) The automorphism group of $(0 \in S_9)$ permutes these nine components. See Looijenga and Wahl (1986, sec.6) for another description.

For $m \geq 6$, the deformation theory of cusps is much harder, see Gross et al. (2015). Degenerate cusps are all smoothable; Stevens (1998).

2.27 (Deformations of hypersurface singularities) For general references, see Artin (1976); Looijenga (1984); Arnol′d et al. (1985); Hartshorne (2010).

Let $0 \in X \subset \mathbb{A}^n_{\mathbf{x}}$ be a hypersurface singularity defined by an equation $(f(\mathbf{x}) = 0)$. Choose polynomials p_i that give a basis of

$$k[[x_1, \dots, x_n]]/\Big(f, \tfrac{\partial f}{\partial x_1}, \dots, \tfrac{\partial f}{\partial x_n}\Big). \tag{2.27.1}$$

If $(0 \in X)$ is an isolated singularity, then the quotient has finite length, say N. In this case, the miniversal deformation of $(0 \in X)$ is given by

$$\begin{array}{ccccc} X & \hookrightarrow & (f(\mathbf{x}) + \sum_i t_i p_i(\mathbf{x}) = 0) & \hookrightarrow & \mathbb{A}^n_{\mathbf{x}} \times \mathbb{A}^N_{\mathbf{t}} \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \in & \mathbb{A}^N_{\mathbf{t}} & = & \mathbb{A}^N_{\mathbf{t}}. \end{array}$$

In particular, the miniversal deformation space $\mathrm{Def}(X)$ is smooth.

If the quotient in (2.27.1) has infinite length, then it is best to think of the resulting infinite dimensional deformation space as an inverse system of deformations over Artinian rings whose embedding dimension goes to infinity.

The next step (2.25.3) in the classification is to describe all μ_m-actions, but it is more transparent to consider reductive commutative groups. These are of the form $G \times \mathbb{G}_m^r$ where G is a finite, commutative group and $\mathbb{G}_m = \mathrm{GL}(1)$ the multiplicative group of scalars, see Humphreys (1975, sec.16).

2.28 (Commutative groups acting on Du Val singularities) The action of a reductive commutative group on $\mathbb{A}^n$ can be diagonalized. Thus let $S \subset \mathbb{A}^3$ be a Du Val singularity that is invariant under a diagonal group action on $\mathbb{A}^3$. It is easy to work through any one of the standard classification methods (for instance, the one in Kollár and Mori (1998, 4.24)) to obtain the following normal forms. In each case we describe first the maximal connected group actions and then the maximal nonconnected group actions.

Main series: $\mathbb{G}_m$-actions.

A_n $(xy + z^{n+1} = 0)$ and $\mathbb{G}_m^2$ acts with character $(1, -1, 0), (0, n+1, 1)$.
D_n $(x^2 + y^2 z + z^{n-1} = 0)$ and $\mathbb{G}_m$ acts with character $(n-1, n-2, 2)$.
E_6 $(x^2 + y^3 + z^4 = 0)$ and $\mathbb{G}_m$ acts with character $(6, 4, 3)$.
E_7 $(x^2 + y^3 + yz^3 = 0)$ and $\mathbb{G}_m$ acts with character $(9, 6, 4)$.
E_8 $(x^2 + y^3 + z^5 = 0)$ and $\mathbb{G}_m$ acts with character $(15, 10, 6)$.

Twisted versions: $\mu_r \times \mathbb{G}_m$-actions.

A_n $(x^2+y^2+z^{n+1} = 0)$. If $n+1$ is odd, then $\mathbb{G}_m$ acts with character $(n+1, n+1, 2)$ and μ_2 acts with character $(0, 1, 0)$. If $n+1$ is even, then $\mathbb{G}_m$ acts with character $(\frac{n+1}{2}, \frac{n+1}{2}, 1)$ and μ_2 acts with character $(0, 1, 0)$.
D_n $(x^2 + y^2 z + z^{n-1} = 0)$, $\mathbb{G}_m$ acts with character $(n-1, n-2, 2)$ and μ_2 acts with character $(1, 1, 0)$.
D_4 $(x^2+y^3+z^3 = 0)$, $\mathbb{G}_m$ acts with character $(3, 2, 2)$ and μ_3 acts with character $(0, 1, 0)$.
E_6 $(x^2 + y^3 + z^4 = 0)$ and $\mathbb{G}_m$ acts with character $(6, 4, 3)$ and μ_2 acts with character $(1, 0, 0)$.

Example 2.29 (Locally stable deformations of surface quotient singularities) Let $(0 \in S)$ be a surface quotient singularity with Du Val cover $(0 \in \tilde{S}) \to (0 \in S)$. By (2.23), the classification of locally stable deformations of all such $(0 \in S)$ is equivalent to classifying all cyclic group actions on Du Val singularities $(0 \in \tilde{S})$ that are free outside the origin and whose action on $\omega_{\tilde{S}} \otimes k(0)$ is faithful. This is straightforward, though somewhat tedious, using (2.28). Alternatively, one can use the classification of finite subgroups of GL(2) as in Brieskorn (1967/1968).

Thus the miniversal locally stable deformation space, which we denote by $\mathrm{Def}_{KSB}(S)$ (6.64), is the fixed point set of the corresponding cyclic group action on $\mathrm{Def}(\tilde{S})$, hence it is also smooth.

A_n**-series** $(xy+z^{n+1} = 0)/\frac{1}{m}(1, (n+1)c-1, c)$ for any m where $((n+1)c-1, m) = 1$. These are equivariantly smoothable only if $m|(n+1)c$.
D_n**-series** $(x^2 + y^2 z + z^{n-1} = 0)/\frac{1}{2k+1}(n-1, n-2, 2)$ where $(2k+1, n-2) = 1$. These are not equivariantly smoothable, but, for instance, if $2k+1|n-1$, they deform to the quotient singularity $\mathbb{A}^2/\frac{1}{2k+1}(-1, 2)$.

E_6-series $(x^2+y^3+z^4=0)/\frac{1}{m}(6,4,3)$ for $(m,6)=1$. For $m>1$ all equivariant deformations are trivial, save for $m=5$, when there is a one-parameter family $(x^2+y^3+z^4+\lambda yz=0)/\frac{1}{5}(1,4,3)$.

E_7-series $(x^2+y^3+yz^3=0)/\frac{1}{m}(9,6,4)$ for $(m,6)=1$. For $m>1$ all equivariant deformations are trivial, save for $m=5$ and $m=7$, when there are one-parameter families $(x^2+y^3+yz^3+\lambda xz=0)/\frac{1}{5}(4,1,4)$ and $(x^2+y^3+yz^3+\lambda z=0)/\frac{1}{7}(2,6,4)$.

E_8-series $(x^2+y^3+z^5=0)/\frac{1}{m}(15,10,6)$ for $(m,30)=1$. For $m>1$ all equivariant deformations are trivial, save for $m=7$, when there is a one-parameter family $(x^2+y^3+z^5+\lambda yz=0)/\frac{1}{7}(1,3,6)$.

A_n-twisted $(x^2+y^2+z^{n+1}=0)/\frac{1}{4m}(n+1,n+1+2m,2)$ for any $(2m,n+1)=1$. These are never equivariantly smoothable.

D_4-twisted $(x^2+y^3+z^3=0)/\frac{1}{18k+9}(9k+6,1,6k+4)$. All equivariant deformations are trivial.

Example 2.30 (Quotients of simple elliptic and cusp singularities) Let $(0\in S)$ be a simple elliptic, cusp or degenerate cusp singularity with minimal resolution (or semi-resolution) $f\colon T\to S$ and exceptional curves $C=\sum C_i$. Then $\omega_T(C)\simeq f^*\omega_S$, which gives a canonical isomorphism $\omega_S\otimes k(0)\simeq H^0(C,\omega_C)$. Since C is either a smooth elliptic curve or a cycle of rational curves, $\operatorname{Aut}(C)$ is infinite, but a finite index subgroup acts trivially on $H^0(C,\omega_C)$.

For cusps and for most simple elliptic singularities this leaves only μ_2-actions. The corresponding quotients are listed in (2.21.4.3), see Simonetti (2022) for their deformations. When the elliptic curves have extra automorphisms, one can have μ_3,μ_4 and μ_6-actions as in (2.21.4.4).

The following is one of the simplest degenerate cusp quotients.

Example 2.31 (Deformations of the double pinch point) Let $(0\in S)$ be the *double pinch point* singularity, defined by $(\bar S=\mathbb{A}^2,\bar D=(xy=0),\tau=(-1,-1))$. Here ω_S is not locally free, but $\omega_S^{[2]}$ is,

$$S\simeq\tilde S/\tfrac{1}{2}(1,1,1),\quad\text{where}\quad\tilde S=(z^2-x^2y^2=0)\subset\mathbb{A}^3.$$

A local generator of $\omega_{\tilde S}$ is given by $z^{-1}dx\wedge dy$, which is anti-invariant. Thus ω_S has index 2 and $\tilde S\to S$ is the index 1 cover. Thus every locally stable deformation of S is obtained as the μ_2-quotient of an equivariant deformation of $\tilde S$. By (2.27) the miniversal deformation space is given by

$$(z^2-x^2y^2+u_0+u_1xy+\textstyle\sum_{i\ge1}v_ix^{2i}+\sum_{j\ge1}w_jy^{2j}=0)/\tfrac{1}{2}(1,1,1).$$

When $u_0=u_1=v_1=w_1=0$, we get equimultiple deformations to μ_2-quotients of cusps.

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The slc deformations of pairs (X, Δ) are more complicated, even if Δ is a $\mathbb{Z}$-divisor. One difficulty is that $\omega_S(D)$ is locally free for every pair

$$(S, D) := (\mathbb{A}^2, (xy = 0))/\tfrac{1}{n}(1, q)$$

since $\frac{dx}{x} \wedge \frac{dy}{y}$ is invariant. Thus we would need to describe the deformations of every such pair (S, D) by hand. The following is one of the simplest examples, and it already shows that the answer is likely to be subtle.

Example 2.32 (Deformations of $(\mathbb{A}^2, (xy = 0))/\frac{1}{n}(1, 1)$) Flat deformations of the quotient singularity $H_n := \mathbb{A}^2/\frac{1}{n}(1, 1)$ are quite well understood; see Pinkham (1974). H_n can be realized as the affine cone over the rational normal curve $C_n \subset \mathbb{P}^n$ and all local deformations are induced by deformations of the projective cone $C_p(C_n) \subset \mathbb{P}^{n+1}$. If $n \neq 4$ then the deformation space is irreducible and the smooth surfaces in it are minimal ruled surfaces of degree n in $\mathbb{P}^{n+1}$. (For $n = 4$, there is another component, corresponding to the Veronese embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$.)

Since $(xy)^{-1}dx \wedge dy$ is invariant under the group action, it descends to a 2-form on H_n with poles along the curve $D_n := (xy = 0)/\frac{1}{n}(1, 1)$. Thus $K_{H_n} + D_n \sim 0$ and the pair (H_n, D_n) is lc. Our aim is to understand which deformations of H_n extend to a deformation of the pair (H_n, D_n).

Claim 2.32.1 Fix $n \geq 7$ and let $\pi\colon X \to \mathbb{A}^1$ be a general smoothing of H_n. Then the divisor D_n cannot be extended to a divisor D_X such that $\pi\colon (X, D_X) \to \mathbb{A}^1$ is locally stable. However, there are special smoothings $\pi\colon X' \to \mathbb{A}^1$ for which such a divisor D'_X exists.

Proof For $m \in \mathbb{N}$, let $\mathbb{F}_m$ denote the ruled surface $\operatorname{Proj}_{\mathbb{P}^1}(\mathscr{O}_{\mathbb{P}^1} + \mathscr{O}_{\mathbb{P}^1}(-m))$. Let $E_m \subset \mathbb{F}_m$ denote the section with self intersection $-m$ and $F \subset \mathbb{F}_m$ denote a fiber. Note that $K_{\mathbb{F}_m} \sim -(2E_m + (m + 2)F)$.

For $a \geq 1$ set $A_{ma} := E + (m + a)F$. Then A_{ma} is very ample with self intersection $n := m + 2a$ and it embeds $\mathbb{F}_m$ into $\mathbb{P}^{n+1}$ as a surface of degree n. Denote the image by S_{ma}. A general hyperplane section of S_{ma} is a rational normal curve $C_n \subset \mathbb{P}^n$. Consider the affine cones $X_{ma} := C_a(S_{ma})$ and $H_n := C_a(C_n)$. We can choose coordinates such that

$$X_{ma} \subset \mathbb{A}^{n+2}_{x_1,\dots,x_{n+2}} \quad \text{and} \quad H_n = (x_{n+2} = 0).$$

The last coordinate projection gives $\pi\colon X_{ma} \to \mathbb{A}^1$ which is a flat deformation (in fact a smoothing) of H_n. By Kollár (2013b, 3.14.5),

$$\begin{aligned} H^0(X_{ma}, \mathscr{O}_{X_{ma}}(-K_{X_{ma}})) &= \textstyle\sum_{i\in\mathbb{Z}} x_0^i \cdot H^0(S_{ma}, \mathscr{O}_{S_{ma}}(-K_{S_{ma}} + iA_{ma})) \\ &= \textstyle\sum_{i\in\mathbb{Z}} x_0^i \cdot H^0(S_{ma}, \mathscr{O}_{S_{ma}}((2 + i)E_m + (m + 2 + im + ia)F)). \end{aligned}$$

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The lowest degree terms in the sum depend on m and a. For $i < -2$, we get 0. For $i = -2$, we have

$$H^0(S_{ma}, \mathscr{O}_{S_{ma}}((2-m-2a)F)) = H^0(S_{ma}, \mathscr{O}_{S_{ma}}((2-n)F)).$$

This is 0, unless $n = 2$, that is, when X is the quadric cone in $\mathbb{A}^3$. Then D_2 is a Cartier divisor H_2 and so every deformation of H_2 extends to a deformation of the pair (H_2, D_2). Thus assume next that $n \geq 3$.

For $i = -1$ we have the summand $H^0(S_{ma}, \mathscr{O}_{S_{ma}}(E_m + (2-a)F))$. This is again 0 if $a \geq 3$, but for $a = 1$ we get a pencil $|E_m + F|$ (whose members are pairs of intersecting lines) and for $a = 2$ we get a unique member E_m (which is a smooth conic in $\mathbb{P}^{n+1}$). This shows the following.

Claim 2.32.2 For $a = 1, 2$ and any $m \geq 0$, the anticanonical class of the 3-fold X_{ma} contains a (possibly reducible) quadric cone $D \subset X_{ma}$ and $\pi\colon (X_{ma}, D) \to \mathbb{A}^1$ is locally stable. □

For $a \geq 3$, we have to look at the next term $H^0(S_{ma}, \mathscr{O}_{S_{ma}}(2E_m + (m+2)F))$ for a nonzero section. The corresponding linear system consists of reducible curves of the form $E_m + G_m$ where $G_m \in |E_m + (m+2)F|$. These curves have 2 nodes and arithmetic genus 1. Let $B \subset X_{ma}$ denote the cone over any such curve. Then (X_{ma}, B) is log canonical, but $\pi\colon (X_{ma}, B) \to \mathbb{A}^1$ is not locally stable since the restriction of B to H_n consists of $n+2$ lines through the vertex. Thus we have proved:

Claim 2.32.3 For $a \geq 3$ and $m \geq 0$, the anticanonical class of X_{ma} does not contain any divisor D for which $\pi\colon (X_{ma}, D) \to \mathbb{A}^1$ is locally stable. □

Note finally that the surfaces S_{ma} with $n = m+2a$ form an irreducible family. General points correspond to the largest possible value $a = \lfloor (n-1)/2 \rfloor$. The surfaces with $a \leq 2$ correspond to a closed subset, which is a two-dimensional subspace of the versal deformation space of H_n.

2.3 Examples of Locally Stable Families

The aim of this section is to investigate, mostly through examples, fibers of locally stable morphisms. If (S, Δ) is slc then, for any smooth curve C, the projection $\pi\colon (S \times C, \Delta \times C) \to C$ is locally stable with fiber (S, Δ). Thus, in general we can only say that fibers of locally stable morphisms are exactly the slc pairs.

The question becomes, however, quite interesting, if we look at special fibers of locally stable morphisms whose general fibers are "nice," for instance smooth or canonical. The main point is thus to probe the difference between arbitrary slc pairs and those slc pairs that occur on locally stable degenerations of smooth varieties. We focus on two main questions.

Question 2.33 Let $f: X \to T$ be a locally stable morphism over a pointed curve $(0 \in T)$ such that X_t is smooth for $t \neq 0$.
(2.33.1) Is X_0 CM (10.4)?
(2.33.2) Are the irreducible components of X_0 CM?
(2.33.3) Is the normalization of X_0 CM?

Question 2.34 Let $f: (X, \Delta) \to T$ be a locally stable morphism over a pointed curve $(0 \in T)$ such that X_t is smooth and Δ_t is snc for $t \neq 0$.
(2.34.1) Do the supports of $\{\Delta_t : t \in T\}$ form a flat family of divisors?
(2.34.2) Are the sheaves $\mathscr{O}_{X_0}(mK_{X_0} + \lfloor m\Delta_0 \rfloor)$ CM?
(2.34.3) Do the sheaves $\{\mathscr{O}_{X_t}(mK_{X_t} + \lfloor m\Delta_t \rfloor) : t \in T\}$ form a flat family?

A normal surface is always CM, and the (local analytic) irreducible components of an slc surface are CM. The latter follows from the classification of slc surfaces given in Kollár (2013b, sec.2.2). Starting with dimension 3, there are lc singularities that are not CM. The simplest examples are cones over abelian varieties; see (2.35). On the other hand, canonical and log terminal singularities are CM and rational (p.xv) in characteristic 0 by (11.18).

Let us note next that the answer to (2.33.1) is positive, that is, X_0 is CM. Indeed, X is canonical by (2.15) and hence CM by (11.18). Therefore, X_0 is also CM. A more complete answer to (2.33.1), without assuming that X_t is smooth or canonical for $t \neq 0$, is given in (2.66).

For locally stable families of pairs, the boundary provides additional sheaves whose CM properties are important to understand; this motivates (2.34). Unlike for (2.33), the answers to all of these are negative already for surfaces. The first convincing examples were discovered by Hassett (2.41). As a consequence, we see that we cannot think of the deformations of (S, Δ) as a flat deformation of S and a flat deformation of Δ that are compatible in certain ways. In general it is imperative to view (S, Δ) as a single object. See, however, Section 2.7 for many cases where viewing (S, Δ) as a pair does work well.

Our examples will be either locally or globally cones and we need some basic information about them.

2.35 (Cones) Let X be a projective scheme with an ample line bundle L. The *affine cone* over X with conormal bundle L is

$$C_a(X, L) := \operatorname{Spec}_k \oplus_{m \geq 0} H^0(X, L^m).$$

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Away from the vertex $v \in C_a(X, L)$, the cone is locally isomorphic to $X \times \mathbb{A}^1$, but the vertex is usually more complicated. If X is normal then so is $C_a(X, L)$ and its canonical class is Cartier (resp. $\mathbb{Q}$-Cartier) iff $\mathcal{O}_X(K_X) \sim L^m$ for some $m \in \mathbb{Z}$ (resp. $\mathcal{O}_X(rK_X) \sim L^m$ for some $r, m \in \mathbb{Z}$ with $r \neq 0$).

The following results are straightforward; see Kollár (2013b, sec.3.1).

(2.35.1) $H_v^{i+1}(C_a(X, L), \mathcal{O}_{C_a(X,L)}) \simeq \oplus_{m\in\mathbb{Z}} H^i(X, \mathcal{O}_X(L^m))$ for every i.

Over a field of char 0, assume that X has rational singularities.

(2.35.2) If $-K_X$ is ample then $C_a(X, L)$ is CM and has rational singularities. If $-K_X$ is nef (for instance, $K_X \equiv 0$), then $C_a(X, L)$ is CM $\Leftrightarrow H^i(X, \mathcal{O}_X) = 0$ for $0 < i < \dim X$, and $C_a(X, L)$ has rational singularities $\Leftrightarrow H^i(X, \mathcal{O}_X) = 0$ for $0 < i \leq \dim X$.

Next let (X, Δ) be a projective, slc pair and L an ample line bundle on X. Let $\Delta_{C_a(X,L)}$ denote the $\mathbb{R}$-divisor corresponding to Δ on $C_a(X, L)$. Assume that $K_X + \Delta \sim_{\mathbb{Q}} r \cdot L$ for some $r \in \mathbb{R}$. Then $(C_a(X, L), \Delta_{C_a(X,L)})$ is

(2.35.3) terminal iff $r < -1$ and (X, Δ) is terminal,

(2.35.4) canonical iff $r \leq -1$ and (X, Δ) is canonical,

(2.35.5) klt iff $r < 0$ (that is, $-(K_X + \Delta)$ is ample) and (X, Δ) is klt,

(2.35.6) dlt iff either $r < 0$ and (X, Δ) is dlt or $(X, \Delta) \simeq (\mathbb{P}^n, (\prod x_i = 0))$ and the cone is $(\mathbb{A}^{n+1}, (\prod x_i = 0))$.

(2.35.7) lc iff $r \leq 0$ (that is, $-(K_X + \Delta)$ is nef) and (X, Δ) is lc,

(2.35.8) slc iff $r \leq 0$ and X is slc.

Aside The failure of (2.35.2) in positive characteristic has significant consequences for the moduli problem; see Section 8.8.

2.36 (Deformation to cones II) Let $X \subset \mathbb{P}^n$ be a closed subscheme and $H \subset \mathbb{P}^n$ a hyperplane. Thinking of $\mathbb{P}^n \subset \mathbb{P}^{n+1}$ as the hyperplane at infinity, let $C_p(X) \subset \mathbb{P}^{n+1}$ be the projective cone over X with vertex v.

If $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(r)) \to H^0(X, \mathcal{O}_X(r))$ is surjective for every r, then $C_p(X) \setminus X$ is the affine cone $C_a(X)$.

Let $|L|$ be the pencil of hyperplanes in $\mathbb{P}^{n+1}$ that contain $H \subset \mathbb{P}^n$. If $v \notin L_t$ then projection from v shows that $C_p(X) \cap L_t \simeq X$.

There is a unique $L_0 \in |L|$ such that $v \in L_0$. Then $C_p(X) \cap L_0$ is isomorphic to $C_p(X \cap H)$ away from v. If X is pure and $\dim(X \cap H) = \dim X - 1$, then the two are isomorphic iff $H^1(X, \mathcal{O}_X(r)) = 0$ for every r; see (2.35.1) or Kollár (2013b, 3.10).

If all these hold then blowing up H we get a flat morphism $\pi\colon B_H C_p(X) \to \mathbb{P}^1$. There is a unique fiber of π that is isomorphic to $C_p(X \cap H)$, all other fibers are isomorphic to X.

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Example 2.37 (Counterexample to 2.33.2) Let $Q_0 \subset \mathbb{P}^4$ be the quadric cone $(xy - uv = 0)$. Let $|A|$ and $|B|$ be the two families of planes on Q_0 and $H \sim A + B$ the hyperplane class. Let $S_1 \in |2A + H|$ be a general member. Note that S_1 is smooth away from the vertex of Q_0 and at the vertex it has two local analytic components intersecting at a single point. In particular, S_1 is nonnormal and non-CM. (The easiest way to see these is to blow up a plane $B_1 \in |B|$. Then $B_{B_1}Q_0 \to Q_0$ is a small resolution whose exceptional set E is a smooth rational curve. The birational transform of $|2A + H|$ is a very ample linear system whose general member is a smooth surface that intersects E in two points. This is the normalization of the surface S_1.)

Let B_1, B_2 be planes in the other family. Then $X_0 := S_1 + B_1 + B_2 \sim 3H$, thus X_0 is a $(2) \cap (3)$ complete intersection in $\mathbb{P}^4$. We can thus write X_0 as the limit of a smooth family of $(2) \cap (3)$ complete intersections X_t. The general X_t is a smooth K3 surface.

On the other hand, X_0 can also be viewed as a general member of a flat family whose special fiber is $A_1 + A_2 + B_1 + B_2 + H$. The latter is slc by (2.35), thus X_0 is also slc. Hence $\{X_t : t \in T\}$ is a locally stable family such that X_t is a smooth K3 surface for $t \neq 0$. Moreover, the irreducible component $S_1 \subset X_0$ is not CM.

In this case, the source of the problem is easy to explain. At its singular point, S_1 is analytically reducible. The local analytic branches of S_1 and the normalization of S_1 are both smooth.

One can, however, modify this example to get analytically irreducible non-CM examples, albeit in dimension 3. To see this, let

$$Y_0 := C(X_0) = C(S_1) + C(B_1) + C(B_2) \subset \mathbb{P}^5$$

be the cone over X_0. It is still a $(2) \cap (3)$ complete intersection, thus we can write Y_0 as the limit of a smooth family of $(2) \cap (3)$ complete intersections Y_t. The general Y_t is a smooth Fano 3-fold.

By (2.35), Y_0 is slc, thus $\{Y_t : t \in T\}$ is a stable family such that Y_t is a smooth 3-fold for $t \neq 0$. Since S_1 is irreducible, the cone $C(S_1)$ is analytically irreducible at its vertex. It is nonnormal along a line and non-CM.

One can check that the normalization of $C(S_1)$ is CM.

Example 2.38 (Counterexample to 2.33.3) As in (2.37), let $Q_0 \subset \mathbb{P}^4$ be the singular quadric $(xy - uv = 0)$. On it, take a divisor

$$D_0 := A_1 + A_2 + \tfrac{1}{2}(B_1 + \cdots + B_4) + \tfrac{1}{2}H_4$$

where the A_i are planes in one family, the B_i are planes in the other family and H_4 is a general quartic section.

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Note that (Q_0, D_0) is lc (2.35) and $2D_0$ is the intersection of Q_0 with an octic hypersurface. We can thus write (Q_0, D_0) as the limit of a family (Q_t, D_t) where Q_t is a smooth quadric and $2D_t$ a smooth octic hypersurface section of Q_t.

Let us now take the double covers of Q_t ramified along $2D_t$ (11.24). We get a family of $(2) \cap (8)$ complete intersections $X_t \subset \mathbb{P}(1^5, 4)$. The general X_t is smooth with ample canonical class. The special fiber is irreducible, slc, but not normal along $A_1 + A_2$, which is the union of two planes meeting at a point.

Let $\pi\colon \bar{X}_0 \to Q_0$ denote the projection of the normalization of X_0. Then

$$\pi_* \mathcal{O}_{\bar{X}_0} = \mathcal{O}_{Q_0} + \mathcal{O}_{Q_0}(4H - A_1 - A_2).$$

It is easy to compute that $\mathcal{O}_{Q_0}(4H - A_1 - A_2)$ is not CM (see, for instance, Kollár (2013b, 3.15)), so we conclude that $\bar{X}_0$ is not CM.

It is also interesting to note that the preimage of $A_1 + A_2$ in $\bar{X}_0$ is the union of two elliptic cones meeting at their common vertex. These are quite complicated lc centers.

Example 2.39 (Counterexample to 2.33.2–3) Here is an example of a locally stable family of smooth projective varieties $\{Y_t\colon t \in T\}$ such that

(2.39.1) the canonical class K_{Y_t} is ample and Cartier for every t,

(2.39.2) Y_0 is slc and CM,

(2.39.3) the irreducible components of Y_0 are normal, but

(2.39.4) one of the irreducible components of Y_0 is not CM.

Let Z be a smooth Fano variety of dimension $n \geq 2$ such that $-K_Z$ is very ample, for instance $Z = \mathbb{P}^2$. Set $X := \mathbb{P}^1 \times Z$ and view it as embedded by $|-K_X|$ into $\mathbb{P}^N$ for suitable N. Let $C(X) \subset \mathbb{P}^{N+1}$ be the cone over X.

Let $M \in |-K_Z|$ be a smooth member and consider the following divisors

$$D_0 := \{(0:1)\} \times Z,\ D_1 := \{(1:0)\} \times Z, \quad \text{and} \quad D_2 := \mathbb{P}^1 \times M.$$

Note that $D_0 + D_1 + D_2 \sim -K_X$. Let $E_i \subset C(X)$ denote the cone over D_i. Then $E_0 + E_1 + E_2$ is a hyperplane section of $C(X)$ and $\big(C(X), E_0 + E_1 + E_2\big)$ is lc by (2.35). For some $m > 0$, let $H_m \subset C(X)$ be a general intersection with a degree m hypersurface. Then

$$\big(C(X), E_0 + E_1 + E_2 + H_m\big)$$

is snc outside the vertex and is lc at the vertex. Set $Y_0 := E_0 + E_1 + E_2 + H_m$. Since $\mathcal{O}_{C(X)}(Y_0) \sim \mathcal{O}_{C(X)}(m+1)$, as in (2.36), we can view Y_0 as an slc limit of a family of smooth hypersurface sections $Y_t \subset C(X)$.

The cone over X is CM by (2.35), hence its hyperplane section $E_0 + E_1 + E_2 + H_m$ is also CM. However, E_2 is not CM. To see this, note that E_2 is the cone over $\mathbb{P}^1 \times M$ and, by the Künneth formula,

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$$H^i(\mathbb{P}^1 \times M, \mathscr{O}_{\mathbb{P}^1\times M}) = H^i(M, \mathscr{O}_M) = \begin{cases} k & \text{if } i = 0, n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Thus E_2 is not CM by (2.35).

Example 2.40 (Easy counterexamples to 2.34) There are some obvious problems with all of the questions in (2.34) if the D_t contain divisors with different coefficients. For instance, let C be a smooth curve and $D', D'' \subset \mathbb{A}^1 \times C =: S$ two sections of the 1st projection π_1. Set $D := \frac{1}{2}(D' + D'')$. Then $\pi_1\colon (S, D) \to \mathbb{A}^1$ is a stable family of one-dimensional pairs. For general t, the sections D', D'' intersect C_t at two different points and then $\mathscr{O}_{C_t}(K_{C_t} + \lfloor D_t \rfloor) \simeq \mathscr{O}_C(K_C)$. If, however, D', D'' intersect C_t at the same point $p_t \in C_t$, then $\mathscr{O}_{C_t}(K_{C_t} + \lfloor D_t \rfloor) \simeq \mathscr{O}_C(K_C)(p_t)$.

Similarly, the support of D_t is 2 points for general t, but only one point for special values of t.

One can correct for these problems in relative dimension 1 by a more careful bookkeeping of the different parts of the divisor D_t. However, starting with relative dimension 2, no correction seems possible, except when all the coefficients are $> \frac{1}{2}$ (2.81).

The following example is due to Hassett (unpublished).

Example 2.41 (Counterexample to 2.34.1–3) We start with the already studied example of deformations of the cone $S \subset \mathbb{P}^5$ over the degree 4 rational normal curve (1.42), but here we add a boundary to it. Fix $r \geq 1$ and let D_S be the sum of $2r$ lines. Then $(S, \frac{1}{r}D_S)$ is lc and $(K_S + \frac{1}{r}D_S)^2 = 4$.

As in (1.42), there are two different deformations of the pair (S, D_S).

(2.41.1) First, set $P := \mathbb{P}^2$ and let D_P be the sum of r general lines. Then $(P, \frac{1}{r}D_P)$ is lc (even canonical if $r \geq 2$) and $(K_P + \frac{1}{r}D_P)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^5$ to the Veronese surface gives a family $f\colon (X, D_X) \to \mathbb{P}^1$ with general fiber (P, D_P) and special fiber (S, D_S). We can concretely realize this as deforming $(P, D_P) \subset \mathbb{P}^5$ to the cone over a general hyperplane section. Note that for any general D_S there is a choice of lines D_P such that the limit is exactly D_S.

The total space (X, D_X) is the cone over (P, D_P) (blown up along a curve) and X is $\mathbb{Q}$-factorial. Thus by (11.18) the structure sheaf of an effective divisor on X is CM. In particular, D_S is a flat limit of D_P. Since the D_P is a plane curve of degree r, we conclude that

$$\chi(\mathscr{O}_{D_S}) = \chi(\mathscr{O}_{D_P}) = -\tfrac{r(r-3)}{2}.$$

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(2.41.2) Second, set $Q := \mathbb{P}^1 \times \mathbb{P}^1$ and let A, B denote the classes of the two rulings. Let D_Q be the sum of r lines from the A-family. Then $(Q, \frac{1}{r}D_Q)$ is canonical and $(K_Q + \frac{1}{r}D_Q)^2 = 4$. The usual smoothing of $S \subset \mathbb{P}^5$ to $\mathbb{P}^1 \times \mathbb{P}^1$ embedded by $H := A + 2B$ gives a family $g\colon (Y, D_Y) \to \mathbb{P}^1$ with general fiber (Q, D_Q) and special fiber (S, D_S). We can concretely realize this as deforming $(Q, D_Q) \subset \mathbb{P}^5$ to the cone over a general hyperplane section.

The total space (Y, D_Y) is the cone over (Q, D_Q) (blown up along a curve) and Y is not $\mathbb{Q}$-factorial. However, $K_Q + \frac{1}{r}D_Q \sim_{\mathbb{Q}} -H$, thus $K_Y + \frac{1}{r}D_Y$ is $\mathbb{Q}$-Cartier and $(Y, S + \frac{1}{r}D_Y)$ is lc by inversion of adjunction (11.17) and so is $(Y, \frac{1}{r}D_Y)$.

In this case, however, D_S is not a flat limit of D_Q for $r > 1$. This follows, for instance, from comparing their Euler characteristic:

$$\chi(\mathscr{O}_{D_S}) = -\tfrac{r(r-3)}{2} \quad \text{and} \quad \chi(\mathscr{O}_{D_Q}) = r.$$

(2.41.3) Because of their role in the canonical ring, we are also interested in the sheaves $\mathscr{O}(mK + \lfloor \frac{m}{r}D \rfloor)$.

Let H_P be the hyperplane class of $P \subset \mathbb{P}^5$ (that is, $\mathscr{O}_{\mathbb{P}^2}(2)$) and write $m = br + a$ where $0 \le a < r$. Then $mK_P + \lfloor \frac{m}{r}D_P \rfloor + nH_P \sim (2n - 2m - a)L$, so

$$\chi(P, \mathscr{O}_P(mK_P + \lfloor \tfrac{m}{r}D_P \rfloor + nH_P)) = \binom{2n-2m-a+2}{2} = \binom{2n-2m+2}{2} - a(2n - 2m + 1) + \binom{a}{2}.$$

Again by (11.18), $\mathscr{O}_X(mK_X + \lfloor \frac{m}{r}D_X \rfloor)$ is CM, hence its restriction to the central fiber S is $\mathscr{O}_S(mK_S + \lfloor \frac{m}{r}D_S \rfloor)$ as in (2.75). In particular,

$$\chi(S, \mathscr{O}_S(mK_S + \lfloor \tfrac{m}{r}D_S \rfloor + nH_S)) = \binom{2n-2m+2}{2} - a(2n - 2m + 1) + \binom{a}{2}.$$

The other deformation again behaves differently. Write $m = br + a$ where $0 \le a < r$. Then, for $H_Q \sim A + 2B$, we see that

$$mK_Q + \lfloor \tfrac{m}{r}D_Q \rfloor + nH_Q \sim (n - m - a)A + (2n - 2m)B,$$

and therefore

$$\chi(Q, \mathscr{O}(mK_Q + \lfloor \tfrac{m}{r}D_Q \rfloor + nH_Q)) = \binom{2n-2m+2}{2} - a(2n - 2m + 1).$$

From this we conclude that the restriction of $\mathscr{O}_Y(mK_Y + \lfloor mD_Y \rfloor)$ to the central fiber S agrees with $\mathscr{O}_S(mK_S + \lfloor mD_S \rfloor)$ only if $a \in \{0, 1\}$, that is when $m \equiv 0, 1 \mod r$. The if part was clear from the beginning. Indeed, if $a = 0$ then $\mathscr{O}_Y(mK_Y + \lfloor mD_Y \rfloor) = \mathscr{O}_Y(mK_Y + mD_Y)$ is locally free and if $a = 1$ then

$$\mathscr{O}_Y(mK_Y + \lfloor mD_Y \rfloor) = \mathscr{O}_Y(K_Y) \otimes \mathscr{O}_Y\big((m-1)K_Y + (m-1)D_Y\big)$$

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is $\mathscr{O}_Y(K_Y)$ tensored with a locally free sheaf. Both of these commute with restrictions. In the other cases we only get an injection

$$\mathscr{O}_Y(mK_Y + \lfloor mD_Y \rfloor)|_S \hookrightarrow \mathscr{O}_S(mK_S + \lfloor mD_S \rfloor)$$

whose quotient is a torsion sheaf of length $\binom{a}{2}$ supported at the vertex.

In the next example, nonflatness appears in codimension 3.

Example 2.42 On $\mathbb{P}^5$ denote coordinates by $x_1, x_2, x_3, x_1', x_2', x_3'$. Set

$$S := (x_1x_1' = x_2x_2' = x_3x_3' = 0) \subset \mathbb{P}^5.$$

It is a reducible K3 surface, a union of eight planes.

Pick constants a_1, a_2, a_3 and a_1', a_2', a_3' such that $a_ia_j' \neq a_ja_i'$ for $i \neq j$. Set

$$X := (\textstyle\sum a_ix_ix_i' = \sum a_i'x_ix_i' = 0) \subset \mathbb{P}^5.$$

By direct computation, X is singular only at the six coordinate vertices, and it has ordinary double points there. Furthermore, $S \sim -K_X$.

Set $Y := X \cap (\sum(x_i + x_i') = 0) \subset \mathbb{P}^4$ and $C := Y \cap S$. Then Y is a smooth, degree 4 Del Pezzo surface and $C \sim -2K_Y$. Thus $(Y, \frac{1}{2}C)$ is a log CY pair. Let $(X_0, \frac{1}{2}S_0) \subset \mathbb{P}^5$ denote the cone over $(Y, \frac{1}{2}C)$. Deformation to the cone (2.36) gives $\pi : (\mathbf{X}, \frac{1}{2}\mathbf{S}) \to \mathbb{A}^1$, whose central fiber is $(X_0, \frac{1}{2}S_0)$. The other fibers are isomorphic to $(X, \frac{1}{2}S)$.

Note that S contains the pair of disjoint planes $P := (x_1 = x_2 = x_3 = 0)$ and $P' := (x_1' = x_2' = x_3' = 0)$. Their specializations P_0, P_0' meet only at the vertex. This is a nonflat deformation of $P \cup P'$.

Example 2.43 (Counterexample to 2.34.1) As in (2.39), let Z be a smooth Fano variety of dimension $n \geq 2$ such that $-K_Z$ is very ample. Set $X := \mathbb{P}^1 \times Z$, but now view it as embedded by global sections of $\mathscr{O}_{\mathbb{P}^1}(1) \otimes \mathscr{O}_Z(-K_Z)$ into $\mathbb{P}^N$ for suitable N. Let $C(X) \subset \mathbb{P}^{N+1}$ be the cone over X.

Fix $r \geq 1$ and let D_r be the sum of r distinct divisors of the form $\{\text{point}\} \times Z \subset X$. Let $H \subset X$ be a general hyperplane section. Then $H \sim_{\mathbb{Q}} -(K_X + \frac{1}{r}D_r)$, that is, $(X, \frac{1}{r}D_r)$ is (numerically) anticanonically embedded. Thus, by (2.35), $(C(H), \frac{1}{r}C(H \cap D_r))$ is lc and there is a locally stable family with general fiber $(X, \frac{1}{r}D_r)$ and special fiber $(C(H), \frac{1}{r}C(H \cap D_r))$.

However, $C(H \cap D_r)$ is not a flat deformation of D_r. Indeed, if $D_{ri} (\simeq Z)$ is any irreducible component of D_r, then $C(H \cap D_{ri})$ is a flat deformation of D_{ri}. Thus $\amalg_i C(H \cap D_{ri})$ is a flat deformation of $D_r = \amalg_i D_{ri}$. Note further that $\amalg_i C(H \cap D_{ri})$ is the normalization of $C(H \cap D_r)$, and the normalization map is r: 1 over the vertex of the cone. Thus

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$$\chi(D_r, \mathcal{O}_{D_r}) = \sum_i \chi(D_{ri}, \mathcal{O}_{D_{ri}}) = \sum_i \chi(C(H \cap D_{ri}), \mathcal{O}_{C(H\cap D_{ri})})$$
$$\geq \chi(C(H \cap D_r), \mathcal{O}_{C(H\cap D_r)}) + (r-1).$$

Therefore, $C(H \cap D_r)$ cannot be a flat deformation of D_r for $r > 1$. We pick up at least $r-1$ embedded points.

Example 2.44 (Counterexample to 2.34.3) Set $X := C_a(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1, a))$ for some $0 < a < n+1$. Let $D \subset X$ be the cone over a smooth divisor in $|\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1, n+1-a)|$. Then (X, D) is canonical and $K_X + D$ is Cartier.

Let $\pi\colon (X, D) \to \mathbb{A}^1$ be a general projection. Then π is locally stable and its central fiber is the cone $X_0 = C_a(H, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1, a)|_H)$ where $H \in |\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(1, a)|$ is a smooth divisor.

We claim that if $2a > n+1$ then $\mathcal{R}_m\colon \omega^{[m]}_{X/\mathbb{A}^1}\big|_{X_0} \to \omega^{[m]}_{X_0}$ is not surjective for $m \gg 1$. Indeed, $\mathcal{R}_m$ is a sum, for $r \geq 0$ of the restriction maps

$$H^0(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}(r-2m, ra-(n+1)m)) \to H^0(H, \mathcal{O}(r-2m, ra-(n+1)m)|_H),$$

and $\mathcal{R}_m$ is surjective iff $H^1(\mathbb{P}^1 \times \mathbb{P}^n, \mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^n}(r-2m, ra-(n+1)m)) = 0$ for every $r \geq -1$. Choose $r = 2m-2$. By the Künneth formula, this group is

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(-2)) \otimes H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(2a(m-1) - m(n-1))).$$

This is nonzero iff $2a \geq \frac{m}{m-1}(n-1)$.

The following example, related to Patakfalvi (2013), shows that the relative dualizing sheaf does not commute with base change in general.

Example 2.45 Let S be a smooth, projective surface with K_S ample and $p_g = q = 0$. Let C be a smooth, projective curve with K_C ample. For $[L] \in \mathrm{Pic}^\circ(C)$ set $L_X := \omega_{S\times C} \otimes \pi_C^* L$, where π_C is the projection to C. Note that $H^0(S \times C, L_X) = 0$ and, for $m \geq 2$, $h^0(S \times C, L_X^m) = \chi(S \times C, L_X^m)$ is independent of L. Thus the cones $X_L := \operatorname{Spec} \oplus_{m\geq 0} H^0(S \times C, L_X^m)$ form a flat family over $\mathbf{Pic}^\circ(C)$.

By (2.35), K_{X_L} is Cartier iff $L \simeq \mathcal{O}_C$ and $\mathbb{Q}$-Cartier iff $[L] \in \mathrm{Pic}^\circ(C)$ is a torsion point.

2.4 Stable Families

Next we define the notion of stable families over regular, one-dimensional base schemes and establish the valuative criteria of separatedness and properness.

Definition 2.46 Let $f\colon (X,\Delta) \to C$ be a family of pairs (2.2) over a regular one-dimensional scheme C of characteristic 0. We say that $f\colon (X,\Delta) \to C$ is *stable* if f is locally stable (2.3), proper and $K_{X/C} + \Delta$ is f-ample.

Note that if f is locally stable then $K_X + \Delta$ is $\mathbb{R}$-Cartier, so f-ampleness makes sense. By (2.10), being stable is preserved by base change $C' \to C$.

More generally, whenever the notion of local stability is defined later over a scheme S, then $f\colon (X,\Delta) \to S$ is called *stable* if the conditions in the definition are satisfied. (Thus we have to make sure that local stability implies that $K_{X/S} + \Delta$ makes sense and is $\mathbb{R}$-Cartier.)

The relationship between locally stable morphisms and stable morphisms parallels the connection between smooth varieties and their canonical models.

Proposition 2.47 *Let $f\colon (Y,\Delta) \to B$ be a locally stable, proper morphism over a one-dimensional regular scheme B of characteristic* 0*. Assume that the generic fibers are normal, of general type and f has a relative canonical model $f^c\colon (Y^c,\Delta^c) \to B$. Then $f^c\colon (Y^c,\Delta^c) \to B$ is stable.*

Furthermore, taking the relative canonical model commutes with flat base changes $\pi\colon B' \to B$.

Proof First, $K_{Y^c} + \Delta^c$ is f^c-ample by definition (1.38) and (Y^c,Δ^c) is lc.

Let $b \in B$ be any closed point and Y_b (resp. Y^c_b) the fibers over b. Since f is locally stable, $(Y, Y_b + \Delta_Y)$ is lc. Since any fiber is f-linearly trivial, we conclude using Kollár (2013b, 1.28) that $(Y^c, Y^c_b + \Delta^c)$ is also lc. Thus f^c is locally stable, hence stable.

In characteristic 0, being locally stable is preserved by base change (2.10), thus the last assertion follows from (11.40). □

Remark 2.48 In most cases, the fibers of f^c are *not* the canonical models of the fibers of f; see Section 1.5 and (5.10).

A significant exception is when $\Delta = 0$ and Y_b has canonical singularities. Then (Y, Y_b) is canonical by (11.17) and so is (Y^c, Y^c_b) by Kollár and Mori (1998, 3.51). Thus Y^c_b also has canonical singularities by (11.17), it is thus the canonical model of Y_b.

2.49 (Separatedness and Properness) Let C be a regular scheme of dimension 1, and $C^\circ \subset C$ an open and dense subscheme. Let $f^\circ\colon (X^\circ,\Delta^\circ) \to C^\circ$ be a stable morphism. We aim to prove the following two properties.

Separatedness $f^\circ\colon (X^\circ,\Delta^\circ) \to C^\circ$ has at most one extension to a stable morphism $f\colon (X,\Delta) \to C$.

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Properness There is a finite surjection $\pi\colon B \to C$ such that the pull back

$$\pi^* f^\circ\colon (X^\circ \times_C B, \Delta^\circ \times_C B) \to B^\circ := \pi^{-1}(C^\circ)$$

extends to a stable morphism $f_B\colon (X_B, \Delta_B) \to B$.

Next we show that separatedness holds in general and properness holds in characteristic 0. In both cases, the proof relies on theorems that we state in general forms in Section 11.3.

Proposition 2.50 (Separatedness for stable maps) *Let $f_i\colon (X^i, \Delta^i) \to B$ be two stable morphisms over a one-dimensional, regular scheme B. Let*

$$\phi\colon (X^1_{k(B)}, \Delta^1_{k(B)}) \simeq (X^2_{k(B)}, \Delta^2_{k(B)})$$

be an isomorphism of the generic fibers. Then ϕ extends to an isomorphism

$$\Phi\colon (X^1, \Delta^1) \simeq (X^2, \Delta^2).$$

Proof Note that ϕ always extends to an isomorphism over an open, dense subset $B^\circ \subset B$. We can now apply (11.40), whose assumptions are satisfied by (2.15). □

Example 2.50.1 Regularity of B is needed here. As a simple example, let $\bar{B}$ be a smooth curve and $\bar{f}\colon \bar{S} \to \bar{B}$ a smooth, projective family of curves of genus ≥ 2. Assume that we have points $b_1, b_2 \in \bar{B}$ such that the fibers $C_i := \bar{f}^{-1}(b_i)$ are isomorphic. Let B be the nodal curve obtained by identifying b_1 and b_2. We can then descend the family to $f\colon S \to B$ using an isomorphism $C_1 \simeq C_2$. The number of different choices is $|\operatorname{Aut}(C_1)|$. Thus the family over $\bar{B} \setminus \{b_1, b_2\}$ may have several stable extensions over the nodal curve B.

Remark 2.50.2 As a consequence of (2.50) we obtain that $\mathbf{Aut}(X, \Delta)$ is finite for a stable pair (X, Δ) in arbitrary characteristic, using (2.16.2). We prove a more general form of it in (8.64).

Theorem 2.51 (Valuative-properness for stable maps) *Let C be a smooth curve over a field of characteristic 0 and $C^\circ \subset C$ an open and dense subset. Let $f^\circ\colon (X^\circ, \Delta^\circ) \to C^\circ$ be a stable morphism.*

Then there is a finite surjection $\pi\colon B \to C$ such that the pull back

$$f_B^\circ := \pi^* f^\circ\colon (X^\circ \times_C B, \Delta^\circ \times_C B) \to \pi^{-1}(C^\circ)$$

extends to a stable morphism $f_B\colon (X_B, \Delta_B) \to B$.

Proof We begin with the case when X° is normal. Start with $f^\circ\colon (X^\circ, \Delta^\circ) \to C^\circ$ and extend it to a proper flat morphism $f_1\colon (X_1, \Delta_1) \to C$ where X_1 is normal. In general (X_1, Δ_1) is no longer lc.

By Kollár (2013b, 10.46), there is a log resolution (p.xvi) $g_1\colon Y_1 \to X_1$ such that $(g_1^{-1})_*\Delta_1 + \operatorname{Ex}(g_1) + Y_{1c}$ is an snc divisor for every $c \in C$. In general, the fibers of $f_1 \circ g_1\colon Y_1 \to C$ are not reduced, hence $g_1\colon (Y_1, (g_1^{-1})_*\Delta_1 + \operatorname{Ex}(g_1)) \to C$ is not locally stable.

Let B be a smooth curve and $\pi\colon B \to C$ a finite surjection. Let $X_2 \to X_1 \times_C B$ and $Y_2 \to Y_1 \times_C B$ denote the normalizations and $g_2\colon Y_2 \to X_2$ the induced morphism. Let Δ_2 be the pull back of $\Delta_1 \times_C B$ to X_2. Note that

$$f_2 \circ g_2\colon (Y_2, (g_2^{-1})_*\Delta_2 + \operatorname{Ex}(g_2)) \to B$$

is a log resolution over the points where π is étale, but Y_2 need not be smooth. However, by (2.52), $(Y_2, (g_2^{-1})_*\Delta_2 + \operatorname{Ex}(g_2) + \operatorname{red} Y_{2b})$ is lc for every $b \in B$.

By (2.53), one can choose $\pi\colon B \to C$ such that every fiber of $f_2 \circ g_2$ is reduced. With such a choice, $f_2 \circ g_2$ is locally stable.

If the generic fiber $(X_g^\circ, \Delta_g^\circ)$ is klt, then, using (2.15) and after shrinking C°, we may assume that (X°, Δ°) is klt. Pick $0 < \varepsilon \ll 1$. Then $(Y_2, \Delta_2 + (1-\varepsilon)\operatorname{Ex}(g_2))$ is also klt and so it has a canonical model $f_B\colon (X_B, \Delta_B) \to B$ by (11.28.1), which is stable by (2.47).

We are almost done, except that, by construction, $f_B\colon (X_B, \Delta_B) \to B$ is isomorphic to the pull-back of $f^\circ\colon (X^\circ, \Delta^\circ) \to C^\circ$ only over a possibly smaller dense open subset. However, by (2.50), this implies that this isomorphism holds over the entire C°.

The argument is the same if (X°, Δ°) is lc, but we need to take the canonical model of $(Y_2, \Delta_2 + \operatorname{Ex}(g_2))$. Here we rely on (11.28.2).

Next we show how the slc case can be reduced to the lc case.

Let $\bar X^\circ \to X^\circ$ be the normalization with conductor $\bar D^\circ \subset \bar X^\circ$. As we noted in (2.12), we get a stable morphism

$$\bar f^\circ\colon (\bar X^\circ, \bar\Delta^\circ + \bar D^\circ) \to C^\circ. \tag{2.51.4}$$

By the already completed normal case, we get $B \to C$ such that the pull-back of (2.51.4) extends to a stable morphism

$$\bar f_B\colon (\bar X_B, \bar\Delta_B + \bar D_B) \to B. \tag{2.51.5}$$

Finally, (2.54) shows that (2.51.5) is the normalization of a stable morphism $f_B\colon (X_B, \Delta_B) \to B$, which is the required extension of f_B°. □

We have used the following three lemmas during the proof.

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Lemma 2.52 *Let C be a smooth curve over a field of characteristic 0, $f\colon X \to C$ a flat morphism and Δ an $\mathbb{R}$-divisor on X. Assume that $(X, \operatorname{red} X_c + \Delta)$ is lc for every $c \in C$. Let B be a smooth curve, $g\colon B \to C$ a quasi-finite morphism, $g_Y\colon Y \to X \times_C B$ the normalization and $\Delta_Y := g_Y^*\Delta$.*

Then $(Y, \operatorname{red} Y_b + \Delta_Y)$ is lc for every $b \in B$.

Proof Pick $c \in C$ and let $b_i \in B$ be its preimages. By the Hurwitz formula

$$K_Y + \Delta_Y + \textstyle\sum_i \operatorname{red} Y_{b_i} = g_X^*(K_X + \Delta + \operatorname{red} X_c).$$

By assumption, $(X, \Delta + \operatorname{red} X_c)$ is lc for every $c \in C$. Hence, by (11.23.3), $(Y, \Delta_Y + \sum_i \operatorname{red} Y_{b_i})$ is also lc. □

Lemma 2.53 *Let $f\colon X \to T$ be a flat morphism from a normal scheme to a one-dimensional regular scheme T. Let S be another one-dimensional regular scheme and $\pi\colon S \to T$ a quasi-finite morphism. Let $Y \to X \times_T S$ be the normalization and $f_Y\colon Y \to S$ the projection. Assume that π is tamely ramified and, for every $s \in S$, the multiplicity of every irreducible component of $X_{\pi(s)}$ divides the ramification index of π at s.*

Then every fiber of $f_Y\colon Y \to S$ is reduced.

Proof The claim is local, so pick points $0_S \in S$ and $0_T := \pi(0_S) \in T$ with local parameters $t \in \mathscr{O}_T$ and $s \in \mathscr{O}_S$.

We want to study how the multiplicities of the irreducible components of the fiber over 0_T change under base extension. We can focus on one such irreducible component and pass to any open subset of X that is not disjoint from the chosen component. By Noether normalization (10.51), we can think of X as a hypersurface $X \subset \mathbb{A}^n_T$ defined by an equation $f \in \mathscr{O}_T[x_1, \dots, x_n]$. The central fiber X_0 is defined by $\bar f = 0$ where $\bar f$ is the mod t reduction of f. By focusing at a generic point of X_0, after an étale coordinate change we may assume that $\bar f = x_1^m$ where m is the multiplicity of X_0. We can thus write $f = x_1^m - t \cdot u(\mathbf{x}, t)$. Since X is normal (hence regular) at the generic point of X_0, we see that u is not identically zero along X_0.

We can write $\pi^* t = s^e v(s)$ where e is the ramification index of π at 0_S and v is a unit at 0_S. Consider now the fiber product $X_S := X \times_T S \to S$. It is defined by the equation $x_1^m = s^e \cdot u(\mathbf{x}, s^e v(s)) \cdot v(s)$. Note that X_S is not normal along $(s = x_1 = 0)$ if $m, e > 1$.

Constructing the normalization is especially simple if e is a multiple of m. Write $e = md$ and set $x_1' := xs^{-d}$. Then we get $Y \subset \mathbb{A}^n_S$ (with coordinates $x_1', x_2, \dots, x_n$) defined by $x_1'^{\,m} = u(x_1' s^d, x_2, \dots, x_n, s^e v(s)) \cdot v(s)$, and the central fiber Y_0 is defined by the equation $x_1'^{\,m} = u(0, x_2, \dots, x_n, 0) \cdot v(0)$, where the right-hand side is not identically zero.

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If the characteristic of $k(0_S)$ does not divide m, then the projection $Y_0 \to \mathbb{A}^{n-1}_{x_2,\dots,x_n}$ is generically étale and Y_0 is smooth at its generic points. In this case, Y is the normalization of X_S (at least generically along Y_0) and the central fiber of $Y \to S$ has multiplicity 1. □

Aside 2.53.1 If $p := \operatorname{char} k(0_S)$ divides m, then $Y_0 \to \mathbb{A}^{n-1}_{x_2,\dots,x_n}$ is inseparable. If $u(0, x_2, \dots, x_n, 0)$ is not a pth power over the algebraic closure of $k(0_S)$, then Y_0 is geometrically integral, hence generically nonsingular. In this case, Y is the normalization of X_S and the central fiber of $Y \to S$ has multiplicity 1.

If $u(0, x_2, \dots, x_n, 0)$ is a pth power, then Y_0 is not generically reduced. In this case, Y need not be normal and further blow-ups may be needed to reach the normalization. The situation is rather complicated, even for families of curves. A weaker result is in (2.60).

At the end of the proof of (2.51), we needed to construct an slc pair from its normalization. The following is a special case of (11.41), whose assumptions hold by (2.15).

Lemma 2.54 *Let B be a smooth curve over a field of characteristic 0 and $B^\circ \subset B$ a dense open subset. Let $f^\circ : (X^\circ, \Delta^\circ) \to B^\circ$ be a stable morphism. Let $\bar{f}^\circ : (\bar{X}^\circ, \bar{\Delta}^\circ + \bar{D}^\circ) \to B^\circ$ be the normalization with conductor $\bar{D}^\circ \subset \bar{X}^\circ$.*

Assume that $\bar{f}^\circ$ extends to a stable morphism $\bar{f} : (\bar{X}, \bar{\Delta} + \bar{D}) \to B$.

Then f° also extends to a stable morphism $f : (X, \Delta) \to B$. □

As we noted in (2.16), it is not known whether being locally stable is preserved by base change in positive characteristic. However, the next result shows that this holds for all families obtained as in (2.51).

Theorem 2.55 *Let $h : C' \to C$ be a quasi-finite morphisms of regular schemes of dimension 1 and $f : X \to C$ a proper morphism from a regular scheme X to C whose fibers are geometrically reduced, simple normal crossing divisors. Then $X' := X \times_C C'$ has canonical singularities and*

$$\oplus_{m\geq 0} f'_* \omega^{\otimes m}_{X'/C'} \simeq h^* \oplus_{m\geq 0} f_* \omega^{\otimes m}_{X/C}. \tag{2.55.1}$$

Proof Note that (2.55.1) is just the claim that push-forward commutes with flat base change $h : C' \to C$. The substantial part is the assertion that X' has canonical singularities, hence the proj of $\oplus_{m\geq 0} f'_* \omega^{\otimes m}_{X'/C'}$ is also the relative canonical model of any resolution of X'.

Pick a point $x \in X$ and set $c = f(x)$. We may assume that C and C' are the spectra of DVRs with local parameters t and s. Thus the Henselisation of (x, X)

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can be given as a hypersurface $(x_1 \cdots x_m = t) \subset (0, \mathbf{A}^n_C)$, where $\mathbf{A}^n_C$ denotes the Henselisation of $\mathbb{A}^n_C$ at $(0, 0)$.

If $h^*t = \phi(s)$ then (x', X') can be given as a hypersurface

$$(x_1 \cdots x_m = \phi(s)) \subset (0, \mathbf{A}^n_{C'}). \tag{2.55.2}$$

Thus the main claim is that the singularity defined by (2.55.2) is canonical.

If we are over a field, then (2.55.2) defines a toric singularity. We check that although there is no torus action on the base C, we can compute the simplest blow-ups suggested by toric geometry and everything works out as expected.

(Note that, although the pair $(\mathbb{A}^n_k, (x_1 \cdots x_n = 0))$ is lc, this is not a completely toric question. We need to understand all exceptional divisors over $\mathbb{A}^n_k$, not just the toric ones; see Kollár (2013b, 2.11).) □

Lemma 2.56 *Let T be a DVR with local parameter t, residue field k and $\mathbf{A}^n_T$ the Henselisation of $\mathbb{A}^n_T$ at $(0, 0)$. Let $m \le n$ and e be natural numbers and ϕ a regular function on $\mathbf{A}^n_T$. Set*

$$X := X(m, n, e, \phi) = (x_1 \cdots x_m = t^e + t^{e+1}\phi(x_1, \dots, x_n)) \subset (0, \mathbf{A}^n_T), \tag{2.56.1}$$

and let D be the divisor $(t = 0) \subset X$. Then the pair (X, D) is log canonical and X is canonical.

Proof If $\operatorname{char} k = 0$, this immediately follows from (2.10), so the main point is that it also holds for any DVR.

If $m = 0$ or $e = 0$, then X is empty and we are done. Otherwise we can set $x'_m := x_m(1 + t\phi)^{-1}$ to get the simpler equation $x_1 \cdots x_m = t^e$. For inductive purposes we introduce a new variable s and work with

$$\begin{aligned} &X := (x_1 \cdots x_m - s^e = x_{m+1} \cdots x_{m+r}s - t = 0) \subset (0, \mathbf{A}^{n+1}_T), \\ &D := (t = 0), \quad \text{where } 0 \le r \le n - m. \end{aligned} \tag{2.56.2}$$

The case $r = 0$ corresponds to (2.56.1). We use induction on m and e.

Let E be an exceptional divisor over X and v the corresponding valuation. Assume first that $v(x_1) \ge v(s)$. We blow up $(x_1 = s = 0)$. In the affine chart where $x'_1 := x_1/s$, we get the new equations

$$x'_1x_2 \cdots x_m - s^{e-1} = x_{m+1} \cdots x_{m+r}s - t = 0$$

defining (X', D'). A local generator of $\omega_{X/T}(D)$ is

$$\frac{1}{t} \cdot \frac{dx_2 \wedge \cdots \wedge dx_n}{x_2 \cdots x_{m+r}}, \tag{2.56.3}$$

which is unchanged by pull-back.

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Such operations reduce e, until we reach a situation where $\nu(x_i) < \nu(s)$ for every i. If $\nu(x_i) = 0$ for some i and $i \neq m$ then x_i is nonzero at the generic point of $\operatorname{center}_X E$. Thus we can set $x'_m := x_i x_m$ and reduce the value of m. Thus we may assume that $\nu(x_i) > 0$ for $i = 1, \dots, m$. Since $\sum \nu(x_i) = e \cdot \nu(s)$, we conclude that $e < m$. If $e \geq 2$ then we may assume that $\nu(x_e)$ is the smallest. Set $x'_i = x_i/x_e$ for $i = 1, \dots, e-1$ and $s' := s/x_e$. We get new equations

$$x'_1 \cdots x'_{e-1} x_{e+1} \cdots x_m - (s')^e = x_e x_{m+1} \cdots x_{m+r} s' - t = 0 \tag{2.56.4}$$

defining (X', D') and the value of m dropped. The pull-back of (2.56.3) is

$$\begin{aligned} &\frac{1}{t} \cdot \frac{d(x_e x'_2) \wedge \cdots \wedge d(x_e x'_{e-1}) \wedge dx_e \wedge \cdots \wedge dx_n}{(x_e x'_2) \cdots (x_e x'_{e-1}) x_e \cdots x_{m+r}} \\ &\quad = \frac{1}{t} \cdot \frac{dx'_2 \wedge \cdots dx'_{e-1} \wedge dx_e \wedge \cdots \wedge dx_n}{x'_2 \cdots x'_{e-1} x_e \cdots x_{m+r}}, \end{aligned} \tag{2.56.5}$$

which is again a local generator of $\omega_{X'/T}(D')$.

Eventually we reach the situation where $e = 1$. We can now eliminate s and, after setting $r + m \mapsto m$, rewrite the system as

$$\begin{aligned} X &:= (x_1 \cdots x_m = t) \subset (0, \mathbf{A}^n_T), \\ D &:= (t = 0). \end{aligned} \tag{2.56.6}$$

Now X is regular: this case was treated in Kollár (2013b, 2.11). □

We discuss a collection of other results about extending one-parameter families of varieties or pairs. These can be useful in many situations.

2.57 (Extending a stable family without base change) Let C be a smooth curve over a field of characteristic 0, $C^\circ \subset C$ an open and dense subscheme, and $f^\circ\colon (X^\circ, \Delta^\circ) \to C^\circ$ a stable morphism. Here we consider the question of how to extend f° to a proper morphism $f\colon X \to C$ in a "nice" way without a base change. For simplicity, assume that X° is normal.

We can take any extension of f° to a proper morphism $f_1\colon X_1 \to C$, then take a log resolution of $(X_2, \Delta_2) \to (X_1, \Delta_1)$, and finally the canonical model of (X_2, Δ_2) using (11.28). We have proved:

Claim 2.57.1 There is a unique extension $f\colon (X, \Delta) \to C$ such that (X, Δ) is lc and $K_X + \Delta$ is f-ample. □

This model has the problem that its fibers over $C \setminus C^\circ =: \{c_1, \dots, c_r\}$ can be pretty complicated. A slight twist improves the fibers considerably. Instead of starting with the (X_1, Δ_1), we take a log resolution $(X_2, \Delta_2 + \sum \operatorname{red} X_{2,c_i})$ of $(X_1, \Delta_1 + \sum \operatorname{red} X_{1,c_i})$ and its canonical model over C. We need to apply

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(11.28) to $(X_2, \Delta_2 + \sum \operatorname{red} X_{2,c_i} - \varepsilon \sum X_{2,c_i})$ and use (11.28.2) to obtain the following.

Claim 2.57.2 There is a unique extension $f: (X, \Delta) \to C$ such that $(X, \Delta + \sum \operatorname{red} X_{c_i})$ is lc and $K_X + \Delta + \sum \operatorname{red} X_{c_i}$ is f-ample. By adjunction, in this case $(\operatorname{red} X_{c_i}, \operatorname{Diff} \Delta)$ is slc. □

A variant of this starts with any extension (X_1, Δ_1) and then takes a dlt modification of $(X_1, \Delta_1 + \sum \operatorname{red} X_{1,c_i})$ as in Kollár (2013b, 1.36).

Claim 2.57.3 There is a dlt modification $(Y^\circ, \Delta_Y^\circ) \to (X^\circ, \Delta^\circ)$ and an extension of it to $g: (Y, \Delta_Y) \to C$ such that $(Y, \Delta + \sum \operatorname{red} Y_{c_i})$ is dlt. □

Taking a minimal model of $g: (Y, \Delta_Y + \sum \operatorname{red} Y_{c_i}) \to C$ yields another useful version.

Claim 2.57.4 There is a dlt modification $(Y^\circ, \Delta_Y^\circ) \to (X^\circ, \Delta^\circ)$ and an extension of it to $g: (X, \Delta_X) \to C$ such that $(X, \Delta_X + \sum \operatorname{red} X_{c_i})$ is dlt and $K_X + \Delta_X + \sum \operatorname{red} X_{c_i}$ is f-nef. □

Finally, if we are willing to change X° drastically, Kollár (2013b, 10.46) gives the following.

Claim 2.57.5 There is a log resolution $(Y^\circ, \Delta_Y^\circ) \to (X^\circ, \Delta^\circ)$ and an extension of it to $g: (Y, \Delta_Y) \to C$ such that $(Y, \Delta_Y + \operatorname{red} Y_c)$ is snc for every $c \in C$. □

Let us also mention the following very strong variant of (2.57.5), traditionally called the "semistable reduction theorem." We do not use it, and one of the points of our proof of (2.51) was to show that the much easier (2.52) and (2.53) are enough for our purposes.

Theorem 2.58 (Kempf et al., 1973) *Let C be a smooth curve over a field of characteristic 0, $f: X \to C$ a flat morphism of finite type and D a divisor on X. Then there is a smooth curve B, a finite surjection $\pi: B \to C$ and a log resolution $g: Y \to X \times_C B$ such that for every $b \in B$,*

(2.58.1) $g_*^{-1}(D \times_C B) + \operatorname{Ex}(g) + Y_b$ *is an snc divisor and*

(2.58.2) Y_b *is reduced.* □

The positive or mixed characteristic analogs of (2.58) are not known, but the following result on "semi-stable alterations" holds in general.

Theorem 2.59 (de Jong, 1996) *Let T be a one-dimensional regular scheme and $f: X \to T$ a flat morphism of finite type whose generic fiber is geometrically reduced. Then there is a one-dimensional regular scheme S, a finite* *surjection* $\pi\colon S \to T$ *and a generically finite, separable, proper morphism* $g\colon Y \to X \times_T S$ *such that* Y_s *is a reduced snc divisor for every* $s \in S$. □

The following variant of (2.53) is an easy consequence of (2.59).

Corollary 2.60 *Let* $f\colon X \to T$ *be a flat morphism of finite type from a pure dimensional scheme to a one-dimensional regular scheme* T*. Then there is a one-dimensional regular scheme* S *and a finite morphism* $\pi\colon S \to T$ *such that every fiber of the projection of the normalization* $\overline{X \times_T S} \to S$ *is reduced.* □

2.5 Cohomology of the Structure Sheaf

In studying moduli questions, it is very useful to know that certain numerical invariants are locally constant. In this section, we study the deformation invariance of (the dimension of) certain cohomology groups. The key to this is the Du Bois property of slc pairs. The definition of Du Bois singularities is rather complicated, but fortunately for our applications we need to know only the following two facts.

2.61 (Properties of Du Bois singularities) Let M be a complex analytic variety. Since constant functions are analytic, there is an injection of sheaves $\mathbb{C}_M \hookrightarrow \mathscr{O}_M^{\mathrm{an}}$. Taking cohomologies we get

$$H^i(M, \mathbb{C}) \to H^i(M, \mathscr{O}_M^{\mathrm{an}}).$$

If X is projective over $\mathbb{C}$ and X^{an} is the corresponding analytic variety, then, by the GAGA theorems of Serre (1955–1956), $H^i(X^{\mathrm{an}}, \mathscr{O}_X^{\mathrm{an}}) \simeq H^i(X, \mathscr{O}_X)$.

If X is also smooth, Hodge theory tells us that

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \to H^i(X^{\mathrm{an}}, \mathscr{O}_X^{\mathrm{an}}) \simeq H^i(X, \mathscr{O}_X)$$

is surjective. Du Bois singularities were essentially defined to preserve this surjectivity; see Du Bois (1981) and Steenbrink (1983). (There does not seem to be a good definition of Du Bois singularities in positive characteristic; see however Kollár and Kovács (2020).) Thus we have the following.

Property 2.61.1 (Du Bois (1981)) Let X be a proper variety over $\mathbb{C}$ with Du Bois singularities. Then the natural maps

$$H^i(X^{\mathrm{an}}, \mathbb{C}) \to H^i(X^{\mathrm{an}}, \mathscr{O}_X^{\mathrm{an}}) \simeq H^i(X, \mathscr{O}_X) \quad \text{are surjective.}$$ □

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Next we need to know which singularities are Du Bois. Over a field of characteristic 0, rational singularities are Du Bois; see Kollár (1995b, 12.9) and Kovács (1999), but for our applications the key result is the following.

Property 2.61.2 (Kollár and Kovács (2010, 2020)) Let (X, Δ) be an slc pair over $\mathbb{C}$. Then X has Du Bois singularities. □

These are the only facts we need to know about Du Bois singularities. The main use of (2.61.1) is through a base-change theorem, due to Du Bois and Jarraud (1974); Du Bois (1981).

Theorem 2.62 *Let S be a Noetherian scheme over a field of characteristic 0 and $f\colon X \to S$ a flat, proper morphism. Assume that the fiber X_s is Du Bois for some $s \in S$. Then there is an open $s \in S^\circ \subset S$ such that, for all i,*

(2.62.1) *$R^i f_* \mathscr{O}_X$ is locally free and commutes with base change over S°, and*

(2.62.2) *$s \mapsto h^i(X_s, \mathscr{O}_{X_s})$ is a locally constant function on S°.*

Proof By Cohomology and Base Change, the theorem is equivalent to proving that the restriction maps

$$\phi^i_s\colon R^i f_* \mathscr{O}_X \to H^i(X_s, \mathscr{O}_{X_s}) \tag{2.62.3}$$

are surjective for every i. By the Theorem on Formal Functions, it is enough to prove this when S is replaced by any Artinian local scheme S_n, whose closed point is s.

Thus assume from now on that we have a flat, proper morphism $f_n\colon X_n \to S_n$, $s \in S_n$ is the only closed point, and X_s is Du Bois. Then $H^0(S_n, R^i f_* \mathscr{O}_X) = H^i(X_n, \mathscr{O}_{X_n})$, hence we can identify the ϕ^i_s with the maps

$$\psi^i\colon H^i(X_n, \mathscr{O}_{X_n}) \to H^i(X_s, \mathscr{O}_{X_s}). \tag{2.62.4}$$

By the Lefschetz principle, we may assume that $k(s) \simeq \mathbb{C}$. Then both sides of (2.62.4) are unchanged if we replace X_n by the corresponding analytic space X_n^{an}. Let $\mathbb{C}_{X_n}$ (resp. $\mathbb{C}_{X_s}$) denote the sheaf of locally constant functions on X_n (resp. X_s) and $j_n\colon \mathbb{C}_{X_n} \to \mathscr{O}_{X_n}$ (resp. $j_s\colon \mathbb{C}_{X_s} \to \mathscr{O}_{X_s}$) the natural inclusions. We have a commutative diagram

$$\begin{array}{ccc} H^i(X_n, \mathbb{C}_{X_n}) & \xrightarrow{\alpha^i} & H^i(X_s, \mathbb{C}_{X_s}) \\ \downarrow{\scriptstyle j^i_n} & & \downarrow{\scriptstyle j^i_s} \\ H^i(X_n, \mathscr{O}_{X_n}) & \xrightarrow{\psi^i} & H^i(X_s, \mathscr{O}_{X_s}). \end{array}$$

Note that α^i is an isomorphism since the inclusion $X_s \hookrightarrow X_n$ is a homeomorphism, and j^i_s is surjective since X_s is Du Bois. Thus ψ^i is also surjective. □

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Complement 2.62.5 This proof also works if $f: X \to S$ is a flat, proper morphism of complex analytic spaces and X_s is an algebraic space with Du Bois singularities.

Definition 2.63 A scheme Y is said to be *potentially slc* or *slc-type* if, for every $y \in Y$, there is an effective $\mathbb{R}$-divisor Δ_y on Y such that (Y, Δ_y) is slc at y.

Let $f: X \to S$ be a flat morphism. We say that f has *potentially slc fibers over closed points* if the fiber X_s is potentially slc for every closed point $s \in S$.

One can similarly define the notion *potentially klt,* and so on.

In our final applications, the Δ_s usually come as the restriction of a global divisor Δ to X_s, but we do not assume this.

If (X_s, Δ_s) is slc then X_s is Du Bois by (2.61.2), hence (2.62) implies the following.

Corollary 2.64 *Let S be a Noetherian scheme over a field of characteristic 0, and $f: X \to S$ a proper and flat morphism with potentially slc fibers over closed points. Then, for all i,*

(2.64.1) *$R^i f_* \mathscr{O}_X$ is locally free and compatible with base change, and*

(2.64.2) *if S is connected, then $h^i(X_s, \mathscr{O}_{X_s})$ is independent of $s \in S$.* □

We can derive similar results for other line bundles from (2.64). A line bundle L on X is called *f-semi-ample* if there is an $m > 0$ such that L^m is f-generated by global sections. That is, the natural map $f^*(f_*(L^m)) \to L^m$ is surjective. Equivalently, L^m is the pull-back of a relatively ample line bundle by a suitable morphism $X \to Y$.

Corollary 2.65 *Let S be a Noetherian, connected scheme over a field of characteristic 0 and $f: X \to S$ a proper and flat morphism with potentially slc fibers over closed points. Let L be an f-semi-ample line bundle on X. Then*

(2.65.1) *$R^i f_*(L^{-1})$ is locally free and compatible with base change, and*

(2.65.2) *$h^i(X_s, L^{-1}_{X_s})$ is independent of $s \in S$ for all i.*

Proof The question is local on S, thus we may assume that S is local with closed point s. Chose $m > 0$ such that L^m is f-generated by global sections. Since S is local, L^m is generated by global sections. By (2.13), there is a finite morphism $\pi: Y \to X$ such that $\pi_* \mathscr{O}_Y = \oplus_{r=0}^{m-1} L^{-r}$ and $f \circ \pi: (Y, \pi^{-1}\Delta) \to S$ also has potentially slc fiber over s. Thus, by (2.64),

$$R^i(f \circ \pi)_* \mathscr{O}_Y = \oplus_{r=0}^{m-1} R^i f_*(L^{-r})$$

is locally free and compatible with arbitrary base change. Thus the same holds for every summand. □

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Warning 2.65.4 Note that we assume that L is f-semi-ample, not only that L is semi-ample on fibers over closed points. The Poincaré bundle on $E \times \hat{E}_0 \to \hat{E}_0$ shows that the latter is not enough, where E is an elliptic curve and $\hat{E}_0$ is the localization of its dual at $0 = [\mathscr{O}_E]$.

Corollary 2.66 (Kollár and Kovács, 2010, 2020) *Let S be a Noetherian, connected scheme over a field of characteristic 0 and $f\colon X \to S$ a proper, flat morphism of finite type. Assume that all fibers are potentially slc and X_s is CM for some $s \in S$. Then all fibers of f are CM.*

For an arbitrary flat morphisms $\pi\colon X \to S$, the set of points $x \in X$ such that the fiber $X_{\pi(x)}$ is CM at x is open (10.11), but usually not closed. (Many such examples can be constructed using Kollár (2013b, 3.9–11).) If π is proper, then the set $\{s \in S : X_s \text{ is CM}\}$ is open in S (10.12). Thus the key point of (2.66) is to show that, in our case, this set is also closed.

More generally, under the assumptions of (2.66), if one fiber of f is S_k (10.3.2) for some k, then all fibers of f are S_k; see Kollár and Kovács (2020).

Proof We prove the projective case; see Kollár and Kovács (2020) for the proper one.

Let L be an f-ample line bundle on X. If X_s is CM for some $s \in S$, then, by Kollár and Mori (1998, 5.72), $H^i(X_s, L_{X_s}^{-r}) = 0$ for $r \gg 1$ and $i < \dim X_s$. Thus by (2.65), the same vanishing holds for every $s \in S$. Hence, using Kollár and Mori (1998, 5.72) again, we conclude that X_s is CM for every $s \in S$. □

Theorem 2.67 *(Kollár and Kovács, 2010, 2020) Let S be a Noetherian scheme over a field of characteristic 0 and $f\colon X \to S$ a flat morphism of finite type with potentially slc fibers over closed points. Then $\omega_{X/S}$ is*

(2.67.1) *flat over S with S_2 fibers, and*

(2.67.2) *compatible with base change. That is, for any $g\colon T \to S$, the natural map $g_X^*\omega_{X/S} \to \omega_{X_T/T}$ is an isomorphism, where $g_X\colon X_T := X \times_S T \to X$ is the first projection.*

We give a detailed proof of the projective case; this is sufficient for almost all applications in this book. For the general case, we refer to Kollár and Kovács (2020).

The existence of $\omega_{X/S}$ is easy and, as we see in (2.68.1–3), it holds under rather weak restrictions. Compatibility with base change is not automatic; see Patakfalvi (2013) and (2.45) for some examples.

As we explain in (2.68.4–5), once the definition of $\omega_{X/S}$ is set up right, (2.67.2) and the flatness claim in (2.67.1) become easy consequences of (2.65).

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Once these hold, the fiber of $\omega_{X/S}$ over $s \in S$ is ω_{X_s}; and the latter is S_2 as in Kollár and Mori (1998, 5.69).

2.68 (The relative dualizing sheaf II) The best way to define the relative dualizing sheaf is via general duality theory as in Hartshorne (1966); Conrad (2000) or (Stacks, 2022, tag 0DWE); see also (11.2). It is, however, worthwhile to observe that a slight modification of the treatment in Hartshorne (1977) gives the relative dualizing sheaf in the following cases.

Assumptions S is an arbitrary Noetherian scheme and $f\colon X \to S$ a projective morphism of pure relative dimension n (2.71). (We do not assume flatness.)

2.68.1 (Weak duality for $\mathbb{P}^n_S$) Let $P = \mathbb{P}^n_S$ with projection $g\colon P \to S$ and set $\omega_{P/S} := \wedge^n \Omega_{P/S}$.

The proof of Hartshorne (1977, III.7.1 or III.Exc.8.4) shows that there is a natural isomorphism, called the *trace map*, $t\colon R^n g_* \omega_{P/S} \simeq \mathscr{O}_S$ and, for any coherent sheaf F on X, there is a natural isomorphism

$$g_* \mathcal{H}om_P(F, \omega_{P/S}) \simeq \mathcal{H}om_S(R^n g_* F, \mathscr{O}_S).$$

Note that if S is a point then $g_* \mathcal{H}om_P = \operatorname{Hom}_P$, thus we recover the usual formulation of Hartshorne (1977, III.7.1).

2.68.2 (Construction of $\omega_{X/S}$) Let $f\colon X \to S$ be a projective morphism of pure relative dimension n. We construct $\omega_{X/S}$ first locally over S. Once we establish weak duality, the proof of Hartshorne (1977, III.7.2) shows that a relative dualizing sheaf is unique up to unique isomorphism, hence the local pieces glue together to produce $\omega_{X/S}$. Working locally over S, we can assume that there is a finite morphism $\pi\colon X \to P = \mathbb{P}^n_S$. Set

$$\omega_{X/S} := \mathcal{H}om_P(\pi_* \mathscr{O}_X, \omega_{P/S}).$$

If f is flat with CM fibers over S then $\pi_* \mathscr{O}_X$ is locally free and so is $\pi_* \omega_{X/S}$. Thus $\omega_{X/S}$ is also flat over S with CM fibers and it commutes with base change. We discuss a local version of this in (2.68.7).

2.68.3 (Weak duality for X/S) Let $f\colon X \to S$ be a projective morphism of pure relative dimension n (2.71). Use Hartshorne (1977, Exc.III.6.10) to show that there is a trace map $t\colon R^n f_* \omega_{X/S} \to \mathscr{O}_S$, and for any coherent sheaf F on X there is a natural isomorphism

$$f_* \mathcal{H}om_X(F, \omega_{X/S}) \simeq \mathcal{H}om_S(R^n f_* F, \mathscr{O}_S).$$

If F is locally free, this is equivalent to the isomorphism

$$f_*(\omega_{X/S} \otimes F^{-1}) \simeq \mathcal{H}om_S(R^n f_* F, \mathscr{O}_S).$$

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(Note that $M \mapsto \mathcal{H}om_S(M, \mathscr{O}_S)$ is a duality for locally free, coherent $\mathscr{O}_S$-sheaves, but not for all coherent sheaves. In particular, the torsion in $R^n f_* F$ is invisible on the left-hand side $f_*(\omega_{X/S} \otimes F^{-1})$.)

2.68.4 (Flatness of $\omega_{X/S}$) Let L be relatively ample on X/S. By (3.20), $\omega_{X/S}$ is flat over S iff $f_*(\omega_{X/S} \otimes L^m)$ is locally free for $m \gg 1$. If this holds then $\omega_{X/S}$ is the coherent $\mathscr{O}_X$-sheaf associated to $\bigoplus_{m \geq m_0} f_*(\omega_{X/S} \otimes L^m)$, as a module over the $\mathscr{O}_S$-algebra $\sum_{m \geq 0} f_*(L^m)$.

Applying weak duality with $F = L^{-m}$, we see that these hold if $R^n f_*(L^{-m})$ is locally free for $m \gg 1$. The latter is satisfied in two important cases.

(a) $f: X \to S$ is flat with CM fibers. Then $R^i f_*(L^{-m}) = 0$ for $i < n$ and $m \gg 1$, hence $R^n f_*(L^{-m})$ is locally free of rank $(-1)^n \chi(X_s, L_s^{-m})$ for $m \gg 1$.

(b) $f: X \to S$ is flat with potentially slc fibers. Then $R^n f_*(L^{-m})$ is locally free for $m \geq 0$ by (2.65).

2.68.5 (Base change properties of $\omega_{X/S}$) Let $f: X \to S$ be a projective morphism of pure relative dimension n and L relatively ample. We claim that the following are equivalent:

(a) $\omega_{X/S}$ commutes with base change as in (2.67.2).

(b) $R^n f_*(L^{-m})$ is locally free for $m \gg 0$.

By (2.68.3–4), $\omega_{X/S}$ commutes with base change iff $\mathcal{H}om_S(R^n f_*(L^{-m}), \mathscr{O}_S)$ is locally free and commutes with base change for $m \gg 0$. Finally, show that a coherent sheaf M is locally free iff $\mathcal{H}om_S(M, \mathscr{O}_S)$ is locally free and commutes with base change.

2.68.6 (Warning on general duality) If F is locally free, then we get

$$R^i f_*(\omega_{X/S} \otimes F^{-1}) \times R^{n-i} f_*(F) \to R^n f_* \omega_{X/S} \to \mathscr{O}_S,$$

but this is not a perfect pairing, unless both sheaves on the left are locally free and commute with base change.

2.68.7 (More on the CM case) Let $f: X \to S$ be a projective morphism of pure relative dimension n. We already noted in (2.68.2) that if f is flat with CM fibers over S, then the same holds for $\omega_{X/S}$. We consider what happens if f is not everywhere CM. By (10.11), there is a largest open subset $X^{\mathrm{cm}} \subset X$ such that $f|_{X^{\mathrm{cm}}}$ is flat with CM fibers. Assume for simplicity that $X_s \cap X^{\mathrm{cm}}$ is dense in X_s and $s \in S$ is local. Then, for every $x \in X_s \cap X^{\mathrm{cm}}$ one can choose a finite morphism $\pi: X \to P = \mathbb{P}^n_S$ such that $\pi^{-1}(\pi(x)) \subset X^{\mathrm{cm}}$. Thus $\pi_* \mathscr{O}_X$ is locally free at $\pi(x)$ and so is $\pi_* \omega_{X/S}$. Thus we have proved that the restriction of $\omega_{X/S}$ to X^{cm} is flat over S with CM fibers and commutes with base change.

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This is actually true for all finite type morphisms, one just needs to find a local analog of the projection π (see Section 10.6) and show that (2.68.2.a) holds if π is finite; see Conrad (2000) for details.

Corollary 2.69 *Let S be a connected, Noetherian scheme over a field of characteristic 0 and $f: X \to S$ a proper and flat morphism with potentially slc fibers over closed points. Let L be an f-semi-ample line bundle on X. Then*
(2.69.1) *$R^i f_*(\omega_{X/S} \otimes L)$ is locally free and compatible with base change, and*
(2.69.2) *$h^i(X_s, \omega_{X_s} \otimes L_s)$ is independent of $s \in S$ for all i.*
In particular, for $L = \mathscr{O}_X$ we get that
(2.69.3) *$R^i f_* \omega_{X/S}$ is locally free and compatible with base change, and*
(2.69.4) *$h^i(X_s, \omega_{X_s})$ is independent of $s \in S$ for all i.*

If the fibers X_s are CM, then $H^i(X_s, \omega_{X_s} \otimes L_s)$ is dual to $H^{n-i}(X_s, L_s^{-1})$, so (2.69) follows from (2.65). If the fibers X_s are not CM, the relationship between (2.69) and (2.65) is not so clear. See (8.16) for a more general version.

Proof Let us start with the case $i = 0$. By weak duality (2.68.3),

$$f_*(\omega_{X/S} \otimes L) \simeq \mathcal{H}om_S(R^n f_*(L^{-1}), \mathscr{O}_S),$$

where $n = \dim(X/S)$. By (2.65), $R^n f_*(L^{-1})$ is locally free and compatible with base change, hence so is $f_*(\omega_{X/S} \otimes L)$. Thus (2.69.1) holds for $i = 0$. Next we use this and induction on n to get the $i > 0$ cases.

Choose M very ample on X such that $R^i f_*(\omega_{X/S} \otimes L \otimes M) = 0$ for $i > 0$, and this also holds after any base change. Working locally on S, as in the proof of (2.65), let $H \subset X$ be a general member of $|M|$ such that $H \to S$ is also flat with potentially slc fibers (2.13). The push-forward of the sequence

$$0 \to \omega_{X/S} \otimes L \to \omega_{X/S} \otimes L \otimes M \to \omega_{H/S} \otimes L \to 0$$

gives isomorphisms

$$R^i f_*(\omega_{X/S} \otimes L) \simeq R^{i-1} f_*(\omega_{H/S} \otimes L) \quad \text{for } i \geq 2.$$

Using induction, these imply that (2.69.1) holds for $i \geq 2$.

The beginning of the push-forward is an exact sequence

$$0 \to f_*(\omega_{X/S} \otimes L) \to f_*(\omega_{X/S} \otimes L \otimes M) \to f_*(\omega_{H/S} \otimes L) \to R^1 f_*(\omega_{X/S} \otimes L) \to 0.$$

We already proved that the first three terms are locally free. In general, this does not imply that the last term is locally free, but this implication holds if S is the spectrum of an Artinian ring (2.70).

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In general, pick any point $s \in S$ with maximal ideal sheaf m_s. Set $A_n := \mathscr{O}_{s,S}/m_s^n$ and $X_n := \operatorname{Spec}(\mathscr{O}_X/f^*m_s^n)$. Then $H^1(X_n, (\omega_{X/S} \otimes L)|_{X_n})$ is a free A_n-module by the above considerations, and the restriction maps

$$H^1(X_n, (\omega_{X/S} \otimes L)|_{X_n}) \otimes_{A_n} k(s) \to H^1(X_s, \omega_{X_s} \otimes L_s)$$

are isomorphisms. By the Theorem on Formal Functions, this implies that $R^1 f_*(\omega_{X/S} \otimes L)$ is locally free and commutes with base change. □

2.70 Let (A, m) be a local Artinian ring. Let F be a free A-module and $j\colon A \hookrightarrow F$ an injection. We claim that $j(A)$ is a direct summand of F. Indeed, let $r \geq 1$ be the smallest natural number such that $m^r A = 0$. Note that $m^{r-1}m = 0$. If $j(A) \subset mF$ then $m^{r-1}A = 0$, a contradiction. Thus $j(A)$ is a direct summand of F. By induction this shows that any injection between free A-modules is split. This also implies that if $0 \to M_1 \to \cdots \to M_n \to 0$ is an exact sequence of A-modules and all but one of them are free, then they are all free.

2.71 (Pure dimensional morphisms) A finite type morphism $f\colon X \to S$ is said to have *pure relative dimension n* if, for every integral scheme T and every $h\colon T \to S$, every irreducible component of $X \times_S T$ has dimension $\dim T + n$. We also say that f is *pure dimensional* if it is pure of relative dimension n for some n. It is enough to check this property for all cases when T is the spectrum of a DVR.

Applying the definition when T is a point shows that if f has pure relative dimension n, then every fiber of f has pure dimension n, but the converse does not always hold. For instance, let C be a curve and $\pi\colon \bar{C} \to C$ the normalization. If C is nodal then π does not have pure relative dimension 0 since $\bar{C} \times_C \bar{C}$ contains two isolated points. However, the converse does hold in several important cases.

Claim 2.71.1 Let $f\colon X \to S$ be a finite type morphism whose fibers have pure dimension n. Then f has pure relative dimension n iff it is universally open. Thus both properties hold if f is flat.

Proof Both properties can be checked after base change to spectra of DVRs. In the latter case the equivalence is clear and flatness implies both. □

2.71.2 (Chevalley's criterion) (Grothendieck, 1960, IV.14.4.1) Let $f\colon X \to S$ be a finite type morphism whose fibers have pure dimension n. Assume that S is normal (or geometrically unibranch) and X is irreducible. Then f is universally open.

Proof By an easy limit argument, it is enough to check openness after base change for finite type, affine morphisms $S' \to S$; see Grothendieck (1960, IV.8.10.1). We may thus assume that $S' \subset \mathbb{A}^n_S$ for some n. The restriction of an open morphism to the preimage of a closed subset is also open, thus it is enough to show that the natural morphism $f^{(n)}\colon \mathbb{A}^n_Y \to \mathbb{A}^n_S$ is open for every n. If S is normal then so is $\mathbb{A}^n_S$, thus it is enough to show that all maps as in (2.71.2) are open.

To see openness, let $U \subset X$ be an open set and $x \in U$ a closed point. We need to show that $f(U)$ contains an open neighborhood of $s := f(x)$. Let $x \in W \subset X$ be an irreducible component of a complete intersection of n Cartier divisors such that x is an isolated point of $W \cap X_s$. It is enough to prove that $f(U \cap W)$ contains an open neighborhood of s. After extending $W \to S$ to a proper morphism and Stein factorization, we are reduced to showing that (2.71.2) holds for finite morphisms.

Since $f(U)$ is constructible, it is open iff it is closed under generalization. The latter holds by the going-down theorem. □

2.6 Families of Divisors I

We saw in (2.67) that for locally stable morphisms $g\colon (X, \Delta) \to C$, the relative dualizing sheaf $\omega_{X/C}$ commutes with base change. We also saw in (2.44) that its powers $\omega^{[m]}_{X/C}$ usually do not commute with base change. Here we consider this question for a general divisor D: What does it mean to restrict a divisor D on X to a fiber X_c? How are the two sheaves $\mathscr{O}_X(D)|_{X_c}$ and $\mathscr{O}_{X_c}(D|_{X_c})$ related?

2.72 (Comments on Serre's conditions) For the definition of S_m, see (10.3) or (Stacks, 2022, tag 033P). The following variant will be useful for us.

Let X be a scheme, $Z \subset X$ a closed subset, and F a coherent sheaf on X. We say that F is S_m *along* Z if (10.3.2) holds whenever $x \in Z$.

The following is the key example for us. Let T be a regular one-dimensional scheme, $f\colon X \to T$ a proper morphism, and F a coherent sheaf on X, flat over T. Assume that every fiber F_t is S_m. If $x \in X$ is contained in a closed fiber, then $\operatorname{depth}_x F \geq \min\{m+1, \operatorname{codim}(x, \operatorname{Supp} F)\}$, but for points in the generic fiber we can only guarantee that $\operatorname{depth}_x F \geq \min\{m, \operatorname{codim}(x, \operatorname{Supp} F)\}$. Thus F is not S_{m+1}, but it is S_{m+1} *along closed fibers.*

2.73 (One-parameter families of divisors) Let T be a regular, one-dimensional scheme and $f\colon X \to T$ a flat, proper morphism. For simplicity, assume for now that X is normal. Let D be an effective Weil divisor on X. Under what conditions can we view D as giving a "reasonable" family of Weil divisors on the fibers of f?

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We can view D as a subscheme of X and, if $\operatorname{Supp} D$ does not contain any irreducible component of any fiber X_t, then $f|_D\colon D \to T$ is flat, hence the fibers $D|_{X_t}$ form a flat family of subschemes of the fibers X_t. The $D|_{X_t}$ may have embedded points; ignoring them gives a well-defined effective Weil divisor on the fiber X_t. We will eventually denote it by D_t, but use D_t^{div} or the more precise $D|_{X_t}^{\mathrm{div}}$ if we want to emphasize its construction; see also (2.77).

Understanding the difference between the subscheme $D|_{X_t}$ and the divisor D_t^{div} is the key to dealing with many issues. As a rule of thumb, D defines a "nice" family of divisors iff $D_t^{\mathrm{div}} = D|_{X_t}$ for every $t \in T$.

It can happen that $D \cap X_t$ is contained in $\operatorname{Sing} X_t$ for some t. These are the cases when the correspondence between Weil divisors and rank 1 reflexive sheaves breaks down. Fortunately, this does not happen for locally stable families. Thus we can focus on the cases when D is a relative Mumford divisor (p.xiv).

It is now time to drop the normality assumption and work with divisorial subschemes (4.16.2) in one of the following general settings. (Further generalizations will be considered in Sections 5.4 and 9.3.) We start with the absolute version.

(1.a) X is a pure dimensional, reduced scheme and $H \subset X$ a Cartier divisor. Assume that H is S_2; equivalently, X is S_3 along H (2.72).

(1.b) There is a closed subscheme $Z \subset X$ such that $D|_{X\setminus Z}$ is a Cartier divisor and $\operatorname{codim}_H(H \cap Z) \geq 2$.

(1.c) D is a Mumford divisor along H, that is, $\operatorname{Supp} D$ does not contain any irreducible component of H, and H is regular at generic points of $H \cap D$; see (4.16.4).

In the relative version, we assume the following.

(2.a) T is a regular, one-dimensional, irreducible scheme and $f\colon X \to T$ is a flat, pure dimensional morphism whose fibers are reduced and S_2.

(2.b) There is a closed subscheme $Z \subset X$ such that $D|_{X\setminus Z}$ is a Cartier divisor and $\operatorname{codim}_{X_t}(X_t \cap Z) \geq 2$ for every $t \in T$.

(2.c) D is a relative Mumford divisor (4.68).

Under these conditions, the *divisorial restriction* D_H^{div} (resp. D_t^{div}) is defined as the unique divisorial subscheme (4.16.2) on H (resp. X_t) that agrees with the restriction of the Cartier divisor $D|_{X\setminus Z}$ to $H \setminus Z$ (resp. $X_t \setminus Z$).

Proposition 2.74 *Notation and assumptions as in (2.73.1.a–c). The following conditions are equivalent:*

(2.74.1) *$\mathscr{O}_X(-D)$ is S_3 along $H \cap Z$.*

(2.74.2) *$\mathscr{O}_X(-D)$ is S_3 along H.*

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(2.74.3) *The restriction map* $r_H\colon \mathscr{O}_X(-D)|_H \to \mathscr{O}_H(-D_H^{div})$ *is an isomorphism.*

(2.74.4) *The following sequence is exact:*

$$0 \to \mathscr{O}_X(-D-H) \to \mathscr{O}_X(-D) \to \mathscr{O}_H(-D_H^{div}) \to 0.$$

If D is effective, these are further equivalent to:

(2.74.5) $\mathscr{O}_D$ *has depth* ≥ 2 *at every point of* $H \cap Z$.

(2.74.6) $\mathscr{O}_D$ *is* S_2 *along* H.

(2.74.7) $D \cap H = D_H^{div}$ *(as schemes).*

Proof Since we assume that X is S_3 along H, (2) and (6) hold outside Z. Thus (1) ⇔ (2) and (5) ⇔ (6).

Since $\mathscr{O}_X(-D)$ is S_2, r_H is an injection and an isomorphism outside Z. Since $\mathscr{O}_H(-D_H^{\text{div}})$ is S_2 by definition, it is the S_2-hull of $\mathscr{O}_X(-D)|_H$; see (9.3.4). Thus r_H is surjective ⇔ r_H is an isomorphism ⇔ $\mathscr{O}_X(-D)|_H$ is S_2. This proves (2) ⇔ (3) while (3) ⇔ (4) is clear.

Since $\mathscr{O}_X$ has depth ≥ 3 at codimension ≥ 2 points of H, the sequence

$$0 \to \mathscr{O}_X(-D) \to \mathscr{O}_X \to \mathscr{O}_D \to 0,$$

and an easy lemma (10.28) show that (5) ⇔ (1).

Let s be a local equation of H. Then s is not a zero divisor on $\mathscr{O}_D$ and $\mathscr{O}_{D\cap H} = \mathscr{O}_D/(s)$. Thus (6) ⇔ (7). □

Proposition 2.75 (Relative version) *Using the notation and assumptions of (2.73.2.a–c), let* $0 \in T$ *be a closed point and* $g \in T$ *the generic point.*

(2.75.1) *The conditions (2.74.1–7) are equivalent for* $H = X_0$.

If f is projective and L is f-ample, then these are also equivalent to:

(2.75.2) $\chi(X_0, L_0^m(-D_0^{div})) = \chi(X_g, L_g^m(-D_g))$ *for all* $m \in \mathbb{Z}$.

If $\dim(X_0 \cap Z) = 0$*, then these are further equivalent to:*

(2.75.3) $\chi(X_0, \mathscr{O}_{X_0}(-D_0^{div})) = \chi(X_g, \mathscr{O}_{X_g}(-D_g))$.

Proof The first claim follows from (2.74). If f is projective and $\mathscr{O}_X(-D)$ is flat over T, then

$$\chi(X_g, L_g^m(-D_g)) = \chi(X_0, L^m(-D)|_{X_0}).$$

Hence the difference of the two sides in (2) is $\chi(X_0, L_0^m \otimes Q)$, where Q is the cokernel of $r_0\colon \mathscr{O}_X(-D)|_{X_0} \to \mathscr{O}_{X_0}(-D_0^{\text{div}})$. Thus $Q = 0$ iff equality holds in (2). If $\dim(X_0 \cap Z) = 0$ then Q has 0-dimensional support, thus

$$\chi(X_0, L_0^m \otimes Q) = \chi(X_0, Q) = H^0(X_0, Q),$$

so, in this case, (2) is equivalent to (3). □

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Note that (2.75) shows that one can go rather freely between effective divisors and their ideal sheaves when studying restrictions. Much of the results here on ideal sheaves generalize to arbitrary sheaves; these are worked out in Sections 5.4 and 9.3.

The conditions (2.75) are all preserved by linear equivalence. However, they are not preserved by sums of divisors.

Example 2.76 Consider a family of smooth quadrics $Q \subset \mathbb{P}^3 \times \mathbb{A}^1$ degenerating to the quadric cone Q_0. Take four families of lines L^i, M^i such that $L^1_0, L^2_0, M^1_0, M^2_0$ are four distinct lines in Q_0, $L^1_c \neq L^2_c$ are in one family of lines on Q_c and $M^1_c \neq M^2_c$ are in the other family for $c \neq 0$. Note that

$$(Q, \tfrac{1}{2}(L^1 + L^2 + M^1 + M^2)) \to \mathbb{A}^1$$

is a locally stable family.

Each of the four families of lines L^i, M^i is a flat family of Weil divisors.

For pairs of lines, flatness is more complicated. $L^1 + L^2$ is *not* a flat family (the flat limit has an embedded point at the vertex), but $L^i + M^j$ is a flat family for every i, j. The union of any three of them, for instance $L^1 + L^2 + M^1$ is again a flat family, and so is $L^1 + L^2 + M^1 + M^2$.

Notation 2.77 Let C be a regular, one-dimensional scheme and $f\colon X \to C$ a flat, pure dimensional morphism with reduced, S_2 fibers. Let Δ be a relative Mumford $\mathbb{R}$-divisor (4.68). From now on we use Δ_c to denote the *divisorial fiber* (instead of Δ^{div}_c or $\Delta|^{\mathrm{div}}_{X_c}$ as in (2.73)).

Thus the fiber of a pair (X, Δ) over $c \in C$ is denoted by (X_c, Δ_c).

This notation is harmless for $\mathbb{R}$-divisors, but there is a potential for confusion when used for prime divisors. Then we use the longer $D|_{X_c}$ for the scheme-theoretic fiber and D^{div}_c or $D|^{\mathrm{div}}_{X_c}$ for the divisor-theoretic fiber.

2.78 The main source of divisors D and divisorial sheaves $\mathscr{O}_X(D)$ that satisfy the equivalent conditions of (2.75) is (11.20).

Let (X, Δ) be an slc pair. The conditions of (2.75) are local on X, we can thus assume that $K_X + \Delta \sim_{\mathbb{R}} 0$. Then

$$mK_X + \lfloor m\Delta \rfloor + \{m\Delta\} \sim_{\mathbb{R}} 0 \tag{2.78.1}$$

for any $m \in \mathbb{Z}$. If $\Delta = \sum a_i D_i$ and $\{ma_i\} \leq a_i$ for every i, then $\{m\Delta\} \leq \Delta$, hence $-mK_X - \lfloor m\Delta \rfloor$ satisfies the assumptions of (11.20).

Furthermore, if $B \leq \lfloor \Delta \rfloor$ is an effective $\mathbb{Z}$-divisor, then we can also work with

$$(mK_X + \lfloor m\Delta \rfloor - B) + (\{m\Delta\} + B) \sim_{\mathbb{R}} 0. \tag{2.78.2}$$

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If $a_1 + a_2 = 1$ and the a_i are irrational, then $\{ma_1\} \leq a_1$ and $\{ma_2\} \leq a_2$ hold only for $m = 0$, but (2.78.2) can be useful, relying on (11.50).

However, the numerical conditions $\{ma_i\} \leq a_i$ hold in many other cases; we list some of them in (2.79). These results are generalized to reduced base schemes in (4.33). They influence the definition of various moduli theories in Chapters 6 and 8.

Proposition 2.79 *Let $f\colon (X, \Delta = \sum a_i D_i) \to C$ be a locally stable morphism to a smooth curve over a field of characteristic 0 and $c \in C$ a closed point. Let D be a relative Mumford $\mathbb{Z}$-divisor (4.68). Then*

$$\mathscr{O}_X(D)|_{X_c} \simeq \mathscr{O}_{X_c}(D_c) := \mathscr{O}_{X_c}(D_c^{div}) \qquad (*)$$

in any of the following cases:

(2.79.1) *D is $\mathbb{Q}$-Cartier.*

(2.79.2) *$\Delta = 0$ and $D \sim_{\mathbb{Q}} mK_{X/C}$ for any $m \in \mathbb{Z}$.*

(2.79.3) *$m\Delta$ is a $\mathbb{Z}$-divisor and $D \sim_{\mathbb{Q}} mK_{X/C} + m\Delta$.*

(2.79.4) *$m\Delta$ is a $\mathbb{Z}$-divisor and $D \sim_{\mathbb{Q}} (m+1)K_{X/C} + m\Delta$.*

(2.79.5) *$\Delta = \sum(1 - \frac{1}{r_i})D_i$ for some $r_i \in \mathbb{N}$, and $D \sim_{\mathbb{Q}} mK_{X/C} + \lfloor m\Delta \rfloor$ for any $m \in \mathbb{Z}$.*

(2.79.6) *$\Delta = \sum c_i D_i$, $D \sim_{\mathbb{Q}} mK_{X/C} + \lfloor m\Delta \rfloor$ and $1 - \frac{1}{m} \leq c_i \leq 1$ for every i.*

(2.79.7) *The set $\{m \in \mathbb{N} : (*)$ holds for $D \sim_{\mathbb{Q}} mK_{X/C} + \sum \lfloor ma_i \rfloor D_i\}$ has positive density.*

(2.79.8) *In (1–6) we may replaced D by $D - B$ for any effective relative Mumford $\mathbb{Z}$-divisor $B \leq \lfloor \Delta \rfloor$.*

Proof Let D be a Weil divisor on X as in (2.73.2–4). Assume that there is an effective $\mathbb{R}$-divisor $\Delta' \leq \Delta$ and an $\mathbb{R}$-Cartier $\mathbb{R}$-divisor L such that $D \sim_{\mathbb{R}} \Delta' + L$. Then $\mathscr{O}_X(-D)$ satisfies the equivalent conditions of (2.75) by (11.20).

For (1) we can use $\Delta' = L = 0$, in cases (1–3) we can take $\Delta' = 0$ and $L := -m(K_{X/C} + \Delta)$, and in case (4) we use $\Delta' = \Delta$ and $L := -(m+1)(K_{X/C} + \Delta)$.

For (5–6) we employ $\Delta' = m\Delta - \lfloor m\Delta \rfloor$ and $L := -m(K_{X/C} + \Delta)$. The assumptions on the coefficients of Δ ensure that $\Delta' \leq \Delta$. (Note that if $m\Delta - \lfloor m\Delta \rfloor \leq \Delta$ for every m then in fact every coefficient of Δ is of the form $1 - \frac{1}{r}$ for some $r \in \mathbb{N}$.) Claim (7) follows from (11.50).

Finally, if $B \leq \lfloor \Delta \rfloor$ then $\{m\Delta\} + B \leq \Delta$, giving (8). □

These results are close to being optimal. For instance, under the assumptions of (2.79.3), if n is different from m and $m+1$ then the two sheaves

$$\left(\omega_{X/C}^{[n]}(mD)\right)\Big|_{X_c} \quad \text{and} \quad \omega_{X_c}^{[n]}(mD|_{X_c})$$

are frequently different, see (2.41.3). In general, as shown by (2.44), even the two sheaves $(\omega^{[m]}_{X/C})\big|_{X_c}$ and $\omega^{[m]}_{X_c}$ can be different if $\Delta \neq 0$. However, a considerable generalization of the cases (2.79.5–6) is proved in Section 2.7.

2.7 Boundary with Coefficients $> \frac{1}{2}$

Consider a locally stable morphism $f\colon (X, \Delta = \sum a_i D^i) \to C$ to a smooth curve C. It is very tempting to think of each fiber (X_c, Δ_c) as a compound object $(X_c, D^i_c\colon i \in I, a_i\colon i \in I)$ consisting of the scheme X_c, the divisors D^i_c, and their coefficients a_i. Two issues muddy up this simple picture.

- Different D^i_c may have an irreducible component E_c in common. The definition of a pair treats E_c as a divisor with coefficient $\sum_{i\in I} \operatorname{coeff}_{E_c} D^i_c$. The individual D^i_c do not seem to be part of the data any more.
- Should we just ignore the embedded points of $X_c \cap D^i$?

One could hope that the first is just a matter of bookkeeping, but this does not seem to be the case, as shown by the examples (2.76) and (2.41). In both cases the coefficients in Δ were $\le \frac{1}{2}$.

The aim of this section is to show that these examples were optimal; these complications do not occur if the coefficients in Δ are all $> \frac{1}{2}$. We start with the case when the coefficients are 1.

Given a locally stable map $f\colon (X, \Delta) \to C$, usually the lc centers of the fibers (X_c, Δ_c) do not form a flat family. Indeed, there are many cases when the generic fiber is smooth, but a special fiber is not klt. However, as we show next, the specialization of an lc center on the generic fiber becomes a union of lc centers on a special fiber. Set theoretically, this follows from adjunction (11.17) and (11.12.4), but now we prove this even scheme theoretically.

Theorem 2.80 *Let C be a smooth curve over a field of characteristic 0, $f\colon (X, \Delta) \to C$ a locally stable morphism and $Z \subset X$ any union of lc centers of (X, Δ). Then $f|_Z\colon Z \to C$ is flat with reduced fibers and the fiber Z_c is a union of lc centers of (X_c, Δ_c) for every $c \in C$.*

Proof Z is reduced and every irreducible component of Z dominates C by (2.15). Thus $f|_Z\colon Z \to C$ is flat. We can write its fibers as $Z_c = X_c \cap Z$. Since $X_c + Z$ is a union of lc centers of $(X, X_c + \Delta)$, it is seminormal (11.12.2), so $X_c \cap Z$ is reduced by (11.12.3). The last claim follows from (11.10.3). □

When the coefficients are in $(\frac{1}{2}, 1]$, we start with a simple result.

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2.81 (Restriction and rounding down) Let $f\colon (X, \Delta = \sum_{i\in I} a_i D^i) \to C$ be a locally stable family over a one-dimensional, regular scheme.

By (2.3), (X_c, Δ_c) is slc, hence every component of Δ_c appears with coefficient ≤ 1. For a divisor $A \subset X_c$,

$$1 \geq \operatorname{coeff}_A \Delta_c = \textstyle\sum_{i\in I} a_i \cdot \operatorname{coeff}_A D^i_c.$$

Since the $\operatorname{coeff}_A D^i_c$ are natural numbers, we get the following properties.

(2.81.1) If $a_i > \frac{1}{2}$, then every irreducible component of D^i_c has multiplicity 1.

(2.81.2) If $a_i + a_j > 1$ and $i \neq j$, then the divisors D^i_c and D^j_c have no irreducible components in common.

Next let $\Theta = \sum_j b_j B^j$ be an $\mathbb{R}$-divisor on X. If every irreducible component of B^j_c has multiplicity 1, and the different restrictions have no irreducible components in common, then combining (1–2) we get:

Claim 2.81.3 Assume that $\operatorname{Supp}\Theta \subset \operatorname{Supp}(\Delta^{>1/2})$ (11.1). Then $\operatorname{coeff}(\Theta|_H) \subset \operatorname{coeff}\Theta$ and $\lfloor \Theta|_H \rfloor = \lfloor \Theta \rfloor|_H$. □

Applying this to $\Theta = m\Delta$ gives the following.

Corollary 2.81.4. If $\operatorname{coeff}\Delta \subset (\frac{1}{2}, 1]$ then $\lfloor m\Delta_c \rfloor = \lfloor m\Delta \rfloor_c$ for every m. □

The next result of Kollár (2014) solves the embedded point question when all the occurring coefficients are $> \frac{1}{2}$. Examples (2.41 and 2.42) show that the strict inequality is necessary.

Theorem 2.82 *Let $f\colon (X, \Delta = \sum_{i\in I} a_i D_i) \to C$ be a locally stable morphism to a smooth curve over a field of characteristic 0. Let $J \subset I$ be any subset such that $a_j > \frac{1}{2}$ for every $j \in J$. Set $D_J := \cup_{j\in J} D_j$. Then*

(2.82.1) *$f|_{D_J}\colon D_J \to C$ is flat with reduced fibers,*

(2.82.2) *D_J is S_2 along every closed fiber, and*

(2.82.3) *$\mathscr{O}_X(-D_J)$ is S_3 along every closed fiber.*

Proof Note that each D_i is a log center of (X, Δ) (11.11) and $\operatorname{mld}(D_i, X, \Delta) = 1 - a_i$ by (11.8). Thus $\operatorname{mld}(D_J, X, \Delta) < \frac{1}{2}$.

Let X_c be any fiber of f. Then $(X, X_c + \Delta)$ is slc and

$$\operatorname{mld}(D_i, X, X_c + \Delta) = \operatorname{mld}(D_i, X, \Delta) < \tfrac{1}{2},$$

since none of the D_i is contained in X_c. Each irreducible component of X_c is a log canonical center of $(X, X_c + \Delta)$ (11.10), thus $\operatorname{mld}(X_c, X, X_c + \Delta) = 0$. Therefore, $\operatorname{mld}(D_J, X, X_c + \Delta) + \operatorname{mld}(X_c, X, X_c + \Delta) < \frac{1}{2}$.

We can apply (11.12.3) to $(X, X_c + \Delta)$ with $W = D_J$ and $Z = X_c$ to conclude that $X_c \cap D_J$ is reduced. This proves (1) which implies (2–3) by (2.75). □

For the plurigenera, we have the following generalization of (2.79.5–6).

Theorem 2.83 (Kollár, 2018a) *Let C be a smooth curve over a field of characteristic 0 and $f:(X,\Delta)\to C$ a locally stable morphism with normal generic fiber. Assume that* $\operatorname{coeff}\Delta\subset(\frac{1}{2},1]$. *Then, for every $c\in C$ and $m\in\mathbb{Z}$,*

$$\omega_{X/C}^{[m]}(\lfloor m\Delta\rfloor)|_{X_c}\simeq\omega_{X_c}^{[m]}(\lfloor m\Delta\rfloor_c). \tag{2.83.1}$$

Complement 2.83.2 If $\operatorname{coeff}\Delta\subset[\frac{1}{2},1]$ then (2.83.1) still holds, but needs a more careful case analysis, see Kollár (2018a). Note also that $\lfloor m\Delta\rfloor_c=\lfloor m\Delta_c\rfloor$ if $\operatorname{coeff}\Delta\subset(\frac{1}{2},1]$ by (2.81.3), but they may be different if some coefficients equal $\frac{1}{2}$ and m is odd.

Method of proof If $mK_X+\lfloor m\Delta\rfloor$ is $\mathbb{Q}$-Cartier, then this follows from (2.79.1). Thus we aim to construct a birational modification $X'\to X$ such that $mK_{X'}+\lfloor m\Delta'\rfloor$ is $\mathbb{Q}$-Cartier, and then descend from X' to X.

More generally, let $g\colon Y\to X$ be a proper morphism of normal varieties, F a coherent sheaf on Y, $H\subset X$ a Cartier divisor, and $H_Y:=g^*H$. Assuming that F is S_m along H_Y, we would like to understand when g_*F is S_m along H. If (the local equation of) H_Y is not a zero divisor on F, then the sequence

$$0\to F(-H_Y)\to F\to F|_{H_Y}\to 0 \tag{2.83.3}$$

is exact. By push-forward we get the exact sequence

$$0\to g_*F(-H_Y)\to g_*F\to g_*(F|_{H_Y})\to R^1g_*F(-H_Y), \tag{2.83.4}$$

and $R^1g_*F(-H_Y)\simeq\mathscr{O}_X(-H)\otimes R^1g_*F$. Thus, by (10.28), g_*F is S_m along H if $R^1g_*F=0$, and $g_*(F|_{H_Y})$ is S_{m-1} along H. (In many cases, for instance if g is an isomorphism outside H_Y, these conditions are also necessary.)

We choose $F=\mathscr{O}_{X'}(mK_{X'}+\lfloor m\Delta'\rfloor)$. Then we need that

(5.a) $R^1g_*\mathscr{O}_{X'}(mK_{X'}+\lfloor m\Delta'\rfloor)=0$,

(5.b) $g_*(\mathscr{O}_{X'}(mK_{X'}+\lfloor m\Delta'\rfloor)|_{H_Y})$ is S_2 along H, and

(5.c) $g_*\mathscr{O}_{X'}(mK_{X'}+\lfloor m\Delta'\rfloor)\simeq\mathscr{O}_X(mK_X+\lfloor m\Delta\rfloor)$.

For us, (5.c) will be easy to satisfy. Using a Kodaira-type vanishing theorem, (5.a) needs some semipositivity condition on $(m-1)K_{X'}+\lfloor m\Delta'\rfloor$. By contrast, (11.61) suggests that (5.b) needs some negativity condition on $mK_{X'}+\lfloor m\Delta'\rfloor$.

The next result grew out of trying to satisfy the assumptions of both the relative Kodaira-type vanishing theorem and (11.61). The proof of (2.83) is then given in (2.85).

Proposition 2.84 *Let $(X, S + \Delta)$ be an lc pair where S is $\mathbb{Q}$-Cartier. Let B be a Weil $\mathbb{Z}$-divisor that is Mumford along S (4.68) and Θ an effective $\mathbb{R}$-divisor such that*

(2.84.1) $B \sim_{\mathbb{R}} -\Theta$,

(2.84.2) $\operatorname{Supp}\Theta \le \operatorname{Supp}(\Delta^{(>1/2)})$, *and*

(2.84.3) $\lfloor\Theta\rfloor \le \lfloor\Delta\rfloor$.

Then $\mathscr{O}_X(B)$ is S_3 along S.

Proof Assume first that $\lfloor\Theta\rfloor = 0$. A suitable cyclic cover, as in (11.25), reduces us to the case when S is Cartier. We assume this from now on.

(X, Δ) is also an lc pair and none of its lc centers are contained in S by (11.10.7). If B is $\mathbb{Q}$-Cartier then $\mathscr{O}_X(B)$ is S_3 along S by (11.20), applied with $\Delta' = 0$.

If B is not $\mathbb{Q}$-Cartier, we use (11.32) to obtain $\pi : (X', S' + \Delta') \to (X, S + \Delta)$.

Note that B' is $\mathbb{Q}$-Cartier by (11.32.1), $(X', S' + \Delta')$ is lc and none of the lc centers of $(X', S' + \Delta' - \varepsilon\Theta')$ are contained in $\operatorname{Ex}(\pi)$. In particular, S' is smooth at the generic points of all exceptional divisors of $\pi_S := \pi|_{S'} : S' \to S$. Thus B' is also Mumford along S', hence, as we proved at the beginning, $\mathscr{O}_{X'}(B')$ is S_3 along S'. Thus the sequence

$$0 \to \mathscr{O}_{X'}(B' - S') \to \mathscr{O}_{X'}(B') \to \mathscr{O}_{S'}(B'|_{S'}) \to 0 \tag{2.84.4}$$

is exact by (2.74). Since $R^1\pi_*\mathscr{O}_{X'}(B') = 0$ by (11.32.5), pushing (2.84.4) forward and using (11.32.4) gives an exact sequence

$$0 \to \mathscr{O}_X(B - S) \to \mathscr{O}_X(B) \to (\pi_S)_*\mathscr{O}_{S'}(B'|_{S'}) \to 0. \tag{2.84.5}$$

Again by (2.74), $\mathscr{O}_X(B)$ is S_3 along S iff $(\pi_S)_*\mathscr{O}_{S'}(B'|_{S'})$ is S_2. The latter is equivalent to

$$(\pi_S)_*\mathscr{O}_{S'}(B'|_{S'}) \stackrel{?}{=} \mathscr{O}_S(B|_S). \tag{2.84.6}$$

Now we apply (11.61) with $-N := B'|_{S'} + \Theta'|_{S'}$, which is numerically π_S-trivial. This gives that

$$(\pi_S)_*\mathscr{O}_{S'}(B'|_{S'} + \lfloor\Theta'|_{S'}\rfloor) = \mathscr{O}_S(B|_S). \tag{2.84.7}$$

We are done if $\lfloor\Theta'|_{S'}\rfloor = 0$. This is where assumption (2) enters, in a seemingly innocent way. Indeed, (2.81.3) guarantees that $\lfloor\Theta'|_{S'}\rfloor = \lfloor\Theta'\rfloor|_{S'} = 0$ and $\lfloor\Theta'\rfloor = 0$ by our assumption (3).

The proof is similar if $\lfloor\Theta\rfloor \ne 0$, see Kollár (2018a, prop.28). □

2.85 (Proof of 2.83) We may assume that X is affine and $K_X + \Delta \sim_{\mathbb{R}} 0$. Pick a fiber X_c and let $x \in X_c$ be a point of codimension 1. Then either X_c and X are

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both smooth at x or X_c has a node and $x \notin \operatorname{Supp}\Delta$. Thus $mK_X + \lfloor m\Delta \rfloor$ is Cartier at x, hence a general divisor $B \sim mK_X + \lfloor m\Delta \rfloor$ is Mumford along X_c.

We apply (2.84) to B with $\Theta := m\Delta - \lfloor m\Delta \rfloor$. Thus

$$B \sim mK_X + \lfloor m\Delta \rfloor = m(K_X + \Delta) - \Theta \sim_{\mathbb{R}} -\Theta.$$

By assumption $\Theta \le \lceil \Delta^{(>1/2)} \rceil = \operatorname{Supp}\Delta$. So the assumptions of (2.84) are satisfied and $\mathscr{O}_X(mK_X + \lfloor m\Delta \rfloor) \simeq \mathscr{O}_X(B)$ is S_3 along X_c. By (2.75) this implies (2.83). □

2.8 Local Stability in Codimension ≥ 3

In this section, we prove (2.7). If $K_X + D + \Delta$ is $\mathbb{R}$-Cartier, then (11.17) implies that f is locally stable. The $\mathbb{R}$-divisor case is reduced to the $\mathbb{Q}$-divisor case using (11.47). So from now on we may assume that Δ is a $\mathbb{Q}$-divisor. We need to show that $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier.

We discuss three, increasingly general cases. The last one, treated in (2.88.5), implies (2.7).

2.86 Using the notation of (2.7), assume also that $(X_c, \operatorname{Diff}_{X_c}\Delta)$ is slc. (This holds for flat families of stable pairs.)

After localizing at a generic point of Z_c, we may assume that $Z_c = \{x\}$ is a point. Thus there is an $m > 0$ such that $m(K_X + D + \Delta)$ is a Cartier divisor on $X \setminus \{x\}$, whose restriction to $X_c \setminus \{x\}$ extends to a Cartier divisor on X_c. Since $\operatorname{codim}_{X_c}\{x\} \ge 3$, (2.91) implies that $m(K_X + D + \Delta)$ is a Cartier divisor.

2.87 Here we assume (2.7.2) and apply (11.42) to $(\bar{X}_c, \operatorname{Diff}_{\bar{X}_c}\Delta) \to X_c$. The conclusion is that there is an slc pair (X'_c, Δ'_c) and a finite morphism $\tau\colon X'_c \to X_c$, that is an isomorphism over $X_c \setminus Z_c$.

If X_c is S_2, then $X'_c \simeq X_c$, so $(X_c, \operatorname{Diff}_{X_c}\Delta)$ is slc, as in (2.86).

If X_c is not S_2, then, after localizing, we may assume that τ is an isomorphism, except at a point $x \in X_c$. Since $\tau^{-1}(x) \subset X'_c$ is finite, $m(K_{X'_c} + \operatorname{Diff}_{X'_c}\Delta)$ is trivial in a neighborhood of $\tau^{-1}(x)$ for some $m > 0$. Thus $m(K_{X_c} + \operatorname{Diff}_{X_c}\Delta)$ is trivial in a punctured neighborhood of x.

As before, $m(K_X + D + \Delta)$ is in the kernel of $\operatorname{Pic}^{\mathrm{loc}}(x, X) \to \operatorname{Pic}^{\mathrm{loc}}(x, X_c)$, but (2.91) guarantees its triviality only if $\operatorname{depth}_x X_c \ge 2$.

If $\operatorname{depth}_x X_c = 1$, then typically the kernel of $\operatorname{Pic}^{\mathrm{loc}}(x, X) \to \operatorname{Pic}^{\mathrm{loc}}(x, X_c)$ is a positive dimensional vector space; see Bhatt and de Jong (2014, 1.14) and Kollár (2016a, thm.7) for precise statements. Thus the kernel is p-torsion in char $p > 0$, but torsion free in char 0.

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It is better to discuss this case in the more general setting of the following conjecture, where X_c is replaced by D.

Conjecture 2.88 *Let $(X, D + \Delta)$ be a demi-normal pair, where D is a reduced, $\mathbb{Q}$-Cartier divisor that is demi-normal in codimension 1, whose normalization $(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ is lc. Let $W \subset X$ be a closed subset such that $\operatorname{codim}_D(W \cap D) \geq 3$ and $(X \setminus W, (D + \Delta)_{X \setminus W})$ is slc. Then the following are equivalent:*

(2.88.1) *$(X, D + \Delta)$ is slc in a neighborhood of D.*

(2.88.2) *$(D, \operatorname{Diff}_D \Delta)$ is slc.*

(2.88.3) *$(\bar{D}, \operatorname{Diff}_{\bar{D}} \Delta)$ is lc.*

The main difference between (2.7) and (2.88) is that in the latter we do not assume that $K_X + D + \Delta$ is $\mathbb{R}$-Cartier on $X \setminus D$.

Known implications Note that (1) ⇒ (2) ⇒ (3) follow from (11.17). If $K_X + D + \Delta$ is $\mathbb{R}$-Cartier, then (3) ⇒ (1) also follows from (11.17). The arguments of (2.87) show that (3) ⇒ (2) if D is S_2.

Thus it remains to show that if (3) holds, then $K_X + D + \Delta$ is $\mathbb{R}$-Cartier. As here, the $\mathbb{R}$-divisor case is reduced to the $\mathbb{Q}$-divisor case using (11.47), so from now on we assume that Δ is a $\mathbb{Q}$-divisor.

Special case 2.88.5 Assume that $K_X + D + \Delta$ is $\mathbb{R}$-Cartier on $X \setminus D$. Applying (5.41) gives a small, birational morphism $f\colon Y \to X$ such that $D_Y := f^{-1}(D) \to D$ is birational, $f(\operatorname{Ex}(f)) \subset D$, and it has codimension ≥ 3. There are two ways to get a contradiction from this.

First, note that the relative canonical divisor of $D_Y/\bar{D}$ is ample and is supported on the exceptional divisor of $D_Y \to D$. This cannot happen by (2.90).

Second, we use reduction to char p as in Bhatt and de Jong (2014) or Kollár and Mori (1998, p.14). In char $p > 0$, Bhatt and de Jong (2014, 1.14) shows that $K_{X_p} + D_p + \Delta_p$ is $\mathbb{Q}$-Cartier. By itself, this does not imply that $K_X + D + \Delta$ is $\mathbb{Q}$-Cartier.

However, if $K_X + D + \Delta$ is not $\mathbb{Q}$-Cartier, then $f\colon Y \to X$ is not an isomorphism. So $f_p\colon Y_p \to X_p$ is also not an isomorphism. By (2.89) this implies that $K_{X_p} + D_p + \Delta_p$ is not $\mathbb{Q}$-Cartier, a contradiction. □

Special case 2.88.6 Assume that we are in a situation where the conclusion of (5.41) holds and X is a variety over a field of char 0.

As before, (5.41) gives a small, birational morphism $f\colon Y \to X$ such that $f(\operatorname{Ex}(f)) \cap D$ has codimension ≥ 3. The relative canonical divisor of $D_Y/\bar{D}$ is ample and is supported on the exceptional divisor of $D_Y \to D$. This gives a contradiction using (2.90). □

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Lemma 2.89 *Let* $\pi\colon Y \to X$ *be a proper birational morphism of normal schemes. Assume that* $Z := \operatorname{Ex}(\pi) \subset Y$ *has codimension* ≥ 2*. Let* M_Y *be a* π*-ample line bundle on* Y *and* M_X *a line bundle on* X *such that* $M_Y|_{Y\setminus Z} \simeq \pi^* M_X|_{Y\setminus Z}$*. Then* π *is an isomorphism.*

Proof Since Z has codimension ≥ 2, the assumed isomorphism extends to $M_Y \simeq \pi^* M_X$. If π contracts any curve C, then $0 < (C \cdot M_Y) = (C \cdot \pi^* M_X) = 0$ gives a contradiction. □

We have used the following two theorems. The methods of the proofs would take us in other directions, so we give only some comments and references.

Proposition 2.90 (Kollár, 2013a, Prop.22) *Let* $f\colon Y \to X$ *be a projective, birational morphism of varieties over a field of* char 0*. Let* $D \subset X$ *be a Cartier divisor. Assume that* $f^{-1}(D) \to D$ *is birational and there is a nonzero (but not necessarily effective)* $\mathbb{Q}$*-Cartier divisor* E *on* $f^{-1}(D)$ *such that* $\dim f(\operatorname{Supp} E) \leq \dim D - 3$*. Then* $\operatorname{Ex}(f)$ *has codimension 1 in* Y*.*

Outline of proof The argument is topological over $\mathbb{C}$. Since the claim is algebraic, it would be very good to find a proof that works for arbitrary schemes.

We may assume that $x := f(\operatorname{Supp} E)$ is a point. Let V denote an open neighborhood of $f^{-1}(x) \subset f^{-1}(D)$ that retracts to $f^{-1}(x)$. The assumptions imply that, for $n := \dim D$, the cup product pairing

$$H^2(V, \partial V, \mathbb{Q}) \times H^{2n-2}(V, \mathbb{Q}) \to H^{2n}(V, \partial V, \mathbb{Q}) \tag{2.90.1}$$

is nonzero. If $\operatorname{Ex}(f)$ has codimension ≥ 2, then f is small over a small deformation of D. This can be used to compute that (2.90.1) is zero, giving a contradiction. □

We have the following Grothendieck–Lefschetz-type theorem, where, for a pointed scheme (x, X), we set $\operatorname{Pic}^{\mathrm{loc}}(x, X) := \operatorname{Pic}(\operatorname{Spec}_X \mathscr{O}_{x,X} \setminus \{x\})$.

Theorem 2.91 *Let* $(x \in X)$ *be an excellent, local scheme of pure dimension* ≥ 4 *such that* $\operatorname{depth}_x \mathscr{O}_X \geq 3$*. Let* $x \in D \subset X$ *be a Cartier divisor. Then we have an injective restriction map*

$$r^X_D\colon\ \operatorname{Pic}^{\mathrm{loc}}(x, X) \hookrightarrow \operatorname{Pic}^{\mathrm{loc}}(x, D). \tag{2.91.1}$$

The original version (Grothendieck, 1968, XI.3.16), applies if $\operatorname{depth}_x \mathscr{O}_X \geq 4$. The current form was conjectured in Kollár (2013a) and proved there in the lc case. After Bhatt and de Jong (2014) and Kollár (2016a), the most general version is (Stacks, 2022, tag 0F2B).

The next results are very useful when dealing with Cartier divisors.

2.92 (Flat maps and Cartier divisors) Let $p\colon X \to Y$ be a morphism and D an effective Cartier divisor on Y. Under mild conditions p^*D is an effective Cartier divisor on Y. The converse also holds for flat morphisms.

Claim 2.92.1 Let $(R, m_R) \to (S, m_S)$ be a flat extension of local rings and $I_R \subset R$ an ideal. Then I_R is principal iff I_RS is principal.

Proof One direction is clear. Conversely, assume that I_RS is principal, thus $I_RS/m_SI_RS \simeq S/m_S$. Let $r_1, \dots, r_n$ be generators of I_R. They also generate I_RS, hence at least one of them, say r_1, is not contained in m_SI_RS. Thus $(r_1) \subset I_R$ is a sub-ideal such that $r_1S = I_RS$. Since $(R, m_R) \to (S, m_S)$ is faithfully flat, this implies that $(r_1) = I_R$. □

Pushing forward Cartier divisors is more problematic. For example, consider the natural map $\mathbb{P}^1_{\mathbb{Q}(i)} \to \mathbb{P}^1_{\mathbb{Q}}$. The points (1:1) and ($i$:1) are linearly equivalent, but their scheme-theoretic images have different degrees.

It is better to work with line bundles. Let $\pi\colon X \to Y$ be a finite, flat morphism of degree d. Let L be a line bundle on X. There are two natural ways of getting a line bundle on Y: the determinant of π_*L and the *norm,* denoted by $\operatorname{norm}_{X/Y}(L)$ as in Stacks (2022, tag 0BCX). The two are related by

$$\det(\pi_*L) \simeq (\operatorname{norm}_{X/Y} L) \otimes_Y \det(\pi_*\mathscr{O}_X).$$

The norm gives a group homomorphism $\operatorname{norm}_{X/Y}\colon \operatorname{Pic}(X) \to \operatorname{Pic}(Y)$ and there is a natural isomorphism $\operatorname{norm}_{X/Y}(\pi^*M) \simeq M^d$ for any line bundle M on Y.

Lemma 2.93 (Grothendieck, 1968, XIII.2.1) *Let $(x \in X)$ be a Noetherian, local scheme and $x \in D \subset X$ the support of a Cartier divisor. Assume that $X \setminus Z$ is connected for every closed subset Z of dimension $\le i + 1$. Then $D \setminus Z$ is connected for every closed subset Z of dimension $\le i$.* □

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