FREE LATTICES GENERATED BY PARTIALLY ORDERED SETS AND PRESERVING BOUNDS

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1. Introduction. A construction of the free lattice generated by a partially ordered set P and preserving every least upper bound (lub) and greatest lower bound (glb) of pairs of elements existing in P has been given by Dilworth **(2**, pp. 124–129**)** and, when P is finite, by Gluhov **(5)**.

The results presented here construct the free lattice $\mathrm{FL}(P; \mathfrak{U}, \mathfrak{L})$ generated by the partially ordered set P and preserving

(1) the ordering of P,

(2) those lub's of a family $\mathfrak{U}$ of finite subsets of P which possess lub's in P, and

(3) those glb's of a family $\mathfrak{L}$ of finite subsets of P which possess glb's in P.

$\mathfrak{U}$ and $\mathfrak{L}$ may be chosen so that $\mathrm{FL}(P; \mathfrak{U}, \mathfrak{L})$ becomes the free lattice of Dilworth or the completely free lattice of Dean **(1)** or to provide a new solution for the word problem in finitely presented lattices. The lattice $\mathrm{FL}(P, \mathfrak{U}, \mathfrak{L})$ is obtained as a collection of equivalence classes of lattice words on the elements of P. The equivalence classes arise from an ordering defined on the words. An algorithm is given for determining when two words are comparable. This algorithm is finitistic under suitable conditions on P, $\mathfrak{U}$, and $\mathfrak{L}$; conditions which can be expressed as a decision problem for a class of ideals in P. This latter decision problem has an affirmative solution for a class of partially ordered sets which properly includes finite sets and those which have no non-trivial bounds to be preserved. The ordering may be considered as a natural extension of the techniques employed by Whitman **(7)**.

2. Construction of $\mathrm{FL}(P; \mathfrak{U}, \mathfrak{L})$**.** Let P be a partially ordered set with elements $p, q, \ldots$ and order relation $(<)$. A lattice word A on the elements of P and its length $\lambda(A)$ are defined as usual by recursion:

(i) An element of P standing alone is a word of length one.

(ii) If A and B are words, then the symbols $A \vee B$ and $A \wedge B$ are words each of length $\lambda(A) + \lambda(B)$.

Let $\mathfrak{U}$ be a family of subsets of P such that the following conditions hold:

$\mathfrak{U}1$. If $p < q$ in P, then $\{p, q\} \in \mathfrak{U}$.

$\mathfrak{U}2$. If $S \in \mathfrak{U}$, then S possesses a lub, a_S, in P.

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Dually, let $\mathfrak{L}$ be a family of subsets of P such that the following hold:

$\mathfrak{L}1$. If $p < q$ in P, then $\{p, q\} \in \mathfrak{L}$.

$\mathfrak{L}2$. If $S \in \mathfrak{L}$, then S possesses a glb, b_S, in P.

We wish to define an ordering $(\leqslant)$ on the lattice words which preserves the bounds of the subsets $\mathfrak{U}$ and $\mathfrak{L}$. Note that the ordering of P is incorporated in $\mathfrak{U}$ and $\mathfrak{L}$, but that otherwise $\mathfrak{U}$ and $\mathfrak{L}$ are arbitrary. It will turn out (Theorem 5) that we can guarantee to preserve only the bounds of the finite subsets in $\mathfrak{U}$ and $\mathfrak{L}$.

The definition of the relation $A \leqslant B$ is in three parts. First the case in which $\lambda(B) = 1$, second $\lambda(A) = 1$, and finally the case in which $\lambda(A) > 1$ and $\lambda(B) > 1$. The symbol $(\equiv)$ occurring in the definitions denotes logical identity.

DEFINITION 1. *$A \leqslant p$ if and only if, for some integer n, $A \leqslant p(n)$ where:*

(1.1) *$A \leqslant p(0)$ if and only if $A \equiv q$ and $q < p$ in P.*

Proceeding inductively, $A \leqslant p(n)$ if and only if one or more of the following hold:

(1.2) $A \equiv A_1 \vee A_2$ and *$A_i \leqslant p(n-1)$ for $i = 1$ and 2.*

(1.3) *$A \equiv A_1 \wedge A_2$ and $A_i \leqslant p(n-1)$ for $i = 1$ or 2.*

(1.4) *There exists a set $S \in \mathfrak{L}$ such that $b_S < p$ in P and $A \leqslant s(n-1)$ for all $s \in S$.*

DEFINITION 2. *$p \leqslant A$ if and only if, for some integer n, $p \leqslant A(n)$ where:*

(2.1) *$p \leqslant A(0)$ if and only if $A \equiv q$ and $p < q$ in P.*

Proceeding inductively, $p \leqslant A(n)$ if and only if one or more of the following hold:

(2.2) *$A \equiv A_1 \wedge A_2$ and $p \leqslant A_i(n-1)$ for $i = 1$ and 2.*

(2.3) *$A \equiv A_1 \vee A_2$ and $p \leqslant A_i(n-1)$ for $i = 1$ or 2.*

(2.4) *There exists a subset S in $\mathfrak{U}$ such that $p < a_S$ in P and $s \leqslant A(n-1)$ for all $s \in S$.*

DEFINITION 3. *If $\lambda(A) > 1$ and $\lambda(B) > 1$, then $A \leqslant B$ if and only if one or more of the following hold:*

(3.1) *$A \equiv A_1 \vee A_2$ and $A_i \leqslant B$ for $i = 1$ and 2,*

(3.2) *$A \equiv A_1 \wedge A_2$ and $A_i \leqslant B$ for $i = 1$ or 2,*

(3.3) *$B \equiv B_1 \wedge B_2$ and $A \leqslant B_i$ for $i = 1$ and 2,*

(3.4) *$B \equiv B_1 \vee B_2$ and $A \leqslant B_i$ for $i = 1$ or 2,*

(3.5) *$A \equiv A_1 \wedge A_2$, $B \equiv B_1 \vee B_2$, and for some $p \in P$, $A \leqslant p$ and $p \leqslant B$.*

The most difficult step in establishing that the relation $(\leqslant)$ is a partial ordering of the lattice words is to prove its transitivity. This is Theorem 4; the preceding theorems are either necessary properties of any lattice or of any lattice in which P is embedded. We shall henceforth omit stating dual

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lemmas and theorems. In these results we tacitly assume the words are on the elements of a partially ordered set P with $\mathfrak{U}$ and $\mathfrak{L}$ families of subsets as defined.

We begin with an easy observation which we label for future reference.

LEMMA 1. *If $A \leqslant p(n)$ and $p \prec q$, then $A \leqslant q(m)$ when $m \geqslant n$.*

Proof. It clearly suffices to prove this in the case $m = n + 1$. To do this we form the singleton set $\{p\}$ which belongs to $\mathfrak{L}$ by definition and apply Definition (1.4).

THEOREM 1. *If $q \leqslant p$ as lattice words, then $q \prec p$ in P.*

Proof. Using Definition 1 or Definition 2, the theorem may be restated in the form: "If $q \leqslant p(n)$, then $q \prec p$ in P," which we prove by induction on n. If $n > 0$, from the form of Definition 1 or 2, only (1.4) or (2.4) is applicable. If (1.4) holds we have, for some $S \in \mathfrak{L}$, $b_S \prec p$ and $q \leqslant s(n-1)$ for all $s \in S$. By induction $q \prec s$ in P for all $s \in S$. Hence $q \prec \operatorname{glb} S = b_S$, and so the transitivity of $(\prec)$ yields $q \prec p$.

THEOREM 2. *If $A_1 \vee A_2 \leqslant B$, then $A_i \leqslant B$ for $i = 1$ and 2.*

Proof. The proof is by induction on $\lambda(B)$. For $\lambda(B) = 1$ we prove the stronger statement: "If $A_1 \vee A_2 \leqslant p(n)$, then $A_i \leqslant p(n-1)$," by induction on n. If $A_1 \vee A_2 \leqslant p(n)$ by Definition (1.2), there is nothing to prove. If $A_1 \vee A_2 \leqslant p(n)$ holds by Definition (1.4), then $A_1 \vee A_2 \leqslant s(n-1)$ for all $s \in S$. Since this cannot hold when $n = 1$, the statement holds for $n = 1$. If $n > 1$, then by induction we have $A_i \leqslant s(n-2)$ and hence by Definition (1.4), $A_i \leqslant p(n-1)$ for $i = 1$ and 2.

Now if $\lambda(B) > 1$ and $B \equiv B_1 \vee B_2$, then either Definition (3.1) or (3.4) holds. The former is the desired conclusion. The latter, with induction, yields, for example, $A_i \leqslant B_1$ for $i = 1$ and 2. Thus, from (3.4), we have $A_i \leqslant B$ for $i = 1$ and 2. If $B \equiv B_1 \wedge B_2$, then either (3.1) or (3.3) holds. In the latter case induction yields $A_i \leqslant B_j$, all i and j. Hence, by (3.3) $A_i \leqslant B$ for $i = 1$ and 2.

THEOREM 3. *$A \geqslant A$ for all words A.*

Proof. The proof is by induction on $\lambda(A)$. When $\lambda(A) = 1$, say $A \equiv p$, we have $p \succ p$, hence $p \geqslant p(0)$, or $p \geqslant p$. If $A \equiv A_1 \vee A_2$, by induction we have $A_i \geqslant A_i$, hence by (2.3) or (3.4), whichever is applicable, we conclude that $A_1 \vee A_2 \geqslant A_i$ for $i = 1$ and 2. Finally by (3.1) it follows that $A \geqslant A$. A dual argument holds when $A \equiv A_1 \wedge A_2$.

THEOREM 4. *The relation $(\leqslant)$ is transitive.*

Proof. Let $C \leqslant B$ and $B \leqslant A$. We shall prove $C \leqslant A$ by an induction on $\lambda = \lambda(A) + \lambda(B) + \lambda(C)$. If $\lambda = 3$, then $A \equiv p$, $B \equiv q$, and $C \equiv r$, and using Theorem 1, transitivity in this case follows from the transitivity of

$(\prec)$. If $C \equiv C_1 \vee C_2$ or $A \equiv A_1 \wedge A_2$, then transitivity follows by invoking Theorem 2 or its dual and induction. Hence we suppose hereafter that $\lambda(C) = 1$ or $C \equiv C_1 \wedge C_2$ and dually, $\lambda(A) = 1$ or $A \equiv A_1 \vee A_2$.

Case 1. $\lambda(A) = \lambda(B) = 1$. We prove the stronger statement: "If $C \leqslant q(n)$ and $q \leqslant p$, then $C \leqslant p(n)$," by induction on n. If $n = 0$, then $C \equiv r$, an element of P. From Theorem 1 and the transitivity of $\prec$ the result holds. If $C \equiv C_1 \wedge C_2$ and $C_i \leqslant q(n-1)$, then by induction $C_i \leqslant p(n-1)$, and so the statement holds. Suppose $C \equiv C_1 \wedge C_2 \leqslant q(n)$ holds by Definition (1.4); that is, for some $S \in \mathfrak{L}$, $C \leqslant s(n-1)$ for all $s \in S$, and $b_S \prec q$. Since $q \leqslant p$, $q \prec p$ and so $b_S \prec p$. But now all the conditions for Definition (1.4) are satisfied with respect to C and p; hence $C \leqslant p(n)$. Thus the statement is proved and Theorem 1 holds in Case 1.

Case 2. $\lambda(B) = 1 (B \equiv p)$, $C \equiv C_1 \wedge C_2$, and $A \equiv A_1 \vee A_2$. Transitivity follows in this case by Definition (3.5).

Case 1 and its dual and Case 2 complete the cases when $\lambda(B) = 1$. The cases in which $B \equiv B_1 \vee B_2$ will be treated next; the cases in which $B \equiv B_1 \wedge B_2$ follow by duality.

Case 3. $C \equiv p$ and $B \equiv B_1 \vee B_2$. We prove the stronger statement: "If $p \leqslant B_1 \vee B_2(n)$ and $B \leqslant A$, then $p \leqslant A(n)$," by induction on n. In any event $B_i \leqslant A$ for $i = 1$ and 2 by Theorem 2. The statement is vacuously true if $n = 0$. If $p \leqslant B_i(n-1)$, then, as $B_j \leqslant A$, it follows by induction that $p \leqslant A(n)$. Suppose $p \leqslant B_1 \vee B_2(n)$ holds by Definition (2.4); that is for some $S \in \mathfrak{U}$, $s \leqslant B(n-1)$ for all $s \in S$ and $p \prec a_S$. By induction, $s \leqslant A(n-1)$ for all $s \in S$, and by Definition (2.4), $p \leqslant A(n)$.

Case 4. $C \equiv C_1 \wedge C_2$ and $B \equiv B_1 \vee B_2$. If $C \leqslant B$ holds by (3.2), then for $i = 1$ or 2, $C_i \leqslant B$ and induction on λ yields $C_i \leqslant A$ and so by (3.2), $C \leqslant A$. If $C \leqslant B$ holds by (3.4), then $C \leqslant B_i$ for $i = 1$ or 2. Since $B_i \leqslant A$ for $i = 1$ and 2, an induction on λ yields $C \leqslant A$. Finally, if $C \leqslant B$ holds by (3.5), that is, there is a $p \in P$ such that $C \leqslant p$ and $p \leqslant B$, then by Case 3 it follows that $p \leqslant A$. Hence if $A \equiv A_1 \vee A_2$, it follows from (3.5) that $C \leqslant A$. The only other possibility under our overriding assumptions is that $A \equiv q$. Thus $C \leqslant p$ and $p \leqslant q$; hence by Case 1, $C \leqslant q$ follows.

These exhaust the cases in which $B \equiv B_1 \vee B_2$, and, as we observed above, dual arguments complete the proof of the theorem.

DEFINITION 4. *For lattice words A and B, $A = B$ if and only if $A \leqslant B$ and $B \leqslant A$.*

The relation $(=)$ of Definition 4 is clearly an equivalence relation and the resulting equivalence classes are made into a partially ordered set in the standard fashion. In what follows we shall ignore the classes and refer directly to their representatives, the lattice words.

THEOREM 5. *The lattice words on the elements of P partially ordered by $(\leqslant)$ form a lattice* $\mathrm{FL}(P; \mathfrak{U}, \mathfrak{L})$ *in which a* lub *or* glb *of a finite set of elements S is preserved if $S \in \mathfrak{U}$ or $S \in \mathfrak{L}$ respectively. In particular P is embedded in this lattice.*

Proof. It is easily verified that $A \vee B$ and $A \wedge B$ are the lub and glb respectively of the words A and B under $(\leqslant)$. Second, $p < q$ in P if and only if $p \leqslant q$ in $\mathrm{FL}(P; \mathfrak{U}, \mathfrak{L})$ by Theorem 1. Finally, suppose that for a finite $S \in \mathfrak{U}$, S has a lub, a_S, in P. We shall prove that a_S is the lub of S in the free lattice. Clearly $a_S \geqslant s$ for all $s \in S$. Now suppose that A is any word such that $A \geqslant s$, for all $s \in S$. Since S is finite, we may, in view of Lemma 1, suppose that $A \geqslant s(n)$ for all $s \in S$. But now the condition of Definition (2.4) holds with respect to a_S and A; hence $A \geqslant a_S(n + 1)$ and so $A \geqslant a_S$ in the free lattice. Thus a_S is the lub of S and it is easy to show that if $S = \{s_1, \ldots, s_m\}$ then $a_S = (\ldots(s_1 \vee s_2) \vee s_3)\ldots) \vee s_m$. Dually the glb's are preserved.

In general it is not the case that the bounds of infinite sets are preserved. To find such an example we need only construct a partially ordered set with an infinite sequence $\mathfrak{C} = \{c_1, c_2, \ldots\}$ possessing a lub, p, and enough other elements so that there is a word W satisfying $c_1 \leqslant W(i)$ but not $c_i \leqslant W(i - 1)$ and yet so that $p \leqslant W$ is false. Thus in the free lattice W and p become upper bounds for $\mathfrak{C}$. Such a partially ordered set is pictured in Figure 1.

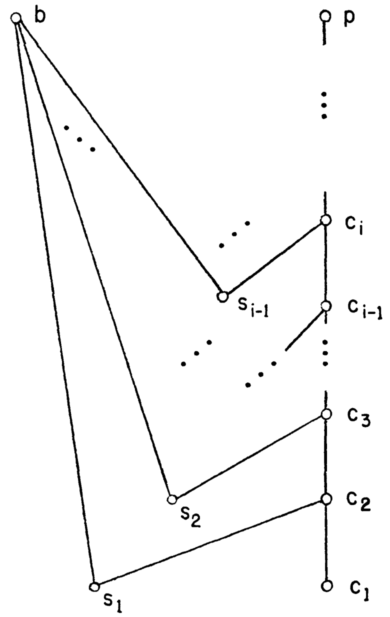

FIGURE 1

In the example P of Figure 1, let $\mathfrak{U}$ consist of all the singleton sets $\{x\}$ for all $x \in P$ and as well the sets

$$\{s_i, c_j\}, \{c_i, c_j\}, \{s_i, p\}, \{c_i, p\}, \{s_i, b\}, \{c_1, c_2, \ldots\}$$

for $i \leqslant j$ and $i, j = 1, 2, \ldots$. The lub's of these sets are to be taken as shown in the figure. $\mathfrak{L}$ is to be taken arbitrarily, except for condition $\mathfrak{L}1$. It is easily verified that $c_i \leqslant c_1 \vee b(i)$. It is also true that for $i \geqslant 1$, $c_i \leqslant c_1 \vee b(i-1)$ is false. To prove this by induction note that $c_i \leqslant b$ and $c_i \leqslant c_1$ are false; hence if $c_i \leqslant c_1 \vee b(i-1)$, it must be the case that $c_i \leqslant a_S$ with $a_S = \text{lub}\, S$ and $x \leqslant c_1 \vee b(i-2)$ for all $x \in S$. If $x = p$ or $x = c_k$ with $k > i - 2$, then by the dual of Lemma 1, $c_{i-1} \leqslant c_1 \vee b(i-2)$, a contradiction. Thus the members of S must be among the following: $b, s_j, j = 1, 2, \ldots$, or c_k with $k \leqslant i - 2$. However, there is no set S in $\mathfrak{U}$ with these members and possessing a lub a_S containing c_i. In a similar fashion it is easy to see that $p \leqslant c_1 \vee b$ does not hold. Thus p is not the lub of $\mathfrak{C}$ in $\text{FL}(P; \mathfrak{U}, \mathfrak{L})$.

The next theorem demonstrates the free character of our lattice.

THEOREM 6. *Let P be a partially ordered set and let $\mathfrak{U}$ and $\mathfrak{L}$ be families of subsets satisfying conditions $\mathfrak{U}1$, $\mathfrak{U}2$, and $\mathfrak{L}1$, $\mathfrak{L}2$ respectively, and let L be a lattice. Let ϕ be a mapping of P onto L such that*

(i) *if $S \in \mathfrak{U}$, then in L, $\phi(a_S) = \text{lub}\{\phi(s) | s \in S\}$;*

(ii) *if $S \in \mathfrak{L}$, then in $\mathfrak{L}$, $\phi(b_S) = \text{glb}\{\phi(s) | s \in S\}$;*

(iii) *the smallest sublattice of L containing $\phi(P)$ is L.*

Then ϕ can be extended to a homomorphism of $\text{FL}(P; \mathfrak{U}, \mathfrak{L})$ *onto L.*

Proof. For brevity, let $P' = \phi(P)$ be the range of ϕ. The words on P' clearly form a sublattice of L, which by hypothesis (iii) must be all of L. Hence the natural extension of ϕ to $\text{FL}(P; \mathfrak{U}, \mathfrak{L})$ obtained by replacing every occurrence of an element p in a word $w(P)$ by $\phi(p)$ to obtain an element $w(P')$ of L yields a mapping of the words on P onto L. We denote the mapping simply by

$$\phi : w(P) \to w(P').$$

To prove that this mapping is a lattice homomorphism we first show that ϕ preserves the ordering. (We use ($\geqslant$) to denote the ordering in L.) We prove that $w(P) \geqslant v(P)$ implies $w(P') \geqslant v(P')$ by induction on

$$\lambda = \lambda[w(P)] + \lambda[v(P)].$$

If $\lambda = 2$, then $w(P) \equiv p$ and $v(P) \equiv q$ and $p \geqslant q$. Hence $p > q$ and $\{p, q\} \in \mathfrak{U}$ with its lub being p. By Condition (i), it follows that $\phi(p) \geqslant \phi(q)$ in L.

If $v(P) \equiv v_1(P) \vee v_2(P)$, then we have $w(P) \geqslant v_i(P)$ for $i = 1$ and 2 by Theorem 2, and so by induction we conclude that $w(P') \geqslant v(P')$. From this and a dual argument we consider henceforth only those cases where $v(P) \equiv v_1(P) \wedge v_2(P)$ or $v(P) \equiv q$ and $w(P) \equiv w_1(P) \vee w_2(P)$ or $w(P) \equiv p$.

Case 1. $v(P) \equiv q$. By Definition 2 this means that $w(P) \geqslant q(n)$ for some n. By induction we prove that this implies $w(P') \geqslant \phi(q)$. If $n = 0$, then $w(P) \equiv p$ and, as above, this entails $\phi(p) \geqslant \phi(q)$ in L.

Suppose that $w(P) \equiv w_1(P) \vee w_2(P)$. If $w_i(P) \geqslant q(n-1)$, then by the induction hypothesis of Case 1, $w_i(P') \geqslant \phi(q)$ in L; hence $w(P') \geqslant \phi(q)$. If, for some $S \in \mathfrak{U}$, $a_S > q$ in P and $s \leqslant w(P)(n-1)$ for all $s \in S$, then, by induction, $\phi(s) \leqslant w(P')$ in L. Hence, in L, $\text{lub}\{\phi(s) \mid s \in S\} \leqslant w(P')$. By Hypothesis (i) of this theorem, this lub is equal to $\phi(a_S)$. Thus $\phi(a_S) \leqslant w(P')$. Finally, $a_S > q$ implies $\phi(a_S) \geqslant \phi(q)$ in L and so $\phi(q) \leqslant w(P')$. This case and its dual complete the cases in which $\lambda[w(P)] = 1$ or $\lambda[v(P)] = 1$.

Case 2. $w(P) \equiv w_1(P) \vee w_2(P)$ and $v(P) = v_1(P) \wedge v_2(P)$. If $w(P) \geqslant v(P)$ is valid by virtue of (3.1), (3.2), (3.3), or (3.4), then an easy application of the induction hypothesis on λ yields $w(P') \geqslant v(P')$. If, for some $p \in P$, $w(P) \geqslant p$ and $p \geqslant v(P)$, then by the previous case and its dual we have $w(P') \geqslant \phi(p) \geqslant v(P')$ in L.

Since order of the lattice words is preserved under ϕ, it follows that equality is also, and so $w(P) \rightarrow w(P')$ is a well-defined mapping of the elements of $\text{FL}(P; \mathfrak{U}, \mathfrak{L})$ onto L. Thus if $u(P) \rightarrow u(P')$ and $v(P) \rightarrow v(P')$, then if $w(P) = u(P) \vee v(P)$, it follows that in L, $w(P') = u(P') \vee v(P')$, and so joins are preserved. Dually, meets are preserved and the proof is complete.

By specializing the choice of $\mathfrak{U}$ and $\mathfrak{L}$, various lattices may be obtained. For example, if $\mathfrak{U}$ and $\mathfrak{L}$ consist only of those pairs $\{p, q\}$ where $p < q$ in P, then the completely free lattice of Dean **(1)** is obtained. If $\mathfrak{U}$ consists precisely of those pairs $\{p, q\}$ which possess a lub in P and, dually, $\mathfrak{L}$ consists of those pairs $\{p, q\}$ which possess a glb, then the free lattice of Dilworth **(2)** is obtained. These remarks follow from the observations that in each case the elements of the lattices are represented by the same set of lattice words, and by Theorem 6, in each case, equality in $\text{FL}(P; \mathfrak{U}, \mathfrak{L})$ implies equality in the special lattices constructed. The converse implication is provided by similar theorems proved in **(1** and **2)** to guarantee the free nature of those lattices.

One further remark may help to clarify the problem. A partially ordered set P may be embedded in a lattice so that all existing lub's and glb's of pairs of elements which exist in P are preserved and yet the embedding need not preserve the lub's and glb's of all finite subsets which may exist in P. Figure 2 shows a partially ordered set P and two lattices. P is embedded in each lattice in such a way as to preserve all existing bounds of pairs of elements. In P the set $\{a, b, c\}$ has v as its least upper bound. In the first lattice this remains true; in the second it does not.

3. Ideals. We now extend the concept of ideals in lattices to ideals in partially ordered sets in a natural way which reflects the bounds in $\mathfrak{U}$ and $\mathfrak{L}$. (A similar notion of ideals in partially ordered sets was used by Worthie Doyle in *An arithmetical theorem for partially ordered sets*, Bull. Am. Math. Soc., *56* (1950), p. 366, Abstract 361.) In this section we use the symbols $\subset$, $\cup$, $\cap$ to denote, respectively, inclusion, union, and intersection of sets.

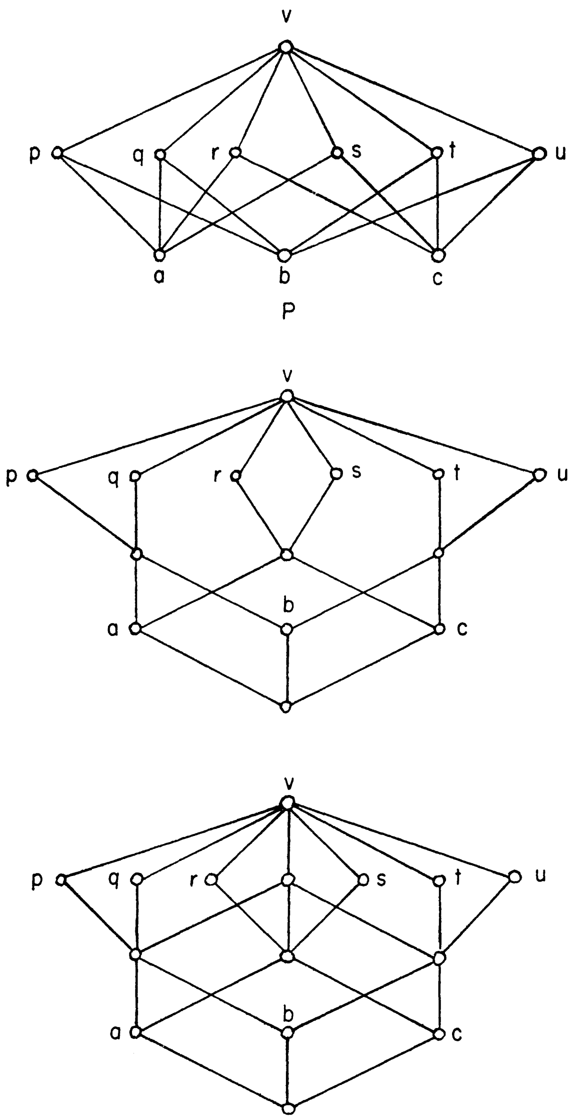

FIGURE 2

DEFINITION 6. *Let P be a partially ordered set and let $\mathfrak{U}$ and $\mathfrak{L}$ be collections of subsets of P satisfying the conditions $\mathfrak{U}1$, $\mathfrak{U}2$ and $\mathfrak{L}1$, $\mathfrak{L}2$, respectively, of Section* 2.

A subset J of P is called a $\mathfrak{U}$-ideal provided that whenever S is a finite subset of J and $S \in \mathfrak{U}$ and $p \prec \operatorname{lub} S = a_S$, then $p \in J$.

Dually a subset M is called an $\mathfrak{L}$-ideal provided that whenever S is a finite subset of J and $S \in \mathfrak{L}$ and $p \succ \operatorname{glb} S = b_S$, then $p \in M$.

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Note that these conditions imply that whenever $q < p$ and p belongs to the $\mathfrak{U}$-ideal J, then $q \in J$. With any lattice word on the elements of P we associate a $\mathfrak{U}$-ideal and an $\mathfrak{L}$-ideal.

DEFINITION 7. *Let W be a word on the elements of P:*

$$J(W) = \{p \mid p \in P \text{ and } p \leqslant W\},$$
$$M(W) = \{p \mid p \in P \text{ and } p \geqslant W\}.$$

It is an easy matter to verify that these are ideals. Let S be a finite subset of $J(W)$, $S \in \mathfrak{U}$ and $p \leqslant a_S$. For all $s \in S$ we have $s \leqslant W$ and, in particular, for some n, $s \leqslant W(n)$. By virtue of the finiteness of S and Lemma 1 we may assume that for some m, $s \leqslant W(m)$ for all $s \in S$. Hence $p \leqslant W(m + 1)$, or $p \in J(W)$. The significance of these definitions lies in the next theorem.

THEOREM 7. *In* FL$(P; \mathfrak{U}, \mathfrak{L})$, *$A \leqslant B$ if and only if one or more of the following hold:*

(1) $A \equiv A_1 \vee A_2$ *and* $A_i \leqslant B$ *for* $i = 1$ *and* 2;
(2) $A \equiv A_1 \wedge A_2$ *and* $A_i \leqslant B$ *for* $i = 1$ *or* 2;
(3) $B \equiv B_1 \wedge B_2$ *and* $A \leqslant B_i$ *for* $i = 1$ *and* 2;
(4) $B \equiv B_1 \vee B_2$ *and* $A \leqslant B_i$ *for* $i = 1$ *or* 2;
(5) $M(A)$ *and* $J(B)$ *have an element of* P *in common.*

Proof. To prove the sufficiency of (1) to (5), note that if $\lambda(A)$ and $\lambda(B)$ are both greater than one, then (1) to (5) are simply a restatement of Definition 3. If $\lambda(A) = 1$ or $\lambda(B) = 1$, then only (5) is applicable, whereupon the transitivity of $(\leqslant)$ yields the desired result. For example, if $A \equiv p$, then if $q \in M(p) \wedge J(B)$, we have $p \leqslant q$ and $q \leqslant B$, and hence $p \leqslant B$.

To prove the necessity, again, if $\lambda(A)$ and $\lambda(B)$ are both greater than one, necessity follows from Definition 3. If $\lambda(A) = 1$ or $\lambda(B) = 1$, then (5) is but a restatement of the hypothesis $A \leqslant B$.

If C is any set of $\mathfrak{U}$-ideals, then the set intersection $\bigcap C$ is a $\mathfrak{U}$-ideal. Since P is itself a $\mathfrak{U}$-ideal, it follows that the set of all $\mathfrak{U}$-ideals, ordered by set inclusion, forms a complete lattice, $L(P, \mathfrak{U})$. Similarly we may construct the lattice $L(P, \mathfrak{L})$ of all $\mathfrak{L}$-ideals. We shall employ the symbols $\triangle$ and ∇ to denote lattice meet and join in these lattices.

It is easy to see that the mapping of P into $L(P, \mathfrak{U})$ given by $p \to J(p)$ is an order isomorphism, and, moreover, one which preserves the bounds $\mathfrak{U}$. Thus if $s \in \mathfrak{U}$ we have $s \leqslant a_S$ for all s in S and so $J(s) \subset J(a_S)$. On the other hand if $J(s) \subset J(x)$, then $s \leqslant x$ and so if $J(x)$ is an upper bound for all $J(s)$, $s \in S$, it follows that $x \geqslant a_S = \text{lub}\, S$ in P; hence $J(x) \supset J(a_S)$. As the example of Figure 1 shows, we cannot guarantee that $J(a_S)$ is the lub of $\{J(s) \mid s \in S\}$ in $F(P, \mathfrak{U})$ unless S is finite. In this way we see that P is embedded in $L(P; \mathfrak{U})$. Dually, the mapping $p \to M(p)$ yields a dual embedding of P in $L(P; \mathfrak{L})$.

The lattice join in $L(P;\mathfrak{U})$ of a set of $\mathfrak{U}$-ideals can be described constructively. The description is but a recasting of Definition 2.

THEOREM 8. *If $\mathfrak{N}$ is a set of $\mathfrak{U}$-ideals, then*

$$\nabla\,\mathfrak{N} = \{x \mid x \in \nabla\,\mathfrak{N}(n);\, n = 0, 1, 2, \ldots\}$$

where

(i) *$x \in \nabla\mathfrak{N}(0)$ if $x \in N$ for some $N \in \mathfrak{N}$;*

(ii) *$x \in \nabla\mathfrak{N}(n)$ if there is a finite set $S \in \mathfrak{U}$ such that $x \leqslant a_S$ and $s \in \nabla\mathfrak{N}(n-1)$ for all $s \in S$.*

Proof. Let C be the set described by the theorem. We begin by showing that C is a $\mathfrak{U}$-ideal. Suppose that T is a finite set all of whose members belong to C, and such that $T \in \mathfrak{U}$. Let $p \leqslant a_T$. By a result analogous to Lemma 1, we may suppose that $t \in C(m)$ for all $t \in T$. Now by Condition (ii) above, it follows that $p \in C(m+1)$. Thus C is an ideal. Since $C \supset N$ for all $N \in \mathfrak{N}$ it follows that $C \supset \bigcup\mathfrak{N}$. Finally, if J is any $\mathfrak{U}$-ideal containing all $N \in \mathfrak{N}$, it is easy to see that $J \geqslant C$. Hence $C = \nabla\mathfrak{N}$.

The lattice $L(P;\mathfrak{U})$ is a compactly generated lattice in the sense of Dilworth and Crawley (**3**, p. 2). Clearly an ideal C is the join, in $L(P;\mathfrak{U})$ of the set $\{J(p) \mid p \in C\}$. It suffices, then, to show that $J(p)$ is a compact element in $L(P;\mathfrak{U})$. Suppose that $J(p) \subset \nabla\mathfrak{F}$ for a family $\mathfrak{F}$ of ideals F. Then $p \in \nabla\mathfrak{F}$. In view of Theorem 8 we may assume that $p \in \nabla\mathfrak{F}(n)$ and by a straightforward induction on n it follows that if $p \in \nabla\mathfrak{F}(n)$, then there is a finite subset $\mathfrak{F}'$ of $\mathfrak{F}$ such that $p \in \nabla\mathfrak{F}'$, i.e., $J(p) \subset \nabla\mathfrak{F}'$.

THEOREM 9. *Let A and B be words on the elements of P. In $L(P,\mathfrak{U})$,*

(1) *if $A \geqslant B$, then $J(A) \supset J(B)$;*

(2) *$J(A \wedge B) = J(A) \bigtriangleup J(B) = J(A) \cap J(B)$.*

Furthermore, if for every $S \in \mathfrak{U}$ there is a finite subset $S' \in \mathfrak{U}$ such that $\operatorname{lub} S' = \operatorname{lub} S$, *then $J(A \vee B) = J(A) \bigtriangledown J(B)$.*

Proof. (1) is a trivial consequence of the definition. $J(A \wedge B) = J(A) \cap J(B)$ is a consequence of (1) and the dual of Theorem 2. The other equality of (2) holds since in $L(P,\mathfrak{U})$ set intersection coincides with lattice meet.

To prove the final statement, note that in view of (1), we need only prove $J(A \vee B) \subset J(A) \bigtriangledown J(B)$. By induction on n we prove that if $p \leqslant A \vee B(n)$, then $p \in J(A) \bigtriangledown J(B)$. This is vacuously true if $n = 0$. The case $p \leqslant A$ or $p \leqslant B$ is trivial; hence we may suppose that for some $S \in \mathfrak{U}$ we have $p \leqslant a_S$ and $s \leqslant A \vee B(n-1)$ for all $s \in S$. By induction we have $s \in J(A) \bigtriangledown J(B)$. By the special hypothesis we may assume that S is finite, and since $J(A) \bigtriangledown J(B)$ is a $\mathfrak{U}$-ideal, it follows that $p \in J(A) \bigtriangledown J(B)$.

4. Decision problems. The first result is an immediate corollary of Theorem 7.

THEOREM 10. *The word problem in* FL$(P; \mathfrak{U}, \mathfrak{L})$ *has an affirmative solution if there is an affirmative solution to the problem of determining whether two ideals of* P *of the form* $M(A)$ *and* $J(B)$ *have a common element.*

COROLLARY 1. *If* P *is finite, then the word problem in* FL$(P; \mathfrak{U}, \mathfrak{L})$ *has an affirmative solution.*

Proof. Since P is finite, it has at most a finite number of (finite) subsets. In a finite number of steps we may test each subset to see which are $\mathfrak{U}$-ideals and $\mathfrak{L}$-ideals. With all the ideals explicitly determined the subsequent tests may be carried out in a finite number of steps.

We remark that to answer a question of the form "Is $A \leqslant B$?," for a specific A and B, it is probably easier to use the recursive nature of the words and the associated ideals given by Theorems 8 and 9.

COROLLARY 2. *If* L *is a finitely presented lattice, then the word problem in* L *has an affirmative solution.*

Proof. We need only observe that from the generators of L and its defining relations we may construct a finite partially ordered set P, and families of subsets $\mathfrak{U}$ and $\mathfrak{L}$ such that FL$(P; \mathfrak{U}, \mathfrak{L})$ is isomorphic with L. Sorkin **(6)** shows how to do this by using the algorithm of Evans to put the defining relations into closed form **(4**, p. 68**)**. In this presentation, which may have more generators and defining relations than the original, each defining relation has the form $x \vee y = z$ or $x \wedge y = z$, where x, y, and z are generators. The definition $x \leqslant y$ if and only if $x \vee y = y$ is a defining relation gives rise to a partial ordering of the generators. (Unless the defining relations are in closed form, this definition of $(\leqslant)$ need not be transitive.) Now put a set $S = \{a, b\}$ in $\mathfrak{U}$ if and only if $a \vee b = c$ is a defining relation $(a_S = c)$. Dually, put $S = \{a, b\}$ in $\mathfrak{L}$ if and only if $a \wedge b = c$ is a defining relation.

This partially ordered set is clearly embedded in L and so, by Theorem 6, L is a homomorphic image of FL$(P; \mathfrak{U}, \mathfrak{L})$. On the other hand, since any defining relation of L holds in FL$(P; \mathfrak{U}, \mathfrak{L})$ any equality between words in L holds in FL$(P; \mathfrak{U}, \mathfrak{L})$. Hence the two lattices are isomorphic.

It is important to point out that in the partially ordered set defined in the proof of Corollary 2 there may be bounds of pairs of elements *not* preserved by $\mathfrak{U}$ and $\mathfrak{L}$. As an example, take the lattice wtih generators a, b, c, d and defining relations $a \vee b = b$, $b \vee d = d$, $a \vee d = d$, $b \vee c = d$, and $c \vee d = d$. By adding the dual relations $a \wedge b = a$, etc., implied by these relations and the absorptive law, and the trivial relations reflecting commutivity and idempotence, these defining relations come into closed form. The associated partially ordered set is drawn in Figure 3. Note that $d = \text{lub}(a, c)$ in the partially ordered set, yet $d \neq a \vee c$ in the free lattice obtained by the algorithm above; indeed $\{a, c\} \notin \mathfrak{U}$.

It is difficult to see just how much can be said about the word problem in an arbitrary FL$(P; \mathfrak{U}, \mathfrak{L})$ beyond the information given in Theorem 10.

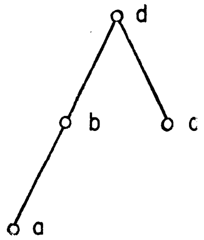

FIGURE 3

It seems to depend heavily on how P is given and how the $\mathfrak{U}$-ideals and $\mathfrak{L}$-ideals can be characterized. The next theorem gives a sufficient condition on $\mathfrak{U}$ to ensure the existence of a bound β depending only on a word A such that if $p \in P$ and $p \leqslant A$, then $p \leqslant A(\beta)$. However, this is not sufficient (even with a dual condition) to guarantee the solution of the word problem. It does suggest that the set of elements $\{p \mid p \leqslant A\}$ is recursive if $\mathfrak{U}$ and $\mathfrak{L}$ are recursive.

THEOREM 11. *Let P be a partially ordered set and $\mathfrak{U}$ be a collection of subsets of P satisfying conditions* $\mathfrak{U}1$ *and* $\mathfrak{U}2$. *Furthermore, let the set*

$$F = \{a_S \mid S \in \mathfrak{U}, a_S \notin S\}$$

be finite with cardinality α. Then for every word A there is an integer β such that for all $p \in P$, $p \leqslant A$ implies $p \leqslant A(\beta)$. Moreover, $1 + (\alpha + 1)\lambda(A)$ is a suitable value for β.

Proof. If $\lambda(A) = 0$, then $A \equiv q \in P$, and so $\beta = 0$ is suitable by Theorem 1. Proceeding by induction on $\lambda(A)$, we consider first the case $A \equiv A_1 \wedge A_2$. If $p \leqslant A_1 \wedge A_2$, then $p \leqslant A_i$ for $i = 1$ and 2. Thus we may assume $p \leqslant A_i(\beta_i)$, where $\beta_i \leqslant 1 + (\alpha + 1)\lambda(A_i)$ for $i = 1$ and 2. Since

$$\lambda(A) \geqslant \lambda(A_i) + 1,$$

it follows that $p \leqslant A(\beta)$, where $\beta = 1 + (\alpha + 1)\lambda(A)$. If $A \equiv A_1 \vee A_2$ and $p \leqslant A_i$ for $i = 1$ or 2, an argument such as that above proves the assertion.

Finally, then, if the theorem is false, it must be that for the case $A \equiv A_1 \vee A_2$ there is a $p \in P$ and an integer n such that $p \leqslant A(n)$, but neither $p \leqslant A(n-1)$, nor $p \leqslant A_1$, nor $p \leqslant A_2$, where $n > 1 + (\alpha + 1)\lambda(A)$. Under the condition on n we shall produce a sequence $a_1, a_2, \ldots, a_{\alpha+1}$ of elements in F such that $a_i \leqslant A(n - i + 1)$, but not $a_i \leqslant A(n - i)$ for any i such that $1 \leqslant i \leqslant \alpha + 1$. From these conditions and Lemma 1 it follows that the $\alpha + 1$ elements are distinct, in contradiction of the cardinality of F.

We begin with the observation that for $A \equiv A_1 \vee A_2$ and an element $q \in P$, if $q \leqslant A(m)$, but not $q \leqslant A(m - 1)$, $q \leqslant A_1$, or $q \leqslant A_2$, then there is an element in F with the same properties as q with respect to A. This is so since, from Definition 2, the only alternative is (2.4); that is, for some set $S \in \mathfrak{U}$ we have $q \leqslant a_S$ and $s \leqslant A(m-1)$ for all $s \in S$. If it were the case that $a_S \in S$ or that $a_S \leqslant A(m-1)$, then by Lemma 1, $q \leqslant A(m-1)$ would follow. Clearly $a_S \leqslant A(m)$. From this argument the existence of a_1 is guaranteed.

Now let us suppose that elements $a_1, a_2, \ldots, a_i$ $(i \leqslant \alpha)$ have been produced with the desired properties. In particular, we may suppose that $a_i = \text{lub}\, S_i$, where $S_i \in \mathfrak{U}$ and $s \leqslant A(n-i)$ for all $s \in S_i$. Since $a_i \leqslant A(n-i)$ is not true, $s_i \leqslant A(n-(i+1))$ must be false for some $s_i \in S_i$. We shall now prove that since $i \leqslant \alpha$, we have neither $s_i \leqslant A_1$ nor $s_i \leqslant A_2$. From these conditions the argument in the preceding paragraph with q replaced by s_i and m replaced by $n-i$ yields the existence of a_{i+1}.

If $s_i \leqslant A_1$, then from the induction hypothesis on the length of the word, we have $s_i \leqslant A_1(1+(\alpha+1)\lambda(A_1))$. On the other hand from

$$n > 1 + (\alpha+1)\lambda(A)$$

and $\lambda(A) \geqslant \lambda(A_1) + 1$, we have $n > 1 + (\alpha+1)[\lambda(A_1)+1]$, or

$$n - (\alpha+1) > 1 + (\alpha+1)\lambda(A_1).$$

Since $i \leqslant \alpha$, $n-(i+1) \geqslant n-(\alpha+1)$ follows and so by Lemma 1 we have $s_i \leqslant A(n-(i+1))$, which is a contradiction.

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