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ABSTRACT

In this article, we present a conjectural formula describing the cokernel of the Albanese map of zero-cycles of smooth projective varieties X over p-adic fields in terms of the Néron–Severi group and provide a proof under additional assumptions on an integral model of X. The proof depends on a non-degeneracy result of Brauer–Manin pairing due to Saito–Sato and on Gabber–de Jong's comparison result of cohomological and Azumaya–Brauer groups. We will also mention the local–global problem for the Albanese cokernel; the abelian group on the 'local side' turns out to be a finite group.

1. Introduction

Let K be a p-adic field with residue field k and X be a smooth projective variety over K. In this paper, the term 'variety' always means a geometrically irreducible scheme over a field.

When X is a curve, Lichtenbaum [Lic69] defined and studied the pairing

$$\mathrm{Pic}(X) \times \mathrm{Br}(X) \to \mathbb{Q}/\mathbb{Z}, \tag{L}$$

which he showed to be non-degenerate, where $\mathrm{Br}(X)$ denotes the étale cohomology group $H^2_{\text{ét}}(X, \mathbb{G}_m)$. Further, with the aid of Tate duality for abelian varieties over p-adic fields, he proved that the cokernel of the map

$$\mathrm{Pic}^0(X) \to J_X(K)$$

is canonically Pontryagin dual to the cokernel of the degree map $\mathrm{Pic}(\overline{X})^{G_K} \to \mathbb{Z}$. Here J_X denotes the Jacobian variety of X and $\overline{X}$ is the base change to an algebraic closure $\overline{K}$ of K. Also, G_K denotes the absolute Galois group of K.

In this paper we present an attempt to generalize Lichtenbaum's result to higher dimensional X. Namely, we study the cokernel of the Albanese map

$$\mathrm{alb}_X : \mathrm{CH}_0(X)^0 \to \mathrm{Alb}_X(K),$$

where $\mathrm{CH}_0(X)^0$ denotes the degree-0 part of the Chow group $\mathrm{CH}_0(X)$.

It is easy to see that this cokernel is torsion and is a birational invariant (i.e. birational morphisms induce isomorphisms) of smooth proper varieties over any field. Over p-adic fields, it is finite by the argument of Saito and Sujatha [SS95]. Except for these basic facts, little is known.

Our main result is the following.

THEOREM 1.1 (see Theorem 3.1 and Remark 3.3). *Let X be a smooth projective variety over a p-adic field K and suppose that X admits a smooth projective model $\mathcal{X}$ over the integer ring $\mathcal{O}_K$ whose Picard scheme $\mathbf{Pic}_{\mathcal{X}/\mathcal{O}_K}$ over $\mathrm{Spec}(\mathcal{O}_K)$ (which exists in this case) is smooth.*
Then the cokernel of the map

$$\mathrm{alb}_X : \mathrm{CH}_0(X)^0 \to \mathrm{Alb}_X(K)$$

is canonically Pontryagin dual to the cokernel of

$$\mathrm{Pic}(\overline{X})^{G_K} \to \mathrm{NS}(\overline{X})^{G_K}.$$

Moreover, the last cokernel is a subquotient of the p-primary torsion subgroup of $\mathrm{NS}(\overline{X})^{G_K}$.

Here NS denotes the Néron–Severi group: for any variety Y over a field, $\mathrm{NS}(Y)$ is defined to be the quotient $\mathrm{Pic}(Y)/\mathrm{Pic}^0(Y)$. The smoothness condition on the Picard scheme is satisfied if $H^2(X, \mathcal{O}_X)$ and $H^2(Z, \mathcal{O}_Z)$ vanish (here we set $Z := \mathcal{X} \otimes_{\mathcal{O}_K} k$).

In the proof of the theorem, we interpret the pairing (L) as the Brauer–Manin pairing introduced in [Man70]:

$$\mathrm{CH}_0(X) \times \mathrm{Br}(X) \to \mathrm{Br}(K). \tag{BM}$$

This pairing has been used to study zero-cycles on varieties, especially over p-adic fields and number fields. By non-degeneracy results on the Brauer–Manin pairing due to Saito and Sato, together with the Tate pairing, we reduce our problem to an injectivity problem concerning Brauer groups. We solve it using arguments of Artin and an existence theorem of Azumaya algebras due to Gabber and de Jong.

There is another surjectivity result, proved separately.

THEOREM 1.2 (see Theorem 4.1). *Let X be a smooth projective variety over K with good reduction. Suppose that the ramification index of K over $\mathbb{Q}_p$ is $< p - 1$. Then the Albanese map for X*

$$\mathrm{alb}_X : \mathrm{CH}_0(X)^0 \to \mathrm{Alb}_X(K)$$

is surjective.

The proof of this theorem is based on knowledge about the behavior of Néron models of abelian varieties under reduction.

Note that given a smooth projective variety X over a *number field K* (here 'projective' can be replaced by 'proper'; § 2), we can use Theorem 1.2 to show that the map

$$\mathrm{CH}_0(X_{K_v})^0 \to \mathrm{Alb}_X(K_v)$$

is surjective for almost all places v of K. Here K_v denotes the completion of K at v. So, we are tempted to consider the map between finite groups

$$\mathrm{Alb}_X(K)/\mathrm{alb}_X \mathrm{CH}_0(X)^0 \to \prod_{v:\,\text{all places}} \mathrm{Alb}_X(K_v)/\mathrm{alb}_{X_{K_v}} \mathrm{CH}_0(X_{K_v})^0.$$

When X is a curve, its injectivity follows from the Hasse principle for the Brauer group. An interesting question is whether or not it is injective in the higher-dimensional case as well.

Another generalization of Lichtenbaum's duality concerning the Picard variety and the Picard group, which is dual to our point of view, was considered comprehensively by van Hamel [vHa04].

2. Preliminaries

2.1 Albanese cokernel: definition and basic properties

We include the definition of the Albanese map for completeness.

Let Y be a smooth proper variety over a field F. Then there exist an abelian variety Alb_Y over F and a morphism $\phi : Y \times_F Y \to \mathrm{Alb}_Y$ which satisfy the following property: $\phi|_{\Delta_Y}$ is the constant map to zero, where $\Delta_Y \subset Y \times Y$ is the diagonal subscheme; given a morphism $\psi : Y \times Y \to A$ into an abelian variety A which is the constant map to zero on the diagonal, there is a unique homomorphism of abelian varieties $f : \mathrm{Alb}_Y \to A$ with $\psi = f \circ \phi$ [Lan59, pp. 45–46].

In this case, Y being proper, the Albanese variety Alb_Y of Y is also characterized by the following property (cf. [Gab01, Lemma 2.3]): every time we choose an extension field L of F and an L-valued point x_0 of Y, $(\mathrm{Alb}_Y)_L$ co-represents the functor

$$\{\text{Abelian varieties over } L\} \to \{\text{Sets}\}$$
$$A \mapsto \mathrm{Hom}_*((Y_L, x_0), (A, 0)),$$

where Hom_* denotes the set of base-point-preserving morphisms of L-schemes. In particular, there is a universal morphism $\phi_{x_0} : Y_L \to (\mathrm{Alb}_Y)_L$ sending x_0 to 0.

By Galois descent, there are a torsor Alb_Y^1/F under Alb_Y and a morphism $\phi' : Y \to \mathrm{Alb}_Y^1$ which has a universal property: given a morphism $\psi' : Y \to A'$ into a torsor A' under an abelian variety A, there are a unique homomorphism $f : \mathrm{Alb}_Y \to A$ and a unique morphism $f' : \mathrm{Alb}_Y^1 \to A'$ which are compatible.

The torsor Alb_Y^1 is determined by an element $[\mathrm{Alb}_Y^1] \in H^1_{\text{ét}}(F, \mathrm{Alb}_Y)$. By the canonical isomorphism $H^1_{\text{ét}}(F, \mathrm{Alb}_Y) \cong \mathrm{Ext}^1_{(\mathrm{Sch}/F)_{\text{ét}}}(\mathbb{Z}, \mathrm{Alb}_Y)$, it corresponds to an extension of group schemes over F:

$$0 \to \mathrm{Alb}_Y \to A_Y \xrightarrow{\deg} \mathbb{Z} \to 0, \tag{1}$$

where A_Y is described explicitly as follows. Denote by Alb_Y^i the ith power of the torsor Alb_Y^1, which corresponds to $i \cdot [\mathrm{Alb}_Y^1] \in H^1_{\text{ét}}(F, \mathrm{Alb}_Y)$. Set $A_Y := \coprod_{i \in \mathbb{Z}} \mathrm{Alb}_Y^i$. The morphisms $\mathrm{Alb}_Y^i \times_F \mathrm{Alb}_Y^j \to (\mathrm{Alb}_Y^i \times_F \mathrm{Alb}_Y^j)/\mathrm{Alb}_Y = \mathrm{Alb}_Y^{i+j}$ and $\mathrm{Alb}_Y^{-i} \cong (\mathrm{Alb}_Y^i)^{-1}$ make A_Y into a group scheme. Define a map $\mathrm{Alb}_Y \to A_Y$ as the canonical open and closed immersion to the 0th component, and $A_Y \to \mathbb{Z}$ to be the constant map to i on Alb_Y^i. Being a group scheme with a quasi-projective neutral component, A_Y has a transfer structure (see [SS03, proof of Lemma 3.2] and [BK10, Lemma 1.3.2]) with respect to which (1) is an exact sequence of étale sheaves with transfers.

We define the map $\mathrm{alb}_Y : \mathrm{CH}_0(Y)^0 \to \mathrm{Alb}_Y(F)$. First we define a map $\mathrm{alb}_Y : Z_0(Y) \to A_Y(F)$. This map is determined by determining the image of each $[x]$, $x \in Y_{(0)}$. Let $F(x)$ be the residue field of x. There is a canonical element $x \in Y(F(x))$. Let $\mathrm{alb}_Y([x])$ be the image of x by

$$Y(F(x)) \xrightarrow{\phi'} \mathrm{Alb}_Y^1(F(x)) \subset A_Y(F(x)) \xrightarrow{\text{transfer}} A_Y(F).$$

By the fact that maps in (1) are compatible with transfer structures, the following diagram commutes:

$$\begin{array}{ccc} Z_0(Y) & \xrightarrow{\deg} & \mathbb{Z} \\ \downarrow{\scriptstyle \mathrm{alb}_Y} & & \| \\ A_Y(F) & \xrightarrow{\deg} & \mathbb{Z} \end{array}$$

Therefore, a map

$$\mathrm{alb}_Y : Z_0(Y)^0 \to \mathrm{Alb}_Y(F)$$

between degree-0 parts is induced. It is known that it factors through $\mathrm{CH}_0(Y)^0$.

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DEFINITION 2.1. The cokernel of the map

$$\mathrm{alb}_Y : \mathrm{CH}_0(Y)^0 \to \mathrm{Alb}_Y(F)$$

is called the *Albanese cokernel* of Y.

The following properties are easy to verify.

PROPOSITION 2.2. *The Albanese cokernel of proper smooth varieties satisfies the following.*

(i) *It is trivial if F is an algebraically closed field or if F is a finite field and Y is projective.*

(ii) *It is trivial if Y is an abelian variety.*

(iii) *Albanese cokernels form a covariant functor from the category of smooth proper varieties to the category of abelian groups.*

(iv) *Albanese cokernels are contravariant with respect to base changes $\mathrm{Spec}\, F' \to \mathrm{Spec}\, F$ (F' is a field) and covariant if F'/F is finite. Consequently, they are torsion.*

(v) *The Albanese cokernel is finite if F is $\mathbb{R}$, a p-adic field or a finitely generated field over a prime field.*

Proof. The assertion (i) in the case F is a finite field follows from Kato–Saito's class field theory [KS83, Proposition 9(1)], where they further describe the kernel of the Albanese map. The assertion (v) in the case F is a finitely generated field follows from the Mordell–Weil theorem together with the fact that it is torsion. The case F is a p-adic field follows from arguments in [SS95, p. 409]. The case $F = \mathbb{R}$ follows from knowledge of the structure of commutative Lie groups. □

PROPOSITION 2.3 (Rigidity). *Let K be a Henselian non-archimedean $\mathbb{R}$-valued valuation field of characteristic 0 and $\hat{K}$ be its completion. Let X be a smooth proper variety. Then the map*

$$\frac{\mathrm{Alb}_X(K)}{\mathrm{alb}_X(\mathrm{CH}_0(X)^0)} \to \frac{\mathrm{Alb}_X(\hat{K})}{\mathrm{alb}_{X_{\hat{K}}}(\mathrm{CH}_0(X_{\hat{K}}))^0}$$

is an isomorphism.

Proof. We define a homotopy invariant presheaf with transfers F on Sm/K (the category of separated smooth schemes of finite type) by

$$Y \mapsto \mathrm{Alb}_X(Y)/\mathrm{Im}\, h_0(X/K)(Y)^0,$$

where $h_0(X/K)$ denotes the functor

$$\mathrm{Sm}/K \to (\text{Abelian groups})$$
$$Y \mapsto \mathrm{Cok}(\mathbb{Z}_{\mathrm{tr}}(X)(Y \times \mathbb{A}^1) \xrightarrow{d} \mathbb{Z}_{\mathrm{tr}}(X)(Y))$$

(d is the difference of the evaluation maps at 0 and $1 \in \mathbb{A}^1$) and

$$h_0(X/K)(Y)^0 := \ker(h_0(X/K)(Y) \xrightarrow{\deg} \mathbb{Z}^{\pi_0(Y)})$$

(see [SS03] for the morphism $h_0(X/K)^0 \to \mathrm{Alb}_X$).

Since F takes torsion values on the spectra of fields, its Zariski sheafification is a torsion abelian presheaf with transfers which is homotopy invariant [Voe00, Corollary 4.19 and Proposition 4.26]. Now the assertion follows by applying the rigidity theorem with respect to the extension $K \subset \hat{K}$ [RØ06, Theorem 1] to the sheaf F_{Zar}. □

PROPOSITION 2.4 (Birational invariance). *Let $Y' \to Y$ be a birational morphism of smooth proper varieties over F. Then it induces an isomorphism on their Albanese cokernels.*

Proof. We know that the Albanese variety is a birational invariant and the map $\mathrm{CH}_0(Y')^0 \to \mathrm{CH}_0(Y)^0$ is surjective. Therefore, the induced map on the Albanese cokernel is a bijection. $\square$

PROPOSITION 2.5 (Hypersurface sections). *Let K be a Henselian discrete valuation field of characteristic 0 with finite residue field. Let $X \subset \mathbb{P}^n$ be a smooth projective variety over K of dimension $\geqslant 3$. Then there is a smooth hypersurface section H of X for which the map*

$$\frac{\mathrm{Alb}_H(K)}{\mathrm{alb}_H \mathrm{CH}_0(H)^0} \to \frac{\mathrm{Alb}_X(K)}{\mathrm{alb}_X \mathrm{CH}_0(X)^0}$$

is an isomorphism.

Proof. First suppose that $K = \hat{K}$ is a p-adic field. Then $\mathrm{alb}_X \mathrm{CH}_0(X)^0 \subset \mathrm{Alb}_X(\hat{K})$ contains an open subgroup U of $\mathrm{Alb}_X(\hat{K})$ isomorphic to a direct sum of finitely many copies of $\mathcal{O}_{\hat{K}}$ (cf. [Mat55, Theorem 7] and [SS95, p. 409]). Therefore, $\mathrm{alb}_X \mathrm{CH}_0(X)^0$ is topologically generated by finitely many zero-cycles $a_1, \ldots, a_n$. We choose finitely many closed points $x_1, \ldots, x_m \in X$ such that $a_1, \ldots, a_n$ have supports on $\{x_1, \ldots, x_m\}$ and choose a smooth hypersurface section H passing through these points (possible by [AK79, Theorem (7)]). As $\dim(X) \geqslant 3$, $\mathrm{Alb}_H \to \mathrm{Alb}_X$ is an isomorphism:

$$\begin{array}{ccc}
\mathrm{CH}_0(H)^0 & \longrightarrow & \mathrm{Alb}_H(\hat{K}) \\
\downarrow & & \downarrow \cong \\
\mathrm{CH}_0(X)^0 & \longrightarrow & \mathrm{Alb}_X(\hat{K})
\end{array}$$

Since $\mathrm{alb}_H \mathrm{CH}_0(H)^0 \subset \mathrm{Alb}_X(\hat{K})$ is an open subgroup of $\mathrm{alb}_X \mathrm{CH}_0(X)^0$ containing $a_1, \ldots, a_n$, it coincides with $\mathrm{alb}_X \mathrm{CH}_0(X)^0$.

Next, we consider the general case. Let $\hat{K}$ be the completion of K. Over $\hat{K}$, we have proved the existence of a hypersurface section H with the desired property. By Proposition 2.3, it suffices to show that such an H can be taken over K. For that it is sufficient to find zero-cycles $a_1, \ldots, a_n$ as above which are defined over K. So, we are going to modify the cycles a_i obtained above to find zero-cycles a_i' of degree 0 defined over K such that the elements $\mathrm{alb}_X(a_i') \in \mathrm{Alb}_X(K) \subset \mathrm{Alb}_X(\hat{K})$ generate the subgroup $\mathrm{alb}_{X_{\hat{K}}}(\mathrm{CH}_0(X_{\hat{K}})^0)$ topologically.

Observe that if we change each of the elements $\mathrm{alb}(a_i) \in \mathrm{Alb}_X(\hat{K})$ by an element of pU (here U is the open subgroup of $\mathrm{alb}_{X_{\hat{K}}}(\mathrm{CH}_0(X_{\hat{K}})^0) \subset \mathrm{Alb}_X(\hat{K})$ mentioned earlier in this proof), the property that they are a set of topological generators of $\mathrm{alb}(\mathrm{CH}_0(X_{\hat{K}})^0)$ does not change.

Now, for the time being, consider an arbitrary zero-cycle

$$a = \sum_{\alpha} [x_\alpha] - \sum_{\beta} [x_\beta]$$

on X (or on $X_{\hat{K}}$) of degree 0. They define a point

$$\mathbf{x}(a) = ((x_\alpha)_\alpha, (x_\beta)_\beta) \in \prod_{\alpha} X(K(x_\alpha)) \times \prod_{\beta} X(K(x_\beta)).$$

We have the Albanese map

$$\prod_{\alpha} X(K(x_\alpha)) \times \prod_{\beta} X(K(x_\beta)) \xrightarrow[\sum_{\alpha}(\text{transfer}) - \sum_{\beta}(\text{transfer})]{} \quad \begin{array}{l} \prod_{\alpha} \mathrm{Alb}_X^1(K(x_\alpha)) \times \prod_{\beta} \mathrm{Alb}_X^1(K(x_\beta)), \\ \mathrm{Alb}_X(K) \end{array}$$

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where the addition and subtraction take place in $A_X(K)$, and the map goes into the neutral component Alb_X because of the degree-0 assumption. This map sends $\mathbf{x}(a)$ to $\mathrm{alb}_X(a)$. The map is continuous for the valuation topology on both sides because the transfer map is based on scheme morphisms [SS03, Lemma 3.2].

Getting back to our situation, let us write

$$a_i = \sum_{\alpha \in A_i} [x_\alpha] - \sum_{\beta \in B_i} [x_\beta].$$

Then $\mathbf{x}(a_i)$ is a point in $\prod_{\alpha \in A_i} X(\hat{K}(x_\alpha)) \times \prod_{\beta \in B_i} X(\hat{K}(x_\beta))$. The observation made above and the continuity imply that we may replace each of our a_i (a zero-cycle on $X_{\hat{K}}$) by a zero-cycle represented by a point sufficiently close to $\mathbf{x}(a_i)$.

Since the extension $\hat{K}(x_\alpha)/\hat{K}$ is separable, the integral closure K_α of K in $\hat{K}(x_\alpha)$ is dense in $\hat{K}(x_\alpha)$, so that we have $(K_\alpha)\hat{} = \hat{K}(x_\alpha)$ (and similarly for β). Then, by an approximation theorem [BLR10, § 3.6, Corollary 10], the subset

$$X(K_\alpha) \subset X(\hat{K}(x_\alpha))$$

is dense (and similarly for β). So, we can take a point in $\prod_{\alpha \in A_i} X(K_\alpha) \times \prod_{\beta \in B_i} X(K_\beta)$ sufficiently close to $\mathbf{x}(a_i)$, and we let a_i' be the zero-cycle on X represented by the point. Then the elements

$$\mathrm{alb}_X(a_i') \in \mathrm{Alb}_X(\hat{K})$$

generate the subgroup $\mathrm{alb}_{X_{\hat{K}}}(\mathrm{CH}_0(X_{\hat{K}})^0)$ topologically. This completes the proof. $\qquad\square$

2.2 Automorphisms and Brauer groups

2.2.1.   Suppose that $f : Y \to X$ is a Galois covering of schemes. We have the Hochschild–Serre spectral sequence

$$E_2^{i,j} = H^i(\mathrm{Gal}(Y/X), H^j_{\text{ét}}(Y, \mathbb{G}_m)) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{G}_m),$$

from which we get a homomorphism (where we write $\mathrm{Br}(Y/X) := \ker(\mathrm{Br}(X) \to \mathrm{Br}(Y))$)

$$E_1^2 = \mathrm{Br}(Y/X) \to E_2^{1,1} = H^1(\mathrm{Gal}(Y/X), \mathrm{Pic}(Y)).$$

We will denote it by $\phi_{Y/X}$.

2.2.2.   We recall that an Azumaya algebra on a scheme Y is by definition a sheaf $\mathcal{A}$ of $\mathcal{O}_Y$-algebras (not necessarily commutative) which is a locally free $\mathcal{O}_Y$-module of finite rank and such that the canonical morphism

$$\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{A}^{\mathrm{op}} \to \mathcal{E}nd_{\mathcal{O}_Y\text{-mod.}}(\mathcal{A})$$

is an isomorphism. Two Azumaya algebras $\mathcal{A}$ and $\mathcal{A}'$ are said to be equivalent if there are vector bundles $\mathcal{V}$ and $\mathcal{V}'$ on Y and an isomorphism of algebras $\mathcal{A} \otimes_{\mathcal{O}_Y} \mathcal{E}nd(\mathcal{V}) \cong \mathcal{A}' \otimes_{\mathcal{O}_Y} \mathcal{E}nd(\mathcal{V}')$. The set of equivalence classes of Azumaya algebras forms a torsion abelian group by tensor products. Let us denote this group by $\mathrm{Br}_{\mathrm{Az}}(Y)$. There is a canonical injection $\mathrm{Br}_{\mathrm{Az}}(Y) \hookrightarrow \mathrm{Br}(Y)_{\mathrm{tors}}$, where $(-)_{\mathrm{tors}}$ indicates the torsion subgroup. This is not surjective in general. Nevertheless, we have the following result.

THEOREM 2.6 (de Jong [dJo, Theorem 1.1]). *Let X be a quasi-compact separated scheme which admits an ample line bundle. Then the map $\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}(X)_{\mathrm{tors}}$ is bijective.*

Note that if moreover X is regular and noetherian, we have $\mathrm{Br}_{\mathrm{Az}}(X) = \mathrm{Br}(X)$ since the Brauer group is known to be torsion for regular noetherian schemes.

2.2.3. Suppose that $f : Y \to X$ is a morphism of schemes and an abstract group G acts on Y (on the right) over X. For $\sigma \in G$, denote by $[\sigma] : Y \to Y$ the corresponding action; we have $[\sigma][\tau] = [\tau\sigma]$. Then G acts on $\mathrm{Pic}(Y)$ on the left by $\mathcal{L} \mapsto [\sigma]^*\mathcal{L}$ for $\sigma \in G$.

In this case we have a homomorphism

$$\phi_{G,Y/X} : \mathrm{Br}_{\mathrm{Az}}(Y/X) \to H^1(G, \mathrm{Pic}(Y))$$

(where $\mathrm{Br}_{\mathrm{Az}}(Y/X) := \ker(\mathrm{Br}_{\mathrm{Az}}(X) \to \mathrm{Br}_{\mathrm{Az}}(Y))$) described as follows.

Let $\omega \in \mathrm{Br}_{\mathrm{Az}}(Y/X)$. Take an Azumaya algebra $\mathcal{A}$ on X which represents ω. There are a vector bundle $\mathcal{E}$ on Y and an isomorphism $f^*\mathcal{A} \cong \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})$; pulling it back by $[\sigma]$ for $\sigma \in G$, we get isomorphisms

$$\mathcal{E}nd_{\mathcal{O}_Y}([\sigma]^*\mathcal{E}) \cong [\sigma]^*f^*\mathcal{A} = f^*\mathcal{A} \cong \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E}). \tag{2}$$

By Morita theory [KO74, IV, Proposition 1.3], there are a line bundle $\mathcal{L}_\sigma$ on Y and an isomorphism $[\sigma]^*\mathcal{E} \cong \mathcal{E} \otimes \mathcal{L}_\sigma$ which induces the isomorphism $\mathcal{E}nd_{\mathcal{O}_Y}([\sigma]^*\mathcal{E}) \cong \mathcal{E}nd_{\mathcal{O}_Y}(\mathcal{E})$; moreover, the choices of $\mathcal{L}_\sigma$ and the isomorphism $[\sigma]^*\mathcal{E} \cong \mathcal{E} \otimes \mathcal{L}_\sigma$ are unique up to a unique isomorphism. The mapping $(G \ni \sigma \mapsto \mathcal{L}_\sigma \in \mathrm{Pic}(Y))$ is, therefore, a 1-cocycle of the G-module $\mathrm{Pic}(Y)$. We define $\phi_{G,Y/X}(\omega)$ to be the element it represents in $H^1(G, \mathrm{Pic}(Y))$. It can be checked that this element does not depend on the choices of $\mathcal{A}$ and $\mathcal{E}$.

When we have a diagram of compatible actions

$$
\begin{array}{ccc}
G' & \xrightarrow{\rho} & G \\[4pt]
\circlearrowleft & & \circlearrowleft \\[4pt]
Y' & \xrightarrow{r'} & Y \\[2pt]
\downarrow & & \downarrow \\[2pt]
X' & \xrightarrow{r} & X
\end{array}
\tag{3}
$$

(i.e. the group G' acts on Y' over X', the group G acts on Y over X and the maps r, r' and ρ are compatible in the obvious sense), we have a commutative diagram of groups

$$
\begin{array}{ccc}
\mathrm{Br}_{\mathrm{Az}}(Y'/X') & \longleftarrow & \mathrm{Br}_{\mathrm{Az}}(Y/X) \\[2pt]
\phi_{G',Y'/X'} \downarrow & & \downarrow \phi_{G,Y/X} \\[2pt]
H^1(G', \mathrm{Pic}(Y')) & \longleftarrow & H^1(G, \mathrm{Pic}(Y))
\end{array}
\tag{4}
$$

where the horizontal maps are the natural functorial ones.

2.2.4. Suppose that $f : Y \to X$ is a Galois covering and an abstract group G acts on Y through a group homomorphism $G \to \mathrm{Gal}(Y/X)$. Then we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{Br}_{\mathrm{Az}}(Y/X) & \subset & \mathrm{Br}(Y/X) \\[2pt]
\phi_{G,Y/X} \downarrow & & \downarrow \phi_{Y/X} \\[2pt]
H^1(G, \mathrm{Pic}(Y)) & \longleftarrow & H^1(\mathrm{Gal}(Y/X), \mathrm{Pic}(Y))
\end{array}
\tag{5}
$$

3. Main theorem

THEOREM 3.1. *Let K be a Henselian discrete valuation field of characteristic 0 with finite residue field k and X be a smooth projective variety over K. Suppose that X admits a smooth projective model $\mathcal{X}$ over the integer ring $\mathcal{O}_K$ whose Picard scheme $\mathbf{Pic}_{\mathcal{X}/\mathcal{O}_K}$ over $\mathrm{Spec}(\mathcal{O}_K)$ is smooth.*

Then the cokernel of the map

$$\mathrm{alb}_X : \mathrm{CH}_0(X)^0 \to \mathrm{Alb}_X(K)$$

is canonically isomorphic to the Pontryagin dual of

$$\mathrm{coker}(\mathrm{Pic}(\overline{X})^{G_K} \to \mathrm{NS}(\overline{X})^{G_K}).$$

Remark 3.2. (1) The Picard scheme exists in this situation (see, for example, [Kle05, Theorem 4.8]).

(2) By the Chebotarev density theorem, the special fiber of $\mathcal{X}$ has a degree-1 zero-cycle. By the smoothness of $\mathcal{X}$ and the fact that K is Henselian, it lifts to a degree-1 zero-cycle on X. From this, we find that the restriction map $\mathrm{Br}(K) \to \mathrm{Br}(X)$ is injective and that its image and the image of the injection $\mathrm{Br}(\mathcal{X}) \hookrightarrow \mathrm{Br}(X)$ have a trivial intersection.

The proof of Theorem 3.1 occupies the rest of § 3, throughout which we keep the notation in the theorem.

In addition, we use the following notation:

$$Z = \mathcal{X} \times_{\mathcal{O}_K} k;$$

$\overline{X}$ (respectively $\overline{Z}$) = the base change to an algebraic closure of the base field;

$$\overline{\mathcal{X}} = \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}.$$

Remark 3.3. Under the hypothesis of Theorem 3.1, the group

$$\mathrm{coker}(\mathrm{Pic}(\overline{X})^{G_K} \to \mathrm{NS}(\overline{X})^{G_K})$$

is a subquotient of $\mathrm{NS}(\overline{X})^{G_K}\{p\}$, the p-primary torsion subgroup of $\mathrm{NS}(\overline{X})^{G_K}$. Indeed, by a Kummer sequence, for each n prime to p we have a commutative diagram

$$
\begin{array}{ccc}
\mathrm{NS}(\overline{X})/n & \hookrightarrow & H^2_{\text{ét}}(\overline{X}, \mathbb{Z}/n(1)) \\
\downarrow{\scriptstyle \mathrm{sp}} & & \cong \uparrow{\scriptstyle \mathrm{cosp}} \\
\mathrm{NS}(\overline{Z})/n & \hookrightarrow & H^2_{\text{ét}}(\overline{Z}, \mathbb{Z}/n(1))
\end{array}
$$

where the map cosp is an isomorphism by the proper smooth base change theorem for étale cohomology. From this, we see that the vertical map sp is injective; it follows that the kernel P of the specialization map $\mathrm{NS}(\overline{X}) \to \mathrm{NS}(\overline{Z})$ is a p-primary torsion abelian group. On the other hand, we have a commutative diagram

$$
\begin{array}{ccc}
 & & P^{G_K} \\
 & & \downarrow \\
\mathrm{Pic}(X) \twoheadrightarrow \mathrm{NS}(X) & \hookrightarrow & \mathrm{NS}(\overline{X})^{G_K} \\
{\scriptstyle (*)}\downarrow & & \downarrow \\
\mathrm{Pic}(Z) \twoheadrightarrow \mathrm{NS}(Z) & \underset{(**)}{\overset{\cong}{\longrightarrow}} & \mathrm{NS}(\overline{Z})^{G_k}
\end{array}
$$

The specialization map $(*)$ is surjective because the Picard scheme is assumed to be smooth and by Hensel's lemma. The map $(**)$ is surjective because of $H^1(k, \operatorname{Pic}^0(\overline{Z})) = 0$ (Lang's theorem). From this diagram, we see that $\operatorname{NS}(X) + P^{G_K} = \operatorname{NS}(\overline{X})^{G_K}$. Therefore, $\operatorname{NS}(\overline{X})^{G_K}/\operatorname{NS}(X)$ is a quotient of P^{G_K}.

3.1 Commutative diagrams

We begin the proof of Theorem 3.1. By the fact that $H^3(K, \mathbb{G}_m) = 0$, the homomorphism $\phi_{\overline{X}/X}$ in §2.2.1 induces an isomorphism

$$\phi_{\overline{X}/X} : \operatorname{Br}(\overline{X}/X)/\operatorname{Br}(K) \xrightarrow{\cong} H^1(K, \operatorname{Pic}(\overline{X})).$$

Denote by ϕ the following composite map:

$$
\begin{array}{ccc}
H^1(K, \operatorname{Pic}^0(\overline{X})) & \longrightarrow & H^1(K, \operatorname{Pic}(\overline{X})) \\
& & \cong \downarrow \phi_{\overline{X}/X}^{-1} \\
& \operatorname{Br}(\overline{X}/X)/\operatorname{Br}(K) & \longrightarrow \operatorname{Br}(X)/[\operatorname{Br}(K) + \operatorname{Br}(\mathcal{X})].
\end{array}
$$

PROPOSITION 3.4 (proved in §3.3). *The Brauer–Manin pairing and the Tate pairing are compatible via the Albanese map and the homomorphism ϕ:*

$$
\begin{array}{ccccc}
\operatorname{CH}_0(X)^0 & \times & \frac{\operatorname{Br}(X)}{\operatorname{Br}(K)+\operatorname{Br}(\mathcal{X})} & \longrightarrow \mathbb{Q}/\mathbb{Z} & \text{(BM)} \\
\operatorname{alb}_X \downarrow & & \uparrow \phi & \| & \\
\operatorname{Alb}_X(K) & \times & H^1(K, \operatorname{Pic}^0(\overline{X})) \longrightarrow \mathbb{Q}/\mathbb{Z} & & \text{(T)}
\end{array}
$$

Here the Tate pairing (T) is known to be a perfect pairing of a compact group and a torsion group [Tat57]. On the other hand, Saito and Sato have recently proved the following result.

THEOREM 3.5 ([SS14, Theorem 1.1.3], applicable by [SS14, Remark 2.1.2]). *The pairing (BM) is non-degenerate on the right.*

Therefore, we conclude the following result.

COROLLARY 3.6. *The group* $\operatorname{coker}(\operatorname{alb}_X)$ *is the Pontryagin dual of* $\ker(\phi)$.

By the construction of ϕ, we have an exact sequence

$$
\begin{aligned}
0 \to{} & \ker(H^1(K, \operatorname{Pic}^0(\overline{X})) \to H^1(K, \operatorname{Pic}(\overline{X}))) \\
& \to \ker \phi \underset{\theta}{\to} \ker\left(\operatorname{Br}(\overline{X}/X)/\operatorname{Br}(K) \to \frac{\operatorname{Br}(X)}{\operatorname{Br}(K) + \operatorname{Br}(\mathcal{X})}\right).
\end{aligned}
\tag{6}
$$

We have $\ker(H^1(K, \operatorname{Pic}^0(\overline{X})) \to H^1(K, \operatorname{Pic}(\overline{X}))) = \operatorname{coker}(\operatorname{Pic}(\overline{X})^{G_K} \to \operatorname{NS}(\overline{X})^{G_K})$ by the short exact sequence

$$0 \to \operatorname{Pic}^0(\overline{X}) \to \operatorname{Pic}(\overline{X}) \to \operatorname{NS}(\overline{X}) \to 0.$$

On the other hand, we have (recall $\overline{\mathcal{X}} := \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{\overline{K}}$)

$$
\ker\left(\frac{\mathrm{Br}(\overline{X}/X)}{\mathrm{Br}(K)} \to \frac{\mathrm{Br}(X)}{\mathrm{Br}(K) + \mathrm{Br}(\mathcal{X})} \right) = \mathrm{Br}(\overline{X}/X) \cap \mathrm{Br}(\mathcal{X}) \quad \text{(by Remark 3.2(2))}
$$
$$
= \mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X}),
$$

where the last equality holds because if an element of $\mathrm{Br}(\mathcal{X})$ is annihilated in $\mathrm{Br}(X_{K'})$ for some finite extension field K' of K, it is annihilated in $\mathrm{Br}(\mathcal{X}_{\mathcal{O}_{K'}})$ too because the map $\mathrm{Br}(\mathcal{X}_{\mathcal{O}_{K'}}) \to \mathrm{Br}(X_{K'})$ is injective as $\mathcal{X}_{\mathcal{O}_{K'}}$ is regular by the smoothness assumption on $\mathcal{X}$.

Therefore, the sequence (6) takes the form

$$
0 \to \frac{\mathrm{NS}(\overline{X})^{G_K}}{\mathrm{Im}\,\mathrm{Pic}(\overline{X})^{G_K}} \to \ker\phi \xrightarrow{\theta} \mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X})
$$

and hence we are reduced to showing that $\theta = 0$.

Denote by sp the specialization maps $\mathrm{Pic}(\overline{X}) \to \mathrm{Pic}(\overline{Z})$, $\mathrm{NS}(\overline{X}) \to \mathrm{NS}(\overline{Z})$ and the maps induced on their cohomology groups.

PROPOSITION 3.7. *The following diagram commutes.*

$$
\begin{array}{ccccc}
H^1(K, \mathrm{Pic}^0(\overline{X})) & \longrightarrow & H^1(K, \mathrm{Pic}(\overline{X})) & \longrightarrow & H^1(K, \mathrm{NS}(\overline{X})) \\
\cup & & \phi_{\overline{X}/X} \uparrow \cong & & \downarrow \mathrm{sp} \\
\ker\phi \xrightarrow{\theta} \mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X}) & \subset & \mathrm{Br}(\overline{X}/X)/\mathrm{K} & & H^1(K, \mathrm{NS}(\overline{Z})) \qquad (7)\\
& \downarrow & & & \uparrow \mathrm{inf} \\
& \mathrm{Br}(\overline{Z}/Z) \xrightarrow[\cong]{\phi_{\overline{Z}/Z}} H^1(k, \mathrm{Pic}(\overline{Z})) & \longrightarrow & H^1(k, \mathrm{NS}(\overline{Z})) \\
\end{array}
$$

Proof. The left-hand square commutes by the definition of θ. For the right, we use the map $\phi_{G_K, \overline{\mathcal{X}}/\mathcal{X}} : \mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X}) \to H^1(K, \mathrm{Pic}(\overline{\mathcal{X}}))$. It was defined in §2.2 on the subgroup $\mathrm{Br}_{\mathrm{Az}}(\overline{\mathcal{X}}/\mathcal{X}) \subset \mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X})$ and, by Theorem 2.6, this subgroup is in fact equal to the entire group.

The commutativity of the following diagram is obvious in the right-hand half. In the left-hand half, it follows from §2.2:

$$
\begin{array}{ccccc}
\mathrm{Br}(\overline{X}/X) & \xrightarrow{\phi_{\overline{X}/X}} & H^1(K, \mathrm{Pic}(\overline{X})) & \longrightarrow & H^1(K, \mathrm{NS}(\overline{X})) \\
\cup & & \downarrow \mathrm{sp} & & \downarrow \mathrm{sp} \\
\mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X}) & \xrightarrow{\phi_{G_K, \overline{\mathcal{X}}/\mathcal{X}}} H^1(K, \mathrm{Pic}(\overline{\mathcal{X}})) \longrightarrow & H^1(K, \mathrm{Pic}(\overline{Z})) & \longrightarrow & H^1(K, \mathrm{NS}(\overline{Z})) \\
\downarrow & & \uparrow \mathrm{inf} & & \uparrow \mathrm{inf} \\
\mathrm{Br}(\overline{Z}/Z) & \xrightarrow{\phi_{\overline{Z}/Z}} & H^1(k, \mathrm{Pic}(\overline{Z})) & \longrightarrow & H^1(k, \mathrm{NS}(\overline{Z})) \\
\end{array}
$$

This shows the commutativity of the right-hand half of the diagram (7). □

3.2 Injectivity

In the diagram (7), the upper path from $\ker\phi$ to $H^1(K, \mathrm{NS}(\overline{Z}))$ is zero. Therefore, in order to prove that $\theta = 0$, it suffices to prove that the composite map from $\mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X})$ to $H^1(K, \mathrm{NS}(\overline{Z}))$ along the lower path is injective.

PROPOSITION 3.8. *Each of the maps*

$$H^1(k, \mathrm{Pic}(\overline{Z})) \to H^1(k, \mathrm{NS}(\overline{Z})) \xrightarrow{\mathrm{inf}} H^1(K, \mathrm{NS}(\overline{Z}))$$

is injective.

Proof. The first one is injective because of Lang's theorem: $H^1(k, \mathrm{Pic}^0(\overline{Z})) = 0$. The fact that the second one is injective follows from the general fact that the inflation map of group cohomology is always injective on H^1. □

For a finite extension K'/K, we denote the residue field of K' by k' and set $\mathcal{X}' = \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$ and $Z' = Z \otimes_k k'$.

Since we have

$$\mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X}) = \bigcup_{K'} \mathrm{Br}(\mathcal{X}'/\mathcal{X}),$$

where K' runs through all the finite subextensions of $\overline{K}/K$, for proving the injectivity of $\mathrm{Br}(\overline{\mathcal{X}}/\mathcal{X}) \to \mathrm{Br}(\overline{Z}/Z)$ it suffices to prove the injectivity of $\mathrm{Br}(\mathcal{X}'/\mathcal{X}) \to \mathrm{Br}(Z'/Z)$ for each K'/K.

PROPOSITION 3.9. *Let K' be a finite extension of K. For each integer $i \geqslant 1$, set $Z_i = \mathcal{X} \otimes_{\mathcal{O}_K} \mathcal{O}_K/\mathfrak{m}_K^i$ and $Z_i' = Z_i \otimes_{\mathcal{O}_K} \mathcal{O}_{K'}$. Then the following hold.*

 (i) *The map $\mathrm{Br}(\mathcal{X}) \to \varprojlim_i \mathrm{Br}(Z_i)$ induced by the restriction maps is injective.*
 (ii) *The map induced by the projection to the first component*

$$\ker\left(\varprojlim_i \mathrm{Br}(Z_i) \to \varprojlim_i \mathrm{Br}(Z_i')\right) \to \ker(\mathrm{Br}(Z) \to \mathrm{Br}(Z_1')) \subset \mathrm{Br}(Z'/Z)$$

is injective.
(iii) *$\mathrm{Br}(\mathcal{X}'/\mathcal{X}) \to \mathrm{Br}(Z'/Z)$ is injective.*

Proof. (iii) follows from (i), (ii) and the next diagram.

$$
\begin{array}{ccccc}
\mathrm{Br}(\mathcal{X}) & \longrightarrow & \varprojlim_i \mathrm{Br}(Z_i) & \longrightarrow & \mathrm{Br}(Z) \\
\downarrow & & \downarrow & & \downarrow \\
\mathrm{Br}(\mathcal{X}') & \longrightarrow & \varprojlim_i \mathrm{Br}(Z_i') & \longrightarrow & \mathrm{Br}(Z')
\end{array}
$$

To prove (i), we use the following result.

LEMMA 3.10 [Gro68, III, Lemme (3.3)]. *Let $f : \mathcal{X} \to Y$ be a proper flat morphism, with Y the spectrum of an excellent Henselian discrete valuation ring. Let $Y = \mathrm{Spec}(A)$, $\mathfrak{m}$ be the maximal ideal of A, $Z_i := Z \otimes_A A/\mathfrak{m}^i$. Suppose further that the projective system $(\mathrm{Pic}(Z_i))_i$ satisfies the Mittag-Leffler condition. Then the canonical morphism*

$$\mathrm{Br}_{\mathrm{Az}}(\mathcal{X}) \to \varprojlim_i \mathrm{Br}_{\mathrm{Az}}(Z_i)$$

is injective.

We can apply this lemma to our situation $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_K)$; the Mittag-Leffler condition is fulfilled by the smoothness assumption on the Picard scheme.

Therefore, in the diagram

$$
\begin{array}{ccc}
\mathrm{Br}_{\mathrm{Az}}(\mathcal{X}) & \xrightarrow{\ (*)\ } & \varprojlim_i \mathrm{Br}_{\mathrm{Az}}(Z_i) \\
\cap_{(**)} & & \cap \\
\mathrm{Br}(\mathcal{X}) & \xrightarrow{\ (***)\ } & \varprojlim_i \mathrm{Br}(Z_i)
\end{array}
$$

the map $(*)$ is injective and the inclusion $(**)$ is in fact an equality by Theorem 2.6. It follows that the map $(***)$ is also injective. This proves Proposition 3.9(i).

We show (ii). From the exact sequence of sheaves on the topological space Z:

$$
0 \longrightarrow \mathfrak{m}^i \mathcal{O}_{Z_{i+1}} \xrightarrow{\ 1+\ } \mathcal{O}_{Z_{i+1}}^* \longrightarrow \mathcal{O}_{Z_i}^* \longrightarrow 1
$$
$$
\wr\| \\
\mathcal{O}_Z
$$

where the isomorphism is specified if we choose a generator of $\mathfrak{m}$. There is a similar one on Z':

$$
0 \longrightarrow \mathfrak{m}^i \mathcal{O}_{Z'_{i+1}} \xrightarrow{\ 1+\ } \mathcal{O}_{Z'_{i+1}}^* \longrightarrow \mathcal{O}_{Z'_i}^* \longrightarrow 1
$$
$$
\wr\| \\
\mathcal{O}_{Z'_1}
$$

From these, one obtains a commutative diagram

$$
\begin{array}{ccccccccc}
\mathrm{Pic}(Z_{i+1}) & \xrightarrow{\psi_1} & \mathrm{Pic}(Z_i) & \xrightarrow{\psi_2} & H^2(Z,\mathcal{O}_Z) & \longrightarrow & \mathrm{Br}(Z_{i+1}) & \longrightarrow & \mathrm{Br}(Z_i) \\
\downarrow & & \downarrow & & \downarrow{\scriptstyle v} & & \downarrow & & \downarrow \\
\mathrm{Pic}(Z'_{i+1}) & \xrightarrow{\psi'_1} & \mathrm{Pic}(Z'_i) & \xrightarrow{\psi'_2} & H^2(Z'_1,\mathcal{O}_{Z'_1}) & \longrightarrow & \mathrm{Br}(Z'_{i+1}) & \longrightarrow & \mathrm{Br}(Z'_i)
\end{array}
$$

The maps ψ_1 and ψ'_1 are surjective by the assumption that the Picard scheme is smooth. Therefore, ψ_2 and ψ'_2 are zero maps. The vertical map v is injective by flat base change theorem for coherent cohomology. So, the group $\mathrm{Br}(Z'_{i+1}/Z_{i+1})$ injects into $\mathrm{Br}(Z'_i/Z_i)$ and hence into $\mathrm{Br}(Z'_1/Z)$.

This completes the proof. $\qquad\square$

In order to prove Theorem 3.1, it remains to show Proposition 3.4. This is done in § 3.3.

Remark 3.11. Proposition 3.9(iii) implies the relation $\mathrm{Br}(\mathcal{X}'/\mathcal{X}) \subset \mathrm{Br}(\mathcal{X}'^{\mathrm{ur}}/\mathcal{X})$, where $\mathcal{X}'^{\mathrm{ur}}$ denotes the base change of $\mathcal{X}$ to the maximal unramified subextension of $\mathcal{O}_{K'}/\mathcal{O}_K$. In view of [SS14, Theorem 1.1.3 and Remark 2.1.2] again, this means that any element of $\mathrm{Br}(X)$ which vanishes at every closed point and in $\mathrm{Br}(\overline{X})$ is trivialized by an unramified extension of K. It would be interesting to ask if this holds without the hypotheses in Theorem 3.1.

3.3 Proof of Proposition 3.4

For schemes Y, we use the big étale site $(\mathrm{Sch}/Y)_{\mathrm{\acute{e}t}}$, whose underlying category is the category of all schemes locally of finite type over Y and the coverings are étale surjective morphisms. Denote by $D((\mathrm{Sch}/Y)_{\mathrm{\acute{e}t}})$ the derived category of complexes of abelian étale sheaves.

For K-schemes $\varphi : Y \to \mathrm{Spec}\, K$, we will consider the groups $\mathrm{Hom}_{D((\mathrm{Sch}/K)_{\mathrm{\acute{e}t}})}(R\varphi_*\mathbb{G}_m, \mathbb{G}_m)$. They form a covariant functor in K-schemes Y.

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PROPOSITION 3.12. *For any finite field extension $\varphi_{K'} : \operatorname{Spec} K' \to \operatorname{Spec} K$, there is a canonical isomorphism*

$$\operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(R\varphi_{K'*}\mathbb{G}_m, \mathbb{G}_m) = \mathbb{Z}.$$

Proof. Since $\varphi_{K'}$ is finite, we have $R\varphi_{K'*}\mathbb{G}_m = \varphi_{K'*}\mathbb{G}_m = R_{K'/K}\mathbb{G}_m$ (the Weil restriction). Therefore, the group $\operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(R\varphi_{K'*}\mathbb{G}_m, \mathbb{G}_m)$ equals the group of homomorphisms

$$R_{K'/K}\mathbb{G}_m \to \mathbb{G}_m$$

of group schemes over K. Giving such a morphism is equivalent to giving a morphism over $\overline{K}$

$$\prod_{\operatorname{Hom}_K(K', \overline{K})} \mathbb{G}_m \to \mathbb{G}_m,$$

which is invariant under the action of G_K, which acts on $\prod \mathbb{G}_m$ by permutation of components and trivially on the right-hand side. Since the set of homomorphisms from the $\overline{K}$-group scheme $\mathbb{G}_m$ to itself is canonically isomorphic to $\mathbb{Z}$, the assertion follows. Explicitly, $1 \in \mathbb{Z}$ corresponds to the norm map $R_{K'/K}\mathbb{G}_{m,K'} \to \mathbb{G}_{m,K}$. $\qquad\square$

Let $\varphi : X \to \operatorname{Spec} K$ as in Theorem 3.1 and $i_x : x \to X$ be a closed point. Write $\varphi_x = \varphi \circ i_x : x \to \operatorname{Spec} K$. From the map $\mathbb{G}_m \to i_{x*}\mathbb{G}_m$, we get a map (the first equality is due to Proposition 3.12)

$$\mathbb{Z} = \operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(R\varphi_{x*}\mathbb{G}_m, \mathbb{G}_m) \to \operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(R\varphi_*\mathbb{G}_m, \mathbb{G}_m).$$

Thus, we get a map

$$\operatorname{cl} : Z_0(X) \to \operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(R\varphi_*\mathbb{G}_m, \mathbb{G}_m),$$

which is known to factor through $\operatorname{CH}_0(X)$ [vHa04, Proposition 3.2].

By functoriality, we have a commutative diagram

$$
\begin{array}{ccc}
Z_0(X) & \xrightarrow{\quad\deg\quad} & \mathbb{Z} = Z_0(\operatorname{Spec} K) \\
{\scriptstyle\operatorname{cl}}\downarrow & & {\scriptstyle\operatorname{cl}}\downarrow\| \\
\operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(R\varphi_*\mathbb{G}_m, \mathbb{G}_m) & \xrightarrow{\deg} & \mathbb{Z} = \operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(\mathbb{G}_m, \mathbb{G}_m)
\end{array}
$$

So, we get an induced map between kernels

$$\operatorname{cl} : Z_0(X)^0 \to \operatorname{Hom}_{D((\operatorname{Sch}/K)_{\text{ét}})}(\tau_{\geqslant 1} R\varphi_*\mathbb{G}_m, \mathbb{G}_m),$$

where we put $Z_0(X)^0 = \ker(Z_0(X) \xrightarrow{\deg} \mathbb{Z})$.

There is a commutative diagram of Yoneda pairings obtained from maps

$$\mathbf{Pic}^0_{X/K} \longrightarrow \mathbf{Pic}_{X/K} \longrightarrow \tau_{\geqslant 1} R\varphi_* \mathbb{G}_m \longleftarrow R\varphi_* \mathbb{G}_m :$$

$$H^2_{\text{ét}}(K, \mathbb{G}_m)$$

$$\downarrow$$

$(\text{BM})'$ $\quad \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(R\varphi_* \mathbb{G}_m, \mathbb{G}_m) \quad \times \quad \mathbb{H}^2_{\text{ét}}(K, R\varphi_* \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(K, \mathbb{G}_m)$

$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow \eta \qquad\qquad\qquad \|$

$\quad\quad \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(\tau_{\geqslant 1} R\varphi_* \mathbb{G}_m, \mathbb{G}_m) \quad \times \quad \mathbb{H}^2_{\text{ét}}(K, \tau_{\geqslant 1} R\varphi_* \mathbb{G}_m) \longrightarrow H^2_{\text{ét}}(K, \mathbb{G}_m)$

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \phi' \qquad\qquad\qquad \|$

$\quad\quad \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(\mathbf{Pic}_{X/K}[-1], \mathbb{G}_m) \quad \times \quad \mathbb{H}^2_{\text{ét}}(K, \mathbf{Pic}_{X/K}[-1]) \longrightarrow H^2_{\text{ét}}(K, \mathbb{G}_m)$

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \qquad\qquad\qquad \|$

$(\text{T})'$ $\quad \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(\mathbf{Pic}^0_{X/K}[-1], \mathbb{G}_m) \quad \times \quad \mathbb{H}^2_{\text{ét}}(K, \mathbf{Pic}^0_{X/K})[-1]) \longrightarrow H^2_{\text{ét}}(K, \mathbb{G}_m)$

By

$$H^2_{\text{ét}}(K, \mathbb{G}_m) = \text{Br}(K) = \mathbb{Q}/\mathbb{Z},$$
$$\mathbb{H}^2_{\text{ét}}(K, R\varphi_* \mathbb{G}_m) = H^2_{\text{ét}}(X, \mathbb{G}_m) = \text{Br}(X) \text{ and}$$
$$\text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(\mathbf{Pic}^0_{X/K}[-1], \mathbb{G}_m) = \text{Alb}_X(K),$$

this is rewritten as

$$\text{Br}(K)$$

$$\downarrow$$

$(\text{BM})'$ $\quad \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(R\varphi_* \mathbb{G}_m, \mathbb{G}_m) \quad \times \quad \text{Br}(X) \longrightarrow \mathbb{Q}/\mathbb{Z}$

$\uparrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \downarrow \eta \qquad\qquad \|$

$(\text{BM})''$ $\quad \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(\tau_{\geqslant 1} R\varphi_* \mathbb{G}_m, \mathbb{G}_m) \quad \times \quad \mathbb{H}^2_{\text{ét}}(K, \tau_{\geqslant 1} R\varphi_* \mathbb{G}_m) \longrightarrow \mathbb{Q}/\mathbb{Z}$

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \phi' \qquad\qquad \|$

$\quad\quad \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(\text{Pic}(\overline{X})[-1], \mathbb{G}_m) \quad \times \quad H^1_{\text{Gal}}(K, \text{Pic}(\overline{X})) \longrightarrow \mathbb{Q}/\mathbb{Z}$

$\downarrow \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \uparrow \qquad\qquad \|$

$(\text{T})'$ $\quad\quad\quad\quad \text{Alb}_X(K) \quad\quad\quad\quad \times \quad H^1_{\text{Gal}}(K, \text{Pic}^0(\overline{X})) \longrightarrow \mathbb{Q}/\mathbb{Z}$

The pairing $(\text{BM})'$ can be seen to give the Brauer–Manin pairing when composed with the map $\text{CH}_0(X) \to \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(R\varphi_* \mathbb{G}_m, \mathbb{G}_m)$. The map η is surjective because $H^3_{\text{Gal}}(K, \mathbb{G}_m) = 0$, so η induces an isomorphism

$$\mathbb{H}^2_{\text{ét}}(K, \tau_{\geqslant 1} R\varphi_* \mathbb{G}_m) = \text{Br}(X)/\text{Br}(K). \tag{8}$$

Hence, the pairing $(\text{BM})''$, when composed with the map

$$Z_0(X)^0 \to \text{Hom}_{D((\text{Sch}/K)_{\text{ét}})}(\tau_{\geqslant 1} R\varphi_* \mathbb{G}_m, \mathbb{G}_m),$$

gives the Brauer–Manin pairing

$$Z_0(X)^0 \times \mathrm{Br}(X)/\mathrm{Br}(K) \to \mathbb{Q}/\mathbb{Z}.$$

The pairing $(\mathrm{T})'$ is the same as the Tate pairing as explained in [Mil06, Remark 3.5 in ch. I].

Thus, we get the commutative diagram

$$
\begin{array}{ccccc}
(\mathrm{BM}) & Z_0(X)^0 & \times & \mathrm{Br}(X)/\mathrm{Br}(K) \longrightarrow \mathbb{Q}/\mathbb{Z} & \\
& & & \quad\uparrow{\scriptstyle\phi'} \qquad\qquad \| & \\
a\downarrow & & & H^1_{\mathrm{Gal}}(K, \mathrm{Pic}(\overline{X})) \longrightarrow \mathbb{Q}/\mathbb{Z} & \qquad(9) \\
& & & \qquad\uparrow\qquad\qquad \| & \\
(\mathrm{T}) & \mathrm{Alb}_X(K) & \times & H^1_{\mathrm{Gal}}(K, \mathrm{Pic}^0(\overline{X})) \longrightarrow \mathbb{Q}/\mathbb{Z} &
\end{array}
$$

We have to show that the map a is equal to the Albanese map and the map ϕ' is the inverse of $\phi_{\overline{X}/X}$. The latter can be checked if one notices that the Hochschild–Serre spectral sequence is realized as the one associated with the tower

$$R\Gamma(\overline{X}, \mathbb{G}_m) \to \tau_{\geqslant 1} R\Gamma(\overline{X}, \mathbb{G}_m) \to \tau_{\geqslant 2} R\Gamma(\overline{X}, \mathbb{G}_m) \to \cdots$$

in the derived category of G_K-modules, together with triangles

$$R^n\Gamma(\overline{X}, \mathbb{G}_m)[-n] \to \tau_{\geqslant n} R\Gamma(\overline{X}, \mathbb{G}_m) \to \tau_{\geqslant n+1} R\Gamma(\overline{X}, \mathbb{G}_m) \xrightarrow{+1}.$$

3.3.1. We will check that the map

$$a : Z_0(X)^0 \to \mathrm{Hom}_{D((\mathrm{Sch}/K)_{\text{ét}})}(\mathbf{Pic}^0_X[-1], \mathbb{G}_m) = \mathrm{Alb}_X(K)$$

in (9) is equal to the Albanese map alb_X.

Note that for any field extension K'/K the map $\mathrm{Alb}_X(K) \to \mathrm{Alb}_X(K')$ is injective, and hence it suffices to show the equality after an arbitrary field extension. In particular, we can assume that X is given a base point $x_0 \in X(K)$.

There is a sheafified version of the map a. Suppose that we are given two morphisms f_1, f_2 from a K-scheme Y to X. Since the composite $Y \xrightarrow{\Gamma_{f_i}} Y \times X \xrightarrow{\pi := \mathrm{pr}_1} Y$ $(i = 1, 2)$ is the identity morphism, the composite $\mathbb{G}_{m,Y} \xrightarrow{\pi^\#} R\pi_* \mathbb{G}_{m, Y \times X} \xrightarrow{\Gamma_i^\#} \mathbb{G}_{m,Y}$ is the identity morphism. Note that $\pi^\#$ induces an isomorphism $\mathbb{G}_{m,Y} \cong \pi_* \mathbb{G}_{m, Y \times X}$. Therefore, the map $\Gamma_{f_1}^\# - \Gamma_{f_2}^\# : R\pi_* \mathbb{G}_{m, Y \times X} \to \mathbb{G}_{m,Y}$ induces a map $\Gamma_{f_1}^\# - \Gamma_{f_2}^\# : \tau_{\geqslant 1} R\pi_* \mathbb{G}_{m, Y \times X} \to \mathbb{G}_{m,Y}$. Composed with $(\mathbf{Pic}^0_X)_Y[-1] \hookrightarrow R^1\pi_* \mathbb{G}_{m, Y \times X}[-1] \to \tau_{\geqslant 1} R\pi_* \mathbb{G}_{m, Y \times X}$, it gives

$$(\mathbf{Pic}^0_X)_Y[-1] \to \mathbb{G}_{m,Y},$$

i.e. an element of $\mathrm{Ext}^1_{(\mathrm{Sch}/Y)_{\text{ét}}}((\mathbf{Pic}^0_X)_Y, \mathbb{G}_{m,Y})$. Thus, we have defined a map of sets

$$(X \times_K X)(Y) \to \mathrm{Ext}^1_{(\mathrm{Sch}/Y)_{\text{ét}}}((\mathbf{Pic}^0_X)_Y, \mathbb{G}_{m,Y}).$$

(Here $\mathrm{Ext}^1_{(\mathrm{Sch}/Y)_{\text{ét}}}(-,-)$ denotes the Ext^1 computed in the category of abelian sheaves on $(\mathrm{Sch}/Y)_{\text{ét}}$. The same applies below.) These are organized to define a map of sheaves

$$b : X \times_K X \to \mathcal{E}xt^1_{(\mathrm{Sch}/K)_{\text{ét}}}(\mathbf{Pic}^0_X, \mathbb{G}_m)(= \mathrm{Alb}_X)$$

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(respectively $b_* : X \to \mathrm{Alb}_X$ if X is pointed, putting $f_2 =$ the constant map to x_0).

To prove the claimed equality of maps, it suffices to show that the map $b : X \times_K X \to \mathrm{Alb}_X$ just defined is the Albanese map. By the universality of Alb_X, it suffices to show that the map b_* for $(X, x_0) = (\mathrm{Alb}_X, 0)$ is the identity map of Alb_X.

Thus, the next general proposition is enough for us to conclude the proof.

PROPOSITION 3.13. *Let A be an abelian variety over a field F and P its dual abelian variety. Then the map*

$$b_* : A \to \mathcal{E}xt^1_{(\mathrm{Sch}/K)_{\text{ét}}}(P, \mathbb{G}_m)$$

(here A is pointed by 0) is the same map as the one induced by the Poincaré sheaf $\mathcal{P}$ on $A \times P$.

Here the Poincaré sheaf $\mathcal{P}$ is a biextension of $A \times P$ by $\mathbb{G}_m$, a $\mathbb{G}_m$-torsor on $A \times P$ which is given a structure of an extension of the A-group $P_A = A \times P$ by $\mathbb{G}_{m,A}$

$$0 \to \mathbb{G}_{m,A} \to \mathcal{P} \to P_A \to 0 \tag{10}$$

and a structure of an extension of the P-group $A_P = A \times P$ by $\mathbb{G}_{m,P}$

$$0 \to \mathbb{G}_{m,P} \to \mathcal{P} \to A_P \to 0, \tag{11}$$

which is characterized by the property that for each F-scheme Y the map

$$
\begin{aligned}
P(Y) \to \mathbf{Pic}^0_A(Y) = \{L \in \mathrm{Pic}(A \times Y, 0 \times Y) \mid \forall y \in Y \quad L_y \text{ is} \\
\text{algebraically equivalent to } 0\} \\
(f : Y \to P) \mapsto \text{pull-back of the } \mathbb{G}_m\text{-torsor } \mathcal{P} \text{ on } A \times P \text{ by the map} \\
\mathrm{id} \times f : A \times Y \to A \times P
\end{aligned}
\tag{12}
$$

is bijective, where for a scheme V and a closed subscheme Z of V, $\mathrm{Pic}(V, Z)$ denotes the group of isomorphism classes of pairs (L, φ), L being a line bundle over V and φ being an isomorphism $\varphi : L|_Z \xrightarrow{\cong} \mathcal{O}_Z$.

In the statement of Proposition 3.13, 'the map induced by the Poincaré sheaf' refers to the map which associates to a morphism $f : Y \to A$ the pull-back of (10) by it. Let us denote the map by $b_{\mathcal{P}}$.

The proof of Proposition 3.13 will be completed in § 3.3.3.

3.3.2. Here we recall a basic compatibility. For any scheme Y, there are canonical isomorphisms

$$H^i_{\text{ét}}(Y, \mathbb{G}_m) \cong \mathrm{Ext}^i_{(\mathrm{Sch}/Y)_{\text{ét}}}(\mathbb{Z}_Y, \mathbb{G}_{m,Y})$$

and the one for $i = 1$ is described as follows: given an extension

$$0 \to \mathbb{G}_{m,Y} \to E \to \mathbb{Z}_Y \to 0,$$

the corresponding $\mathbb{G}_m$-torsor is given by the pull-back of E to $Y \times \{1\} \subset Y \times \mathbb{Z}$. Conversely, given a $\mathbb{G}_m$-torsor E' on Y, the corresponding extension is given by $\coprod_{i \in \mathbb{Z}} E'^i$, E'^i being the ith power of E' as a $\mathbb{G}_m$-torsor.

Now let Y be an arbitrary F-scheme and $f : Y \to A$ an F-morphism. By pull-back, we have a $\mathbb{G}_m$-torsor $\mathcal{P} \times_A Y$ on $Y \times P$ and an extension of Y-groups

$$0 \to \mathbb{G}_{m,Y} \to \mathcal{P} \times_A Y \to P_Y \to 0; \tag{13}$$

equivalently, a morphism in $D((\mathrm{Sch}/Y)_{\text{ét}})$

$$P_Y \to \mathbb{G}_{m,Y}[1]. \tag{14}$$

Suppose that we are given an F-morphism $g : Y \to P$. One checks that the following elements (i)–(iv) of $H^1(Y, \mathbb{G}_m) = \mathrm{Ext}^1(\mathbb{Z}_Y, \mathbb{G}_{m,Y})$ are the same.

(i) The image of g by the connecting homomorphism

$$P_Y(Y) \to H^1(Y, \mathbb{G}_m)$$

arising from (13). Which is the same as the image of g by the map

$$H^0(Y, P_Y) \to H^0(Y, \mathbb{G}_{m,Y}) = H^1(Y, \mathbb{G}_{m,Y})$$

arising from (14).

(ii) The image of $(\mathcal{P}_Y, g)$ by the Yoneda pairing

$$\mathrm{Ext}^1(P_Y, \mathrm{G}_{m,Y}) \times H^0(Y, P_Y) \to H^1(Y, \mathbb{G}_m).$$

Which is the same as the image of $(\mathcal{P}_Y, g)$ by the Yoneda pairing

$$\mathrm{Ext}^1(P_Y, \mathbb{G}_{m,Y}) \times \mathrm{Hom}_{(\mathrm{Sch}/Y)_{\text{ét}}}(\mathbb{Z}_Y, P_Y) \to \mathrm{Ext}^1(\mathbb{Z}_Y, \mathbb{G}_m).$$

(iii) The pull-back of the $\mathbb{G}_m$-torsor $\mathcal{P} \times_A Y$ on $Y \times P$ by $\Gamma_g : Y \to Y \times P$. Which is the same as the pull-back of the extension (13) by the map $g : \mathbb{Z}_Y \to P_Y$.

(iv) The pull-back of the $\mathbb{G}_m$-torsor $\mathcal{P}$ on $A \times P$ by the map $f \times g : Y \to A \times P$.

3.3.3 *Proof of Proposition 3.13.* Suppose that we are given an F-scheme Y and a morphism $f : Y \to A$. Factor f as $Y \xrightarrow{\Gamma_f} A \times Y \xrightarrow{\mathrm{pr}_1} A$ and write $\pi = \mathrm{pr}_2 : A \times Y \to Y$. Over A, there is the Poincaré sheaf $\mathcal{P}$, which is an extension of P_A by $\mathbb{G}_{m,A}$. We see it as an element of $\mathrm{Hom}_{D((\mathrm{Sch}/A)_{\text{ét}})}(P_A, \mathbb{G}_{m,A}[1])$:

$$
\begin{array}{c}
\mathcal{P} \\
\vdots \\
Y \overset{\pi := \mathrm{pr}_2}{\underset{\Gamma_f}{\rightleftarrows}} A \times Y \xrightarrow{\mathrm{pr}_1} A
\end{array}
$$

We have a commutative diagram in $D((\mathrm{Sch}/Y)_{\text{ét}})$

$$
\begin{array}{ccc}
P_Y & \xrightarrow{\quad f^*\mathcal{P} \quad} & \mathbb{G}_{m,Y}[1] \\
{\scriptstyle \Gamma_f^\#}\big\uparrow & & \big\uparrow{\scriptstyle \Gamma_f^\#} \\
R\pi_* P_{A\times Y} & \xrightarrow{R\pi_*(\mathcal{P} \times_F Y)} & R\pi_* \mathbb{G}_{m,A\times Y}[1] \\
{\scriptstyle \pi^\#}\big\uparrow & & \\
P_Y & &
\end{array} \tag{15}
$$

and, if we put $f =$ the constant map to 0, it looks like

$$
\begin{array}{ccc}
P_Y & \xrightarrow{\quad 0 \quad} & \mathbb{G}_{m,Y}[1] \\
{\scriptstyle \Gamma_0^\#}\big\uparrow & & \big\uparrow{\scriptstyle \Gamma_0^\#} \\
R\pi_* P_{A\times Y} & \xrightarrow{R\pi_*(\mathcal{P} \times_F Y)} & R\pi_* \mathbb{G}_{m,A\times Y}[1] \\
{\scriptstyle \pi^\#}\big\uparrow & & \\
P_Y & &
\end{array} \tag{16}
$$

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Note that we have $\Gamma_f^{\#} \circ \pi^{\#} = \Gamma_0^{\#} \circ \pi^{\#} = \text{id} : P_Y \to P_Y$. Therefore, by (15), we have that $b_{\mathcal{P}}(f) := f^* \mathcal{P} \in \text{Ext}^1_{(\text{Sch}/Y)_{\text{ét}}}(P_Y, \mathbb{G}_{m,Y})$ is equal to $\Gamma_f^{\#} \circ \mathcal{P} \times_F Y \circ \pi^{\#}$. From (16), we see that $\Gamma_0^{\#} \circ (\mathcal{P} \times_F Y) \circ \pi^{\#} = 0 : P_Y \to \mathbb{G}_{m,Y}[1]$.

CLAIM 3.14. *The following diagram commutes, where the unnamed maps are the canonical ones:*

$$
\begin{array}{ccc}
R\pi_* \mathbb{G}_{m,A\times Y}[1] \longrightarrow \tau_{\geqslant 1} R\pi_* \mathbb{G}_{m,A\times Y}[1] & & \left(\xrightarrow{\Gamma_f^{\#} - \Gamma_0^{\#}} \mathbb{G}_{m,Y}[1] \right) \\
\Big\uparrow {\scriptstyle (\mathcal{P}\times_F Y)\circ \pi^{\#}} \qquad\qquad \Big\uparrow & & \\
P_Y \xrightarrow{\quad (12) \quad} R^1\pi_* \mathbb{G}_{m,A\times Y} & &
\end{array}
$$

If we prove Claim 3.14, the upper path from P_Y to $\mathbb{G}_{m,Y}[1]$ is equal to $b_{\mathcal{P}}(f)$ and the lower path is equal to $b_*(f)$, so Proposition 3.13 follows.

Claim 3.14 follows if the map

$$P_Y(Y) \to H^0(Y, R\pi_* \mathbb{G}_{m,A\times Y}[1]) = (R^1\pi_* \mathbb{G}_{m,A\times Y})(Y) = \text{Pic}(A \times Y, 0 \times Y)$$

obtained by applying $H^0(Y, -)$ to $(\mathcal{P} \times_F Y) \circ \pi^{\#}$ is the same as (12) for any F-scheme Y.

For this, it suffices to show that the map obtained by applying $H^0(Y, -)$ to $R\pi_*(\mathcal{P} \times_F Y)$ in (15)

$$(\pi_* P_{A\times Y})(Y) \to R^1\pi_* \mathbb{G}_{m,A\times Y}(Y) = \text{Pic}(A \times Y, 0 \times Y)$$

is equal to the canonical inclusion (12)

$$(\pi_* P_{A\times Y})(Y) \subset \text{Pic}(A \times (A \times Y), 0 \times (A \times Y))$$

followed by pull-back by $(\text{diag}_A) \times \text{id}_Y : A \times Y \to A \times A \times Y$.

Replacing $A \times Y$ by an arbitrary F-scheme Y, we are reduced to the following result.

LEMMA 3.15. *For any A-scheme $f : Y \to A$, the following diagram commutes:*

$$
\begin{array}{ccc}
P_Y(Y) & \xrightarrow{\;\mathcal{P}\times_A Y\;} & H^1(Y, \mathbb{G}_m) \\
{\scriptstyle \cap (12)} \Big\downarrow & \nearrow {\scriptstyle \Gamma_f^*} & \\
\text{Pic}(A \times Y, 0 \times Y) & &
\end{array}
$$

But this is clear because by §3.3.2 and (12) both maps send $g : Y \to P$ to the pull-back of the $\mathbb{G}_m$-torsor $\mathcal{P}$ on $A \times P$ to Y by the morphism $(f, g) : Y \to A \times P$. This completes the proof of Proposition 3.13.

Remark 3.16. Proposition 3.13 answers the question raised by van Hamel in [vHa04, Remark 3.4].

4. Another surjectivity result

THEOREM 4.1. *Let K be a Henselian discrete valuation field of characteristic $(0, p)$ (p positive) with residue field k over which any principal homogeneous space under any abelian variety is trivial (e.g. a finite field or a separably closed field). Assume that $v_K(p) < p - 1$, where v_K is the normalized additive valuation of K.*

Let X be a smooth projective variety over K with good reduction. Assume that X has a degree-1 zero-cycle (always true if k is finite or separably closed).

Then the Albanese map

$$\mathrm{CH}_0(X)^0 \to \mathrm{Alb}_X(K)$$

is surjective.

Proof. Since X has a degree-1 zero-cycle, we may assume that X has a rational point by a trace argument.

By a Bertini theorem over discrete valuation rings due to Jannsen and Saito [SS10, Theorem 4.2], there is a smooth curve $C \subset X$ which is obtained by repeated hypersurface sections and contains a rational point and has good reduction.

As is explained in [Gab01, Proposition 2.4], such a hypersurface section has the property that the homomorphism

$$\mathrm{Alb}_C \to \mathrm{Alb}_X$$

is smooth and has a connected kernel N.

Now $\mathrm{Alb}_C(= J_C)$ has good reduction because C has good reduction. We have now an exact sequence of abelian varieties

$$0 \to N \to J_C \to \mathrm{Alb}_X \to 0.$$

By [BLR10, Theorem 4, p. 187], which is applicable by the assumption $v_K(p) < p-1$, the induced sequence of Néron models

$$0 \to \mathcal{N} \to \mathcal{J}_C \to \mathcal{A}lb_X \to 0$$

is exact and $\mathcal{N}$ and $\mathcal{A}lb_X$ also have good reduction. Therefore, it induces an exact sequence of abelian varieties over k

$$0 \to \overline{N} \to \overline{J_C} \to \overline{\mathrm{Alb}_X} \to 0.$$

(Here overline means reduction, not the algebraic closure.)

Now we are going to establish the surjectivity of $J_C(K) \to \mathrm{Alb}_X(K)$; if it is proved our assertion follows from the known fact that $\mathrm{CH}_0(C)^0 \to J_C(K)$ is an isomorphism as C has a rational point. Pick any $a \in \mathrm{Alb}_X(K)$. We show that $N_a = J_C \times_{\mathrm{Alb}_X} a$ has a rational point. The section a naturally extends to a section $a' \in \mathcal{A}lb_X(\mathcal{O}_k)$ and induces a section $\overline{a} \in \overline{\mathrm{Alb}_X}(k)$. We consider the scheme $\mathcal{N}_{a'} = \mathcal{J}_C \times_{\mathcal{A}lb_X} a'$. Its special fiber $\overline{N}_{\overline{a}} = \overline{J_C} \times_{\overline{\mathrm{Alb}_X}} \overline{a}$ is a torsor over the field k under the abelian variety $\overline{N}$. By the assumption on the residue field, it has a rational point. By Hensel's lemma, the rational point lifts to a section of $\mathcal{N}_{a'}$, giving a rational point on N_a. $\quad\square$

REFERENCES

AK79   A. Altman and S. Kleiman, *Bertini theorems for hypersurface sections containing a subscheme*, Comm. Algebra **7** (1979), 775–790.

BK10   L. Barbieri-Viale and B. Kahn, *On the derived category of* 1-*motives*, Astérisque, to appear, Preprint (2010), arXiv:1009.1900.

BLR10   S. Bosch, W. Lütkebohmert and M. Raynaud, *Néron models*, Ergebnisse der Mathematik und ihrer Grenzgebiete, 3 Folge, Bd. 21 (Springer, Berlin–Heidelberg, 2010).

dJo   A. J. de Jong, *A result of Gabber*, Preprint, available at
       http://www.math.columbia.edu/~dejong/.

Gab01 O. Gabber, *On space filling curves and Albanese varieties*, Geom. Funct. Anal. **11** (2001),
       1192–1200.

Gro68 A. Grothendieck, *Le groupe de Brauer I, II, III*, in *Dix exposés sur la cohomologie des schémas*
       (North-Holland, Amsterdam, 1968).

KS83   K. Kato and S. Saito, *Unramified class field theory of arithmetical surfaces*, Ann. of Math. (2)
       118 (1983), 241–275.

Kle05 S. Kleiman, *The Picard scheme*, in *Fundamental algebraic geometry. Grothendieck's FGA
       explained*, Mathematical Surveys and Monographs, vol. 123 (American Mathematical Society,
       Providence, RI, 2005), 235–321.

KO74   M. A. Knus and M. Ojanguren, *Théorie de la descente et algèbres d'Azumaya*, Lecture Notes in
       Mathematics, vol. 389 (Springer, Berlin–Heidelberg, 1974).

Lan59 S. Lang, *Abelian varieties*, Interscience Tracts in Pure and Applied Mathematics, vol. 7
       (Interscience, New York–London, 1959).

Lic69 S. Lichtenbaum, *Duality theorems for curves over p-adic fields*, Invent. Math. **7** (1969), 120–136.

Man70 Y. Manin, *Le groupe de Brauer–Grothendieck en géométrie diophantienne*, in *Actes du Congrès
       International des Mathématiciens,* Nice, 1970, Tome 1 (Gauthier-Villars, Paris, 1971), 401–411.

Mat55 A. Mattuck, *Abelian varieties over p-adic ground fields*, Ann. of Math. (2) **62** (1955), 92–119.

Mil06 J. S. Milne, *Arithmetic duality theorems*, second edition (BookSurge, North Charleston, SC,
       2006).

RØ06   A. Rosenschon and P. A. Østvær, *Rigidity of pseudo pretheories*, Invent. Math. **166** (2006),
       95–102.

SS10   S. Saito and K. Sato, *A finiteness theorem for zero-cycles over p-adic fields. With an appendix
       by U. Jannsen*, Ann. of Math. (2) **172** (2010), 593–639.

SS14   S. Saito and K. Sato, *Zero-cycles on varieties over p-adic fields and Brauer groups*, Ann. Sci.
       Éc. Norm. Supér. (4) **47** (2014), 505–537.

SS95   S. Saito and R. Sujatha, *A finiteness theorem for cohomology of surfaces over p-adic fields and an
       application to Witt groups*, in *K-theory and algebraic geometry: connections with quadratic forms
       and division algebras (Santa Barbara, CA, 1992)*, Proceedings of Symposia in Pure Mathematics,
       vol. 58, part 2 (American Mathematical Society, Providence, RI, 1995), 403–415.

SS03   M. Spieß and T. Szamuely, *On the Albanese map for smooth quasi-projective varieties*,
       Math. Ann. **325** (2003), 1–17.

Tat57 J. Tate, *WC-groups over p-adic fields*, Sém. Bourbaki **156** (1956–1958), 265–277.

vHa04 J. van Hamel, *Lichtenbaum–Tate duality for varieties over p-adic fields*, J. Reine Angew. Math.
       575 (2004), 101–134.

Voe00 V. Voevodsky, *Cohomological theory of presheaves with transfers*, in *Cycles, transfers, and
       motivic homology theories*, Annals of Mathematics Studies, vol. 143 (Princeton University Press,
       Princeton, NJ, 2000).

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