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GROUPS AND COMPLEMENTS OF KNOTS

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We investigate the extent to which knot groups determine knot manifolds and knot types. Let $K_i (i = 1, 2)$ denote a tame knot in S^3, let C_i denote a K_i-knot manifold, and assume that $\pi_1(C_1) \approx \pi_1(C_2)$. The first named author recently showed (in [**6**]) that, if C_1 has no essential annulus, then $C_1 \cong C_2$, and so K_1 and K_2 are equivalent, if K_1 has property P.

On the other hand, if C_1 contains an essential annulus, then K_1 is either composite or cabled (see Lemma 1 of this paper, for example). If K_1 is composite, then so is K_2 (by Lemma 2), and there is a one-to-one correspondence between the prime factors of K_1 and those of K_2 such that corresponding knots are equivalent (Theorem 1). The latter result contrasts with the existence of distinct composite knots having isomorphic groups and with the uniqueness of a knot's prime factors (see [**13**]).

Suppose, however, that K_1 is a (p_1, q_1)-cable knot about the knot k_1; that is, suppose $K_1 = J(p_1, q_1; k_1)$. Then K_2 is also a cable knot (see the remark in the proof of Theorem 3 or see [**17**, pp. 6–7, and Comment (A), p. 11]); say, $K_2 = J(p_2, q_2; k_2)$. By Corollary 2, we have $|p_1| = |p_2|$, the number $|q_1| = |q_2| (\geqq 2)$, the knots k_1 and k_2 have homeomorphic complements, and, if $|p_1| \geqq 3$, then K_1 and K_2 are equivalent; of course, if k_1 is trivial, then K_1 is a torus knot, and K_1 and K_2 are immediately equivalent. We thus come to J. Simon's natural conjecture covering the only case for which we do not know whether the group of a prime knot determines the topological type of the knot's complement. The conjecture is that, if $|p_1| = 1$ or 2, then $C_1 \cong C_2$.

If we not only assume that $\pi_1(C_1) \approx \pi_1(C_2)$ but also that K_1 is prime and all knots have property P, then we can prove that K_1 and K_2 are equivalent; this is an old result of F. Waldhausen and J. Hempel (see [**22**] and [**23**]). Because we need this fact and portions of its proof in Section 4, and because no proof is in print (cf. [**23**]), we give the proof in Section 3; see Theorem 2.

We settle an old question in Corollary 3, by showing that the genus of a knot is a knot-group invariant. In Corollary 1 we give yet another algebraic characterization of knots by showing that, if k_1 and k_2 are any knots and if $|p| \geqq 3$, then k_1 and k_2 are (ambient) isotopic if and only if $J(p, q; k_1)$ and $J(p, q; k_2)$ have isomorphic groups (cf. [**17**, Theorem 3, p. 10] as well as [**25**, Theorem 2.2, p. 264]). Our final result, Corollary 4, gives a bound for the number of distinct composite knots with a given group.

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1. Preliminaries. Throughout this work, all spaces are simplicial complexes; the three-sphere has a fixed orientation; all maps are piecewise linear; all submanifolds, polyhedral; and all knots, oriented. The symbol $\approx$ denotes isomorphism of groups; the symbol $\cong$, homeomorphism of spaces; the symbol $\simeq$, homotopy of maps. The complement C of an open solid torus in S^3 is a *toral solid*; if a core K of the solid torus is knotted, then C is a *K-knot manifold*.

Let A be an annulus, and let a be a nonseparating, properly imbedded arc (a *spanning arc*) in A. Let M be a three-manifold. A map $f : (A, \partial A) \to (M, \partial M)$ is *essential*, if $f_* : \pi_1(A) \to \pi_1(M)$ is monic and if $f(a)$ is not homotopic rel its boundary to an arc in ∂M.

We point out the following two results: (1) if $\pi_2(M) = 0$, if ∂M is incompressible, and if $f_* : \pi_1(A) \to \pi_1(M)$ is monic, then f is essential if and only if f is not homotopic rel ∂A to a map into ∂M [**6**, Proposition 3.2, p. 4]; (2) the annulus theorem states that a compact orientable three-manifold admits an essential imbedding of an annulus if and only if it admits essential map of an annulus [**4**, Theorem 3, p. 230]. Recall that a properly imbedded surface F in M is *boundary parallel*, if there is an imbedding $k : F \times I \to M$ such that $k(F \times \{0\}) = F$ and $k(((\partial F) \times I) \cup (F \times \{1\})) \subseteq \partial M$.

PROPOSITION 1. *If A is a properly imbedded, incompressible annulus in a knot manifold C, then A is not boundary parallel if and only if the inclusion $i : A \to C$ is not homotopic rel ∂A to a map into ∂C, that is, if and only if i is essential.*

Proof. If i is homotopic rel ∂A to a map into ∂C, then [**21**, Proposition 5.4, p. 75] implies immediately that A is boundary parallel. Conversely, if A is boundary parallel, then there exists an annulus $B(\subset \partial C)$ and a homeomorphism N $: A \times I \to C$ such that $H|(A \times \{0\}) = i$ and that $H(A \times \{1\}) = B$. Now $A \cup B \cup H((\partial A) \times I)$ is the boundary of a solid torus $H(A \times I)$ in C, and so one can easily construct a homotopy $J : A \times I \to H(A \times I)$ rel ∂A such that $J|(A \times \{1\}) = i$ and that $J(A \times \{1\}) = B \cup H((\partial A) \times I)$; thus, i is homotopic rel ∂A to a map into ∂C, completing the proof of the proposition.

A knot K is *composite*, if, for each K-knot manifold C, there exist knot manifolds X_1 and X_2 and an annulus A such that $C = X_1 \cup_A X_2$ and such that each component of ∂A bounds a meridional disk in $S^3 - \operatorname{Int} C$. A knot manifold of a composite knot is a *composite-knot manifold*. A knot is *prime*, if it is not composite; correspondingly, we have *prime-knot manifolds*. An annulus A in a manifold M is *essential* (or *essentially imbedded*), if the inclusion map $A \to M$ is essential. The proof of the following proposition is routine, and we shall omit it.

PROPOSITION 2. *In a composite-knot manifold $X_1 \cup_A X_2$, the annulus A is essential.*

Let (μ, λ) be an (oriented) meridian-longitude pair on the boundary of a solid torus V in S^3, let K be a core of V, and let (p, q) be a pair of relatively prime integers. A (suitably oriented) simple closed curve in ∂V homologous to $p\mu + q\lambda$ is a *(p, q)-curve on ∂V* (with respect to (μ, λ)) and, if $|q| \geqq 2$, a *(p, q)-cable knot with core K*. The following lemma is a refinement of Lemma 1.1, on page 262 of [**25**], of Lemma 2.2, on page 4 of [**17**], and of Proposition 1.

LEMMA 1. *Let C be a knot manifold in S^3, let A be a properly imbedded, essential annulus in C, and let K be a core of S^3 − Int C. Then there exist toral solids X_1 and X_2 such that*

(a) $C = X_1 \cup_A X_2$;

(b) *the following three statements are mutually equivalent:* (i) *each of X_1 and X_2 is a knot manifold,* (ii) *each component of ∂A bounds a meridional disk in S^3 −* Int C, *and* (iii) *the knot K is composite;*

(c) *if X_2 is a solid torus and if each component of ∂A is a (p, q)-curve on ∂X_2, then the inclusion-induced homomorphism $\pi_1(A) \rightarrow \pi_1(X_2)$ is not surjective, $|q| \geqq 2$, the knot K is a (p, q)-cable knot about a core of X_2, and each component of ∂A is an $(n, \pm 1)$-curve on ∂C for some n.*

The proof (which we shall omit) of Lemma 1 follows easily from the proof of Lemma 1.1 in [**25**, p. 262], the proof of Lemma 2.2 in [**17**, p. 4], and the statement of Proposition 1.

We now point out an easy consequence of Proposition 2 and Lemma 1; D. Noga first proved the result (see [**11**]), and it is, of course, known in a stronger form [**1**, **7**]. We shall omit the proof.

PROPOSITION 3. *The complement of a composite knot determines the knot's type.*

LEMMA 2. *If two knots have isomorphic groups, then both knots are prime or both knots are composite.*

Proof. Let C' be a knot manifold, let C be a composite-knot manifold, and suppose that $\pi_1(C') \approx \pi_1(C)$. By the definition of a composite knot and by Proposition 2 and Proposition 1, there is an annulus A and knot manifolds C_1 and C_2 such that $C = C_1 \cup_A C_2$ and such that A is properly imbedded, incompressible, and not boundary parallel in C.

Because $\pi_1(C') \approx \pi_1(C)$ and because each of C' and C is aspherical, there exists a homotopy equivalence $f : C' \rightarrow C$; cf. [**10**, p. 93]. The manifold C' is orientable and compact, the annulus A is bicollared in C, the ker $(\pi_j(A) \rightarrow (\pi_j(C)) = \{1\}$ $(j = 1, 2)$ because A is incompressible in C and $\pi_2(A) = \{0\}$, and $\pi_2(C_i - A) = \{0\}$ $(i = 1, 2)$ because each C_i is a knot manifold. Hence, by [**20**, Lemma 1.1, p. 506], there exists a mapping $g : C' \rightarrow C$ with the following properties:

(1) $g \simeq f$;

(2) g is transverse with respect to A;

(3) $g^{-1}(A)$ is a compact orientable surface properly imbedded in C';

(4) if F is any component of $g^{-1}(A)$, then the ker $(\pi_j(F) \to \pi_j(C')) = \{1\}$ $(j = 1, 2)$.

If $g^{-1}(A) = \emptyset$, then either $g_*(\pi_1(C')) \subseteq \pi_1(C_1)$ or $g_*(\pi_1(C')) \subseteq \pi_1(C_2)$. Therefore, $g_*(\pi_1(C'))$ is a *proper* subgroup of $\pi_1(C)$, because

$$\pi_1(C)(= \pi_1(C_1)_{\pi_1*(A)}\pi_1(C_2))$$

is a nontrivial free product with amalgamation. Hence, $g^{-1}(A) \neq \emptyset$; for more details, see [**21**, § 1, p. 265].

Because of properties (4) and (1) of g, the group $\pi_1(F)$ is isomorphic to a subgroup of $\pi_1(A)$. Hence, by property (3), the surface F is either a 2-sphere, a disk, or an annulus.

Property (4) implies that $\pi_2(F) = 0$; thus, F is not a 2-sphere. If F were a disk or a boundary-parallel annulus, then we could replace g by a map $g' : C' \to C$ satisfying properties (1) through (4) and the property that $g'^{-1}(A)$ has fewer components than $g^{-1}(A)$; see [**25**, § 2, p. 265] for details. The surface F is, therefore, an incompressible annulus that is properly imbedded but not boundary parallel in C'.

By Lemma 1, C' is a composite-knot manifold (and we are done) or C' is a cable-knot manifold. By [**17**, pp. 6–7, and Comment (A), p. 11], if $\pi_1(C)$ is a cable-knot group, then C is a cable-knot manifold. But cable knots are prime [**14**], which contradicts our assumption that C is a composite-knot manifold and completes the lemma's proof.

Remark. For most admissible pairs (p, q), the topological type of a (p, q)-cable-knot manifold determines the knot type; see [**7**, Theorem 8, p. 72] or [**16**, Theorem 2, p. 196].

2. Composite-knot groups. In this section, we prove that the group of a composite knot determines the knot's prime factors up to knot type.

THEOREM 1. *Let K and L be knots in S^3, let $K_1, \ldots, K_m$ be K's prime factors, and let $L_1, \ldots, L_n$ be L's prime factors; moreover, we assume that $n > 1$. If $\pi_1(S^3 - K) \approx \pi_1(S^3 - L)$, then K is composite and there is a bijection $\{K_1, \ldots, K_m\} \to \{L_1, \ldots, L_n\}$ such that corresponding knots are equivalent.*

Proof. Let M be a K-knot manifold, and let N be an L-knot manifold. Because L is composite and because $\pi_1(M) \approx \pi_1(N)$, the knot K is also composite (Lemma 2).

Let A be a properly imbedded, essential annulus in N. Because each of M and N is aspherical and because $\pi_1(M) \approx \pi_1(N)$, there exists a homotopy equivalence $f : M \to N$. By [**20**, Lemma 1.1, p. 506], we can assume that $f^{-1}(A)$ is a collection of properly imbedded, incompressible disks and annuli. Because M is aspherical and ∂M is incompressible, we can assume that $f^{-1}(A)$ contains no disks. Moreover, after a homotopy, we can assume that $f^{-1}(A)$ is essential, that is, not boundary parallel (by Proposition 1). Furthermore, because $\pi_1(N)$ is the *nontrivial* free product of two knot groups amalgamated over $\pi_1(A)$, we see that $f^{-1}(A) \neq \emptyset$; cf. [**25**, p. 265].

Now let A' denote a component of $f^{-1}(A)$. In the following commutative diagram, all groups are isomorphic to Z; see [**8**, Theorem (3.68), p. 120].

$$\begin{tikzcd} H_1(A'; Z) \arrow[r, "\lambda^*"] \arrow[d, "(f|A')^*"] & H_1(M; Z) \arrow[d, "f^*"] \\ H_1(A; Z) \arrow[r, "\delta^*"'] & H_1(N; Z) \end{tikzcd}$$

The homomorphisms λ^* and δ^* are inclusion-induced, and f induces f^*; moreover, f^* is an isomorphism, because f is a homotopy equivalence. Also, each of λ^* and δ^* is an isomorphism, because both M and N are composite-knot manifolds and because we can, therefore, apply Lemma 1(b)(ii) to each of them. Hence, $(f|A')^*$ is an isomorphism and, therefore, $(f|A')_* : \pi_1(A') \to \pi_1(A)$ is an isomorphism. Thus, $f|A'$ is a homotopy equivalence and, hence, is homotopic to a homeomorphism $A' \to A$. Because f is transverse with respect to A (by [**20**, Lemma 1.1, p. 506]), we can extend any homotopy of $f|A'$ to a homotopy (that acts as the identity outside a small neighborhood of A') of f; therefore, we can assume that $f|A'$ is a homeomorphism for each component A' of $f^{-1}(A)$.

Our final assumption about f is that we can assume that $f^{-1}(A)$ is connected. We shall not prove this, but we refer the reader to any of [**5**; **15**; **17**; or **25**].

The annulus $f^{-1}(A)$—we shall call it A'—splits M into knot manifolds M' and M''. Likewise, A splits N into N' and N'', and (for a proper choice of primes) $f(M') \subseteq N'$ and $f(M'') \subseteq N''$. Evidently, f_* defines isomorphisms $\pi_1(M') \to \pi_1(N')$ and $\pi_1(M'') \to \pi_1(N'')$. Hence, we have $K = K' \# K''$ and $L = L' \# L''$ such that, by Lemma 1(*b*), the knots K' and L' (and K'' and L'') have isomorphic groups *with isomorphisms preserving meridians.*

Now assume that L' (and, hence, K') is prime. There is an annulus $B \subset \partial M'$ such that $\partial M' = B \cup A'$. If $f|B$ were essential as a map, then there would exist a properly imbedded, essential annulus in N' whose boundaries are meridians of L' (and of L) [**4**, Theorem 1, p. 220]. Thus, by Lemma 1(b)(iii), the knot manifold N' is not a *prime*-knot manifold, which contradicts our assumption. Hence, $f|B$ is not essential, and so it is homotopic to a map into $\partial N'$ rel ∂B $(= \partial A')$, because N' is aspherical and $\partial N'$ is incompressible. By the proof and statement of the homotopy extension theorem for maps of polyhedra (see [**8**, Theorem (1.6. 10), p. 33], for example), one can extend this homotopy of $f|B$ to a homotopy of f that is constant on A'.

Hence, if L' (and thus, K') is prime, then there is a homotopy equivalence $(M', \partial M') \to (N', \partial N')$ preserving meridians. By [**21**], $M' \cong N'$, and the homeomorphism can be chosen to preserve meridians. Therefore, if L' is prime, then K' and L' are equivalent knots. Similarly, K'' and L'' are equivalent, if L'' is prime. Because factorization of knots into prime factors is unique (see [**13**]), one can now apply induction on n to complete the proof.

3. The GK-conjecture.

THEOREM 2. *If all knots have property* P, *then the group of a prime knot determines the knot's type, that is, the* GK-*conjecture is true.*

Proof. Let M be a knot manifold of the *prime* knot K, and let N be an L-knot manifold; we assume that $\pi_1(M) \approx \pi_1(N)$. By Lemma 2, L is also a prime knot. If M admits no proper, essential imbeddings of annuli, then M and N are homeomorphic [**6**, Theorem 10, p. 42]; the assumption that all knots have property P then implies that K and L are equivalent. If K is a torus knot, then so is L [**3**], and it is well known that K and L are equivalent [**12**, p. 61]. Therefore, if M contains a properly imbedded, essential annulus A, then we can assume that K is a (p, q)-cable knot about a *knotted* core k and that L is a (p', q')-cable knot about a knotted core h (by Lemma 1).

Remark. Because one can readily show that the knot manifold of a (nontrivial) cable knot contains a properly imbedded, essential annulus and because (nontrivial) cable knots are prime [**14**, Theorem 4, p. 250], it is now evident that if a knot's group is isomorphic to a cable knot's group, then the original knot must also be a cable. One could also apply Lemma 1.1 of [**20**] to prove this; cf. [**17**, pp. 6–7, and Comment (A), p. 11].

By Lemma 1, there is a k-knot manifold X_1 and a solid torus X_2 such that $M = X_1 \cup_A X_2$. Furthermore, $\pi_1(M) \approx \pi_1(X_1)_{\pi_1*(A)}\pi_1(X_2)$, and this amalgamated free product is nontrivial, by Lemma 1(c).

There is a homotopy equivalence $f : N \rightarrow M$, and we can assume that $f^{-1}(A)$ is an orientable, properly imbedded, incompressible surface [**20**, Lemma 1.1, p. 506]. Because $\pi_1(M)$ is a nontrivial free product amalgamated over $\pi_1(A)$, we see that $f^{-1}(A) \neq \emptyset$. Moreover, one can easily show that $f^{-1}(A)$ is a collection of disjoint annuli $A_1, \ldots, A_n$, and, after a homotopy of f, we can assume that none of these annuli is boundary parallel in N.

We claim that we can also assume that $f|A_i (i = 1, \ldots, n)$ is a homeomorphism. To prove this, consider the following commutative diagram in which each group is isomorphic to Z (see [**8**, Theorem (3.68), p. 120]).

$$\begin{tikzcd} H_1(A_i; Z) \arrow[r, "\lambda_i"] \arrow[d, "(f|A_i)^*"'] & H_1(N; Z) \arrow[d, "f^*"] \\ H_1(A; Z) \arrow[r, "\lambda"'] & H_1(M; Z) \end{tikzcd}$$

Both λ_i and λ are inclusion-induced; the homomorphism f^* is an isomorphism, because f is a homotopy equivalence.

Because L is a nontorus cable knot (and, therefore, also prime), the annulus A_i separates N into a knot manifold W_i and a solid torus V_i (so that $N = W_i \cup_{A_i} V_i$), by Lemma 1; by Lemma 1 and the proof of Lemma 2.2, p. 4, of [**17**], a component b_i of ∂A_i belongs to the same (ambient) isotopy type as L. Because L determines both $|p'|$ and $|q'|$ (recall that L is a (p', q')-cable knot) [**14**, Theorem 5, p. 253], the curve b_i is a (p', q')-curve on $S^3 - \text{Int } W_i$ (for a suitably oriented, meridian-longitude pair). Hence, if z_i denotes a generator of $H_1(A_i; Z)$, if z denotes a generator of $H_1(A; Z)$, if t' denotes a generator of $H_1(N; Z)$, and if t denotes a generator of $H_1(M; Z)$, then $\lambda_i(z_i) = t'^{\pm|p'q'|}$; also $\lambda(z) = t^{\pm|pq|}$, because a component of ∂A is a (p, q)-curve (on $\partial(S^3 - \text{Int } X_2)$).

Thus, if $(f|A_i)^*(z_i) = z^r$, then $\pm r|pq| = |p'q'|$, and so $|pq|$ divides $|p'q'|$. Working with a homotopy inverse $M \to N$ of f, one can show that $|p'q'|$ divides $|pq|$. Consequently, $|p'q'| = |pq|$, and so $r = \pm 1$. Therefore, $(f|A_i)^*$ and, thus, $(f|A_i)_* : \pi_1(A_i) \to \pi_1(A)$, is an isomorphism. It follows that $f|A_i$ is a homotopy equivalence and is, therefore, homotopic to a homeomorphism. Finally, such a homotopy extends to a homotopy of f that is constant off a small product neighborhood of A_i, because f is transverse with respect to A (Lemma 1.1 of [**20**]).

We next claim that we can assume that $f^{-1}(A)$ is connected. The proof is, *mutatis mutandis*, the same as in [**17**, Claims 2–6, pp. 7–9], and we shall omit it.

Denote the only component of $f^{-1}(A)$ by A_1, recall that the annulus A_1 separates N into a knot manifold W_1 and a solid torus V_1 so that $N = W_1 \cup_{A_1} V_1$, and let x_1 be a point in A_1; we have

$$\pi_1(N, x_1) = \pi_1(W_1, x_1)_{\pi_1*(A_1, x_1)}\pi_1(V_1, x_1).$$

Because $f^{-1}(A)$ contains only one component, exactly one of the groups $f_*(\pi_1(W_1, x_1))$ and $f_*(V_1, x_1)$ is contained in $\pi_1(X_1, f(x_1))$ and the other is contained in $\pi_1(X_2, f(x_1))$. Also, $f_*(\pi_1(A_1, x_1)) = \pi_1(A, f(x_1))$. Because each of W_1 and X_1 is a knot manifold and each of V_1 and X_2 is a solid torus, and because f_* is an isomorphism, we have $f_*(\pi_1(W_1, x_1)) = \pi_1(X_1, f(x_1))$ and $f_*(\pi_1(V_1, x_1)) = \pi_1(X_2, f(x_1))$ [**2**, Proposition 2.5, p. 485]. Therefore, $f(W_1) \subseteq X_1$, $f(V_1) \subseteq X_2$, and each of $(f|W_1)_*$ and $(f|V_1)_*$ is an isomorphism.

Now set $B = \partial W_1 \cap \partial N$. Suppose that $f|B : (B, \partial B) \to (X_1, \partial X_1)$ is essential. Then there exists a properly imbedded, essential annulus A' in X_1 such that $\partial A' = \partial A$ [**4**, Theorem 1, p. 220]. The components of ∂A are, by Lemma 1(c), (p, q)-curves on ∂X_1 and $(n, \pm 1)$-curves (for some n) on ∂M. Because A' is essential in X_1 and because $\partial A' = \partial A$, the same curves would be $(n', \pm 1)$-curves on ∂X_1, contradicting the fact that $|q| \geqq 2$. Hence, $f|B$ is inessential.

Because X_1 is aspherical and ∂X_1 is incompressible (in X_1), the inessential map $f|B$ is homotopic to a map into ∂X_1 rel ∂B. This homotopy extends to

$f|W_1$, and, hence, $f|W_1$ is homotopic to a map $f_0 : W_1 \to X_1$ such that $f_0(\partial W_1) \subseteq \partial X_1$ and $f_0|\partial A_1 = f|\partial A_1$. By [**21**, Theorem 6.1, p. 77], there exists a homotopy $f_t : (W_1, \partial W_1) \to (X_1, \partial X_1)$ $(0 \leqq t \leqq 1)$ such that f_1 is a homeomorphism. Evidently, after an appropriate isotopic deformation of $(X_1, \partial X_1)$, we can assume that $f_1|A_1 = f|A_1$.

Now because of our assumption that all knots have property P, we can extend f_1 to an autohomeomorphism of S^3 (assuming merely that a knot's complement determines the knot's type would not ostensibly guarantee such an extension of f_1). Because $f_1|\partial A_1 = f|\partial A_1$, there is a boundary component s_1 of A_1 and a boundary component s of A such that $f_1(s_1) = s$; moreover, s_1 is equivalent (as a knot) to L and s is equivalent to K. Therefore, K and L are equivalent, concluding the proof of Theorem 2.

4. Applications of Theorems 1 and 2. Consider the cable knot $J(p, q; k)$ oriented so that it is homologous to $p\mu + q\lambda$ on ∂V (see the definition of $J(p, q; k)$ following Proposition 2 in Section 1). Orient k so that $J(p, q; k)$ is homologous to $|q|k$ in V, and reorient λ (if necessary) so that it is homologous to k in V; denote λ, with this (possibly new) orientation, by $\bar{\lambda}$. Now reorient μ (if necessary) so that its linking number with k is $+1$; let $\bar{\mu}$ denote μ with this orientation. Then $J(p, q; k)$ is homologous (on ∂V) to either $p\bar{\mu} + |q|\bar{\lambda}$ or $-p\bar{\mu} + |q|\bar{\lambda}$, and we can denote $J(p, q; k)$ either by $K(p, |q|; k)$ or by $K(-p, |q|; k)$, accordingly.

Let $\{K\}$ denote the (ambient) isotopy type of a knot K. Recall that the triple $(\{k\}, p, q)$ characterizes $\{K(p, q; k)\}$, provided that p and q are relatively prime, that $q \geqq 2$, and that, if k is unknotted, then $q \leqq |p|$ [**14**, Theorem 5, p. 253].

Corollary 1. *Let K_1 and K_2 be (oriented) knots in S^3, and let (p, q) be a fixed pair of relatively prime integers such that*

(i) $q \geqq 2$;

(ii) $|p| \geqq 3$ *when either K_1 or K_2 is knotted; and*

(iii) $q < |p|$, *when either K_1 or K_2 is unknotted.*

Then K_1 and K_2 belong to the same (ambient) isotopy type if and only if

$$\pi_1(S^3 - K(p, q; K_1)) \approx \pi_1(S^3 - K(p, q; K_2)).$$

Proof. If $\{K_1\} = \{K_2\}$, then $\{K(p, q; K_1)\} = \{K(p, q; K_2)\}$, and so $\pi_1(S^3 - K(p, q; K_1)) \approx \pi_1(S^3 - K(p, q; K_2)\}$, proving the necessity.

The sufficiency obviously holds, if either K_1 or K_2 is unknotted. Hence, suppose that K_1 is knotted, let N be a $K(p, q; K_1)$-knot manifold, and let M be a $K(p, q; K_2)$-knot manifold. From the proof of Theorem 2, we have a map $f : N \to M$ such that $f|W_1 : W_1 \to X_1$ is a homeomorphism, the space $f(V_1) \subseteq X_2$, the annulus $f(A_1) = A$, and f_* is an isomorphism. Here, W_1 is a K_1-knot manifold, and X_1 is a K_2-knot manifold. Let $(\mu_{K_1}, \lambda_{K_1})$ denote a meridian-longitude pair of K_1 on ∂W_1, and let $(\mu_{K_2}, \lambda_{K_2})$ denote a meridian-

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longitude pair of K_2 on ∂X_1; let s_1 denote a component of ∂A_1, and set $s = f(s_1)$. We assume that λ_{K_1} is homologous to K_1 in $S^3 - \operatorname{Int} W_1$ and that λ_{K_2} is homologous to K_2 in $S^2 - \operatorname{Int} X_1$; we assume, also, that the linking number of μ_{K_i} and K_i is $+1$ $(i = 1, 2)$.

If $x \in s_1$, then the curve s_1, properly oriented, represents $\mu_{K_1}{}^p\lambda_{K_1}{}^q$ in $\pi_1(W_1, x)$. Hence, $f_*(\mu_{K_1}{}^p\lambda_{K_1}{}^q)$ is either $\mu_{K_2}{}^p\lambda_{K_2}{}^q$ or $\mu_{K_2}{}^{-p}\lambda_{K_2}{}^{-q}$ in $\pi_1(X_1, f(x))$. Now we can assume that $f_*(\lambda_{K_1}) = \lambda_{K_2}{}^{\pm 1}$ and that $f_*(\mu_{K_1}) = \mu_{K_2}{}^{\pm 1}\lambda_{K_2}{}^m$, for some m. Thus, $f_*(\mu_{K_1}{}^p\lambda_{K_1}{}^q) = \mu_{K_2}{}^{\pm p}\lambda_{K_2}{}^{mp\pm q}$, and we have either

$$\text{(a) } \mu_{K_2}{}^{\pm p}\lambda_{K_2}{}^{mp\pm q} = \mu_{K_2}{}^p\lambda_{K_2}{}^q \quad \text{or} \quad \text{(b) } \mu_{K_2}{}^{\pm p}\lambda_{K_2}{}^{mp\pm q} = \mu_{K_2}{}^{-p}\lambda_{K_2}{}^{-q}.$$

In case (a), we have $mp + q = q$ or $mp - q = q$ according as $f(\lambda_{K_1}) = \lambda_{K_2}$ or $\lambda_{K_2}{}^{-1}$. If $mp - q = q$, then $|p| \leqq 2$, because p and q are relatively prime; but $|p| \geqq 3$ (by hypothesis (ii)); hence, $mp + q = q$, and so $m = 0$. Analogously, in case (b), we obtain $m = 0$. Consequently, $f|W_1 : W_1 \to X_1$ extends to an autohomeomorphism g of S^3 taking K_1 onto K_2 and, theorefore, K_1 and K_2 are equivalent.

Suppose now that $f_*(\mu_{K_1}) = \mu_{K_2}$ and that $f_*(\lambda_{K_1}) = \lambda_{K_2}{}^{-1}$. Then $f_*(\mu_{K_1}{}^p\lambda_{K_2}{}^q) = \mu_{K_2}{}^p\lambda_{K_2}{}^{-q}$. But $f_*(\mu_{K_1}{}^p\lambda_{K_1}{}^q)$ is either $\mu_{K_2}{}^p\lambda_{K_2}{}^q$ or $\mu_{K_2}{}^{-p}\lambda_{K_2}{}^{-q}$, which means that either $q = 0$ or $p = 0$. Because this is impossible, we cannot have both $f_*(\mu_{K_1}) = \mu_{K_2}$ and $f_*(\lambda_{K_1}) = \lambda_{K_2}{}^{-1}$. Similarly, we cannot have both $f_*(\mu_{K_1}) = \mu_{K_2}{}^{-1}$ and $f_*(\lambda_{K_1}) = \lambda_{K_2}$. Therefore, $g : (S^3, K_1) \to (S^3, K_2)$ preserves the orientation of S^3, because each pair $(\mu_{K_i}, \lambda_{K_i})$ has intersection number $+1$ so that g preserves the intersection number. Thus, $\{K_1\} = \{K_2\}$, which concludes the proof.

COROLLARY 2. *The group of a cable knot $J(p, q; k)$ determines the numbers $|p|$ and $|q|$ and the topological type of k's complement. Moreover, if $|p| \geqq 3$, then the group determines the knot $J(p, q; k)$ itself.*

Proof. By the proof of Theorem 2, we can assume that we have a homotopy equivalence $f : N \to M$ such that

(1) $f|W_1 : W_1 \to X_1$ is a homeomorphism,
(2) $f(V_1) \subseteq X_2$,
(3) $f(A_1) = A$, and
(4) f_* is an isomorphism.

Recall that $K = J(p, q; k)$ and that $L = J(p', q'; h)$; let (μ_h, λ_h) denote a meridian-longitude pair of h on ∂W_1, and let (μ_k, λ_k) denote a meridian-longitude pair of k on ∂X_1. Let s_1 denote a component of ∂A_1, and set $s = f(s_1)$. When s_1 is oriented, it represents $\pm p'\mu_h$ in $H_1(W_1)$; similarly, s represents $\pm p\mu_k$ in $H_1(X_1)$. Because $f(s_1) = s$, we have $|p'| = |p|$.

Now $H_1(V_1, A_1; Z) \approx Z_{|q'|}$ and $H_1(X_2, A; Z) \approx Z_{|q|}$. Because

$$(f|V_1) : (V_1, A_1) \to (X_2, A)$$

is a homotopy equivalence [**24**], we have $H_1(V_1, A_1; Z) \approx H_1(X_2, A; Z)$ [**8**, Theorem 3.6.6, p. 120]. Hence $|q'| = |q|$. Because $W_1 \cong X_2$, we have $S^3 - k \cong S^3 - h$ [**7**, Proposition 10.1, p. 74].

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If k is trivial, then $J(p, q; k)$ is a torus knot and its group determines its type. We now assume that k is knotted. As we have noted, if L is a knot whose group is isomorphic to that of $J(p, q; k)$, then L is a cable knot $J(p', q'; h)$ (see the *remark* in the proof of Theorem 2). We have $|p'| = |p|$ and $|q'| = |q|$.

Let K^{-1} denote the inverse of an oriented knot K, let K^* denote the mirror image of K, and orient k so that it is homologous to λ_k in a tubular neighborhood of k. If $q > 0$, then $J(p, q; k) = K(\pm p, q; k)$; if $J(p, q; k) = K(-p, q; k)$, then $J(p, q; k)$ is equivalent to $K^*(-p, q; k)$ $(= K(p, q; k^*))$. If $q < 0$, then $J(p, q; k) = K(-p, -q; k^{-1})$; but $K^*(-p, -q; k^{-1}) = K(p, -q; k^{*-1})$, and so J is equivalent to $K(p, |q|; k^{*-1})$. Thus, in either case, there is a knot $\hat{k}$ equivalent to k such that $J(p, q; k)$ is equivalent to $K(p, |q|; \hat{k})$. Similarly, there exists a knot h' equivalent to h such that $J(p', q'; h)$ and $K(p', |q'|; h')$ are equivalent.

If $p' = -p$, then $J(p', q'; h)$ is equivalent to $K(p, |q|; h'^*)$, because $K^*(p', |q'|; h') = K(-p', |q'|; h'^*)$ and $|q'| = |q|$. Thus, in any case, there exists a knot $\hat{h}$ equivalent to h such that $J(p', q'; h)$ is equivalent to $K(p, |q|; \hat{h})$.

Because $J(p, q; k)$ and $J(p', q'; h)$ have isomorphic groups, it follows that $K(p, |q|; \hat{k})$ and $K(p, |q|; \hat{h})$ also have isomorphic groups, and so $(\{\hat{k}\}, p, |q|) = (\{\hat{h}\}, p, |q|)$ (by Corollary 1). By [**14**, Lemma 1, p. 247],

$$\{K(p, |q|; \hat{k})\} = \{K(p, |q|; \hat{h})\};$$

therefore, $J(p, q; k)$ and $J(p', q'; h)$ are equivalent, establishing the final conclusion.

COROLLARY 3. *The genus of any knot and the bridge number of any composite knot are knot-group invariants.*

Proof. For a composite knot, the genus is the sum of the factors' genera, and the bridge number is the sum of the factors' bridge numbers minus one; the conclusion, for composite knots, follows from Theorem 1. If the knot manifold C of a prime knot K contains no essential annulus, then $\pi_1(C)$ determines the topological type of C [**6**, Theorem 10, p. 42] and, therefore, the genus of K. If C contains an essential annulus, then K is a cable knot (by Lemma 1), and if K is a torus knot, then its group determines its type and, hence, its genus.

Suppose then that K is the cable knot $J(p, q; k)$ and that G is any knot group such that $\pi_1(S^3 - J(p, q; k)) \approx G$. Then, as noted in the proof of Theorem 2, G is the group of a cable knot, $J(p', q'; h)$, say. By Corollary 2, $|p| = |p'|$, $|q| = |q'|$, and the genus $g(k)$ of k is equal to the genus $g(h)$ of h. But by [14, Theorem 1, p. 247],

$$g(J(p, q; k)) = \frac{(|p| - 1)(|q| - 1)}{2} + |q|g(k), \quad \text{and}$$

$$g(J(p', q'; h)) = \frac{(|p'| - 1)(|q'| - 1)}{2} + |q'|g(h);$$

therefore $g(J(p, q; k)) = g(J(p', q'; h))$, which concludes the corollary's proof.

Remark. If all knots have property P, then a knot's group determines the knot's bridge number, by Corollary 3 and Theorem 2.

COROLLARY 4. *If G is the group of a knot with exactly m prime-knot factors and if $m > 1$, then G is the group of at most 2^{m-1} knot types and 2^{m-1} knot manifolds. Moreover, when the m prime factors are mutually inequivalent and when each of them is noninvertible and nonamphicheiral, then G is the group of exactly 2^{m-1} knot types and 2^{m-1} knot manifolds; hence, 2^{m-1} is the best possible bound.*

Proof. According to Theorem 1, any knot K whose group is G has exactly m prime factors each of which G determines up to equivalence. Because a knot type contains no more than four, distinct, oriented types, and because an oriented knot factors uniquely as a product of primes, the (oriented) knot K belongs to exactly one of at most 4^m *oriented*-knot types; thus, K belongs to exactly one of at most 4^{m-1} knot types, and so G is certainly the group of at most 4^{m-1} knot types and 4^{m-1} knot manifolds.

We now orient K and induct on m. For $m = 2$, let K_1 and K_2 denote the prime factors of K. Then each of $K_1 \# K_2$, $K_1 \# K_2^*$, $K_1 \# K_2^{-1}$, and $K_1 \# (K_2^*)^{-1}$ represents the 4 ($= 4^{2-1}$) (perhaps mutually distinct) knot types whose group is G. But it is easy to see that the groups of $K_1 \# K_2$ and $K_1 \# (K_2^*)^{-1}$ as well as the groups of $K_1 \# K_2^*$ and $K_1 \# K_2^{-1}$ are isomorphic. Hence, when $m = 2$, G is the group of at most 2 ($= 2^{2-1}$) knot types and 2 knot manifolds. If K_1 and K_2 are inequivalent, noninvertible, and nonamphicherial, then it is evident from the proof of Theorem 1 that G is the group of *exactly* 2 knot types and of 2 knot manifolds (by Proposition 3).

We omit the remainder of our inductive argument, because of its similarity to that of the first part (in the preceding paragraph). This concludes the proof of Corollary 4.

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