Canad. Math. Bull. Vol. **51** (1), 2008 pp. 60–66

Følner Nets for Semidirect Products of Amenable Groups

David Janzen

Abstract. For unimodular semidirect products of locally compact amenable groups N and H, we show that one can always construct a Følner net of the form $(A_\alpha \times B_\beta)$ for G, where (A_α) is a strong form of Følner net for N and (B_β) is any Følner net for H. Applications to the Heisenberg and Euclidean motion groups are provided.

Amenable groups are characterized by a hierarchy of invariance properties, one of the most significant being the combinatorial Følner conditions. Følner conditions are often especially useful because they describe amenability in terms of the internal group structure.

The proof that every amenable group has a Følner net is highly non-constructive. Consequently, a significant body of work has been devoted to the explicit construction of Følner nets for certain groups (see [2, §3.6] and [6, Ch. 4, 6] for discussions of this endeavour). In the case where $G = N \times H$ is a direct product of amenable groups N and H, it is not difficult to show that $(A_\alpha \times B_\beta)$ is a Følner net for G if and only if (A_α) is a Følner net for N and (B_β) is a Følner net for H. One would like to know when Følner nets of the form $(A_\alpha \times B_\beta)$ can be constructed for semidirect products of amenable groups as well. It is noted [2, p. 68] that no Følner net of the form $(A_\alpha \times B_\beta)$ exists for the non-unimodular $ax + b$ group, given by the semidirect product $\mathbb{R} \rtimes \mathbb{R}^+$. For this reason we restrict our attention to the unimodular case. Corollary 8 below tells us that when $G = N \rtimes H$ is a unimodular semidirect product of amenable groups, one can always construct a Følner net of the form $(A_\alpha \times B_\beta)$ for G, from a strong type of Følner net (A_α) for N (see Theorem 3 below) and any Følner net (B_β) for H.

Throughout, we let G, N, and H denote locally compact groups with respective Haar measures given by λ_G, λ_N, and λ_H. By a measurable set, we will mean a Borel measurable set. If Λ and Γ are directed sets, we consider $\Lambda \times \Gamma$ as a directed set with the product ordering.

Definition 1 Let (A_α) be a net of measurable subsets of G such that $0 < \lambda_G(A_\alpha) < \infty$. Then (A_α) is called a Følner net if it satisfies

$$\frac{\lambda_G(xA_\alpha \mathbin{\triangle} A_\alpha)}{\lambda_G(A_\alpha)} \to 0$$

uniformly in x on compact sets in G.

Received by the editors October 17, 2005; revised June 30, 2006.
This research was supported by NSERC grant 298444-04 (Dr. Ross Stokke).
AMS subject classification: Primary: 22D05; secondary: 43A07, 22D15, 43A20.

We will make use of the fact that the above condition is equivalent to the uniform convergence on compacta of $\lambda_G(xA_\alpha - A_\alpha)/\lambda_G(A_\alpha) \to 0$.

If $G = N \rtimes H$ is the semidirect product of amenable groups N and H, then G is itself an amenable group. We will denote the action of H on N by $h \cdot n$. Whenever it is convenient to do so, we will identify $G = N \rtimes H$ with $G = NH$ where N is a closed normal subgroup of G, H is a closed subgroup of G, $H \cap N = \{e\}$ and where the action of H on N is via inner automorphisms. We use $*$ to denote the multiplication in the semidirect product group $G = N \rtimes H$, and denote the multiplication in the direct product group $G = N \times H$ by juxtaposition. Thus, for $n, m \in N$ and $h, k \in H$ we can write

$$(n, h) * (m, k) = (n(h \cdot m), hk) = (n, h)(h \cdot m, k). \tag{1}$$

Lemma 2 *If $G = N \rtimes H$ is unimodular, then the following statements hold.*

(i) *For every measurable subset A of N and each $y \in H$, $\lambda_N(y \cdot A) = \lambda_N(A)$.*

(ii) *If (A_α) is a Følner net for N, then*

$$\frac{\lambda_G(x(y \cdot A_\alpha) - y \cdot A_\alpha)}{\lambda_G(A_\alpha)} \to 0$$

uniformly in xy $(x \in N, y \in H)$ on compact sets in G.

Proof From the unimodularity of G we obtain $\lambda_G = \lambda_N \times \lambda_H$ [3, 15.29(a)], which is also the Haar measure for the direct product group $N \times H$. Let A and B be measurable subsets of N and H, respectively, with $0 < \lambda_H(B) < \infty$. Then for any $x \in N$ and $y \in H$ we have

$$\begin{aligned}\lambda_N(A)\lambda_H(B) &= \lambda_N \times \lambda_H((x, y) * (A \times B)) = \lambda_N \times \lambda_H((x, y)(y \cdot A \times B)) \\ &= \lambda_N \times \lambda_H(y \cdot A \times B) = \lambda_N(y \cdot A)\lambda_H(B),\end{aligned}$$

where we have used (1). This yields part (i). For (ii), first observe that because N is normal in G, for any $x \in N$ we have $xy = y\phi_y(x)$, where $\phi_y(x) = y^{-1}xy \in N$. Part (i) now gives

$$\lambda_N(xyA_\alpha y^{-1} - yA_\alpha y^{-1}) = \lambda_N(y(\phi_y(x)A_\alpha - A_\alpha)y^{-1}) = \lambda_N(\phi_y(x)A_\alpha - A_\alpha).$$

Let $K = LM$ with $L \subset N$ and $M \subset H$ compact. Then $\{\phi_y(x) : x \in L, y \in M\} \subset M^{-1}LM \cap N$, which is compact. We are assuming that (A_α) is a Følner net for N, so (ii) follows. ∎

Theorem 3 *Let $G = N \rtimes H$, for amenable groups N and H, be unimodular. Let (B_β) be a Følner net for H and let (A_α) be a Følner net for N. Then $(A_\alpha \times B_\beta)$ is a Følner net for G if and only if (A_α) satisfies*

$$\frac{\lambda_N(y \cdot A_\alpha - A_\alpha)}{\lambda_N(A_\alpha)} \to 0 \tag{2}$$

uniformly in y on compact sets of H.

https://doi.org/10.4153/CMB-2008-008-7 Published online by Cambridge University Press

Proof Suppose that (A_α) satisfies (2). As in the proof of Lemma 2, observe that $(x,y)*(A_\alpha \times B_\beta) = (x,y)(y\cdot A_\alpha \times B_\beta)$. Hence,

$$(3)\quad (x,y)*(A_\alpha \times B_\beta) - (A_\alpha \times B_\beta) \\ = x(y\cdot A_\alpha) \times (yB_\beta - B_\beta) \cup (x(y\cdot A_\alpha) - A_\alpha) \times (yB_\beta),$$

and it follows that

$$\frac{\lambda_G((x,y)*(A_\alpha \times B_\beta) - (A_\alpha \times B_\beta))}{\lambda_G(A_\alpha \times B_\beta)} \\ \leq \frac{\lambda_N(x(y\cdot A_\alpha))\lambda_H(yB_\beta - B_\beta)}{\lambda_N(A_\alpha)\lambda_H(B_\beta)} + \frac{\lambda_N(x(y\cdot A_\alpha) - (A_\alpha))\lambda_H(yB_\beta)}{\lambda_N(A_\alpha)\lambda_H(B_\beta)}.$$

As $\lambda_N(y\cdot A_\alpha) = \lambda_N(A_\alpha)$, the uniform convergence on compacta of the first term on the right is obtained from the assumption that B_β is a Følner net for H. For the second term, we observe that $x(y\cdot A_\alpha) - A_\alpha \subseteq x(y\cdot A_\alpha) - (y\cdot A_\alpha) \cup (y\cdot A_\alpha) - (A_\alpha)$. This yields

$$\frac{\lambda_N(x(y\cdot A_\alpha) - A_\alpha)}{\lambda_N(A_\alpha)} \leq \frac{\lambda_N(x(y\cdot A_\alpha) - (y\cdot A_\alpha))}{\lambda_N(A_\alpha)} + \frac{\lambda_N(y\cdot A_\alpha - A_\alpha)}{\lambda_N(A_\alpha)}.$$

So the implication follows from Lemma 2(ii) and the assumption that (A_α) satisfies (2).

For the converse, note that $(x(y\cdot A_\alpha) - A_\alpha) \times (yB_\beta) \subseteq (x,y)*(A_\alpha \times B_\beta) - (A_\alpha \times B_\beta)$ by (3). Consequently,

$$\frac{\lambda_G((x(y\cdot A_\alpha) - A_\alpha) \times (yB_\beta))}{\lambda_G(A_\alpha \times B_\beta)} \leq \frac{\lambda_G(x,y)*(A_\alpha \times B_\beta) - (A_\alpha \times B_\beta)}{\lambda_G(A_\alpha \times B_\beta)}.$$

The term on the left is just equal to $\lambda_N(x(y\cdot A_\alpha) - A_\alpha)/\lambda_N(A_\alpha)$, so (2) follows from the assumption that $(A_\alpha \times B_\beta)$ is a Følner net for $N \rtimes H$. ■

Using this result, we are able to construct Følner nets for the Heisenberg group, generalized Heisenberg groups, as well as the motion groups on $\mathbb{R}^k$. We note that if G is any one of these groups, then it is exponentially bounded. Hence, [6, Proposition 6.8] implies that a Følner sequence for G can be chosen from among the sets $\{C^n : n \in \mathbb{N}\}$, where C is taken to be any compact neighbourhood of the identity. However, even if we begin with a nice set C, the set C^n does not allow for a clean description in the semidirect product group. For these groups then our approach seems to be much simpler.

Example 4 The regular Heisenberg group H of 3×3 upper triangular matrices with real entries can be expressed as $\mathbb{R}^2 \rtimes \mathbb{R} = H$, with $c \in \mathbb{R}$ acting on $(a,b) \in \mathbb{R}^2$ by $c\cdot(a,b) = (a, b+ac)$. If we take $A_n = [-n,n] \times [-n^2, n^2]$ as a Følner sequence

https://doi.org/10.4153/CMB-2008-008-7 Published online by Cambridge University Press

in $\mathbb{R}^2$ and (B_n) to be any Følner sequence in $\mathbb{R}$, then $(A_n \times B_n)$ is a Følner sequence for H.

To see this, we show that (A_n) satisfies (2). One can easily check that $c \cdot A_n = \{(x, y) : y \in [-n^2 + cx, n^2 + cx] \text{ for } x \in [-n, n]\}$ and hence,

$$\frac{\lambda(c \cdot A_n - A_n)}{\lambda(A_n)} = \frac{(|c|n)(n)}{(2n)(2n^2)} = \frac{|c|}{4n} \quad (c \in \mathbb{R}).$$

This gives the desired uniform convergence in c on compact subsets of $\mathbb{R}$.

Taking $A_n = [-n, n] \times [-n, n]$ and $B_n = [-n, n]$, provides an example of two Følner nets whose Cartesian product does not yield a Følner net for the semidirect product group. Indeed, $\lambda(c \cdot A_n - A_n)/\lambda(A_n) = (|c|n)(n)/(2n)(2n) = |c|/4$ which does not even converge pointwise to 0 in $\mathbb{R}$.

Example 5 A generalized Heisenberg group H_p is given by all matrices of the form

$$\begin{pmatrix} 1 & A & c \\ 0 & I & B \\ 0 & 0 & 1 \end{pmatrix},$$

where $A, B \in \mathbb{R}^p$, $c \in \mathbb{R}$, and I is just the $p \times p$ identity matrix. Here $H_p = \mathbb{R}^{p+1} \rtimes \mathbb{R}^p$ with the action of $h = (a_1, a_2, \dots, a_p) \in \mathbb{R}^p$ on $m = (c, b_1, b_2, \dots, b_p) \in \mathbb{R}^{p+1}$ given by $h \cdot m = (c + \Sigma_{i=1}^p a_i b_i, b_1, b_2, \dots, b_p)$.

Take (B_n) to be any Følner sequence in $\mathbb{R}^p$ (for example $[-n, n] \times [-n, n] \times \cdots \times [-n, n]$) and take $A_n = [-n^2, n^2] \times [-n, n] \times \cdots \times [-n, n]$ as a Følner sequence in $\mathbb{R}^{p+1}$. Again, we show that (A_n) satisfies (2). If $h = (a_1, \dots, a_p) \in \mathbb{R}^p$, then $\lambda(h \cdot A_n - A_n) \leq (|a_1| + |a_2| + \cdots + |a_p|)n(2n)^p$. This gives

$$\frac{\lambda(h \cdot A_n - A_n)}{\lambda(A_n)} \leq \frac{(|a_1| + |a_2| + \cdots + |a_p|)n(2n)^p}{(2n^2)(2n)^p} = \frac{(|a_1| + |a_2| + \cdots + |a_p|}{2n}$$

which yields the uniform convergence in h on compact subsets of $\mathbb{R}^p$.

As with the regular Heisenberg group, one can show that the Følner sequence $A_n = [-n, n] \times [-n, n] \times \cdots \times [-n, n]$ for $\mathbb{R}^{p+1}$ will not yield a Følner sequence of the form $(A_n \times B_n)$ in H_p.

Example 6 We now consider the Euclidean motion group on $\mathbb{R}^k$, $G = \mathbb{R}^k \rtimes SO(k)$. It is not difficult to see that $A_n = \{(x_1, x_2, \dots, x_k) : x_1^2 + x_2^2 + \cdots + x_k^2 \leq n^2\}$ defines a Følner sequence for $\mathbb{R}^k$ which is invariant under the action of $SO(k)$. It follows that (A_n) trivially satisfies (2). Consequently, if (B_n) is any Følner sequence for $SO(k)$ (since $SO(k)$ is compact, we may simply take $B_n = SO(k)$ for all n), we obtain a Følner sequence $(A_n \times B_n)$ for $\mathbb{R}^k \rtimes SO(k)$.

In each of the above examples, we exhibited Følner sequences for N satisfying (2). However, it was noted in Examples 4 and 5 that not every Følner net for N satisfies this property. We now show that Følner nets of this form always exist in N.

Theorem 7 *For amenable groups N and H, let $G = N \rtimes H$ be unimodular. Then there exists a Følner net (A_α) for N such that*

$$\frac{\lambda_N(y \cdot A_\alpha - A_\alpha)}{\lambda_N(A_\alpha)} \to 0$$

uniformly in y on compact sets of H.

Proof Define a continuous group action of $G = NH$ on N by

$$s \cdot x = nhxh^{-1} \quad (s = nh, x, n \in N, h \in H).$$

Letting

$$(s \cdot f)(x) = f(s^{-1} \cdot x) \quad (s \in G, x \in N, f \in L^1(N)), \tag{4}$$

$L^1(N)$ becomes a left Banach G-module (see [6, p. 42] for the definition). It is clear that $s \cdot f \geq 0$ whenever $f \geq 0$, and it follows from Lemma 2(i) that the action is isometric on $L^1(N)$. Thus, in the sense of [9, Definition 1.1], (4) defines a positive action of G on $L^1(N)$. Given $\alpha = (\epsilon, K)$, where $\epsilon > 0$ and K is a compact subset of G, it suffices to find a measurable subset A_α of N such that $0 < \lambda_N(A_\alpha) < \infty$ and

$$\lambda_N(s \cdot A_\alpha \Delta A_\alpha) < \epsilon \lambda_N(A_\alpha) \quad (s \in K). \tag{5}$$

The locally compact group G is amenable, so by [9, Corollary 1.11; Proposition 1.13], there exists a positive norm one function ϕ in $L^1(N)$ such that $\|s \cdot \phi - \phi\|_1 < \epsilon$, $(s \in K)$. (There is a misprint in [9, Proposition 1.13 (2)]: $\mathcal{M}_*$ should be $(\mathcal{M}_*)_1^+$.) The standard proof of the Følner condition (FC) from Reiter's condition now yields (5). For example, in the proofs of [2, Theorem 3.6.3, Lemma 3.6.4], one only needs to replace ${}_s\phi$ by $s \cdot \phi$ and sA by $s \cdot A = \{s \cdot a : a \in A\}$. ■

Combining the results of Theorems 3 and 7 yields the following corollary.

Corollary 8 *Every unimodular semidirect product of amenable groups N and H possesses a Følner net of the form $(A_\alpha \times B_\beta)$ where (A_α) is a Følner net for N and (B_β) a Følner net for H.*

One might wonder whether a similar result can be obtained for semidirect products of semigroups. The situation is somewhat different since neither the strong nor the weak Følner conditions characterize left amenability for semigroups. (A semigroup S satisfies the weak Følner condition (WFC) if, for every finite subset F of S and any $\epsilon > 0$, there exists a finite subset A of S satisfying $|xA - A| < \epsilon|A|$ for each $x \in F$. For the strong Følner condition (SFC), $xA - A$ is replaced by $A - xA$, or equivalently by $A \triangle xA$.) It is known that if S satisfies SFC, then it is left amenable and if it is left amenable, then it satisfies WFC, but that none of these implications is reversible (a nice discussion of these results is contained in [6, 4.22]). In [5], it is shown that if a semidirect product of semigroups $S = U \rtimes T$ satisfies SFC, then U and T also satisfy SFC. One might then ask the following.

Question 9 For a semigroup $S = U \rtimes T$ satisfying SFC (resp. WFC), under what conditions can one obtain a strong (resp. weak) Følner net for S of the form $(A_\alpha \times B_\beta)$ from strong (resp. weak) Følner nets (A_α) and (B_β) for U and T respectively?

For more on Følner conditions and semidirect products of semigroups, the reader is referred to [5, 11] and the references therein.

Related to Følner nets (see Remark 12 below) are what we shall presently call AI-nets (asymptotically invariant nets).

Definition 10 Let (U_α) be a net of compact sets which comprises a base for the neighbourhood system at the identity in G. We call (U_α) an AI-net if

$$\frac{\lambda_G(xU_\alpha x^{-1} \triangle U_\alpha)}{\lambda_G(U_\alpha)} \to 0$$

uniformly in x on compact subsets of G.

AI-nets were studied in [9], where it was noted that the (regular) Heisenberg group H always possesses an AI-sequence. However, no explicit construction of an AI-sequence for H was given. We conclude this note with such a construction.

Example 11 Let $H = \mathbb{R}^2 \rtimes \mathbb{R}$ be the Heisenberg group. Then

$$U_n = [-1/n^2, 1/n^2] \times [-1/n, 1/n] \times [-1/n^2, 1/n^2]$$

is an AI-net for H.

For $(a, b, c), (x, y, z) \in H$, one has $(a, b, c)^{-1} = (-a, -b + ac, -c)$ and so

$$(a, b, c), (x, y, z)(a, b, c)^{-1} = (x, y + cx - az, z).$$

By taking $(x, y, z) \in U_n$, one can check that

$$\lambda\big((a, b, c)U_n(a, b, c)^{-1} - U_n\big) \leq \frac{2(|a| + |c|)}{n^2}\frac{2}{n^2}\frac{2}{n^2} = \frac{8(|a| + |c|)}{n^6}.$$

Thus

$$\frac{\lambda((a, b, c)U_n(a, b, c)^{-1} - U_n)}{\lambda(U_n)} \leq \frac{8(|a| + |c|)}{n^6}\frac{n^5}{8} = \frac{|a| + |c|}{n}.$$

This implies the desired uniform convergence on compacta.

Remark 12 With regard to the cohomology of the Fourier algebra $A(G)$ and the group algebra $L^1(G)$, the importance of Følner nets in combination with AI-nets for G can be found in [4, 10]. Indeed, [10, Remark 2.6] gives a simple formula for constructing an approximate diagonal for $L^1(G)$ from Følner and AI-nets for G. Let (A_α) be a Følner net for G, and let (ξ_α) be the associated $L^2(G)$-normalized characteristic functions. Then the net of norm one positive-definite functions $u_\alpha(s) = \langle \pi_2(s)\xi_\alpha, \xi_\alpha \rangle$ $(s \in G)$, where π_2 is the left regular representation of G, gives a bounded approximate identity for $A(G)$. Thus, the proof of [4, Lemma 3.4] shows how to construct a nice operator approximate diagonal for $A(G)$ from Følner and AI-nets for G (also see [1, 7], and [8, §7.4]). If H is the Heisenberg group, Examples 4 and 11 thus allow for the explicit construction of an approximate diagonal for $L^1(H)$ and an operator approximate diagonal for $A(H)$.

Acknowledgements This research was completed as part of an undergraduate summer research project at the University of Winnipeg under the supervision of Dr. Ross Stokke. The author would like to thank Dr. Stokke for his encouragement and many helpful comments as well as the proof of Theorem 7 and an improvement to Lemma 2(ii).

References

[1] O. Aristov, V. Runde, and N. Spronk, *Operator biflatness of the Fourier algebra and approximate indicators for subgroups.* J. Funct. Anal. **209**(2004), no. 2, 367–387.
[2] F. P. Greenleaf, *Invariant means on topological groups and their applications.* Van Nostrand Mathematical Studies 16, Van Nostrand, New York, 1969.
[3] E. Hewitt and K. A. Ross, *Abstract Harmonic Analysis. I. Structure of Topological Groups. Integration Theory, Group Representations.* Grundlehren der Mathematischen Wissenschaften 115, Springer-Verlag, Berlin, 1963.
[4] M. Ilie and N. Spronk, *Completely bounded homomorphisms of the Fourier algebra.* J. Funct. Anal. **225**(2005), no. 2, 480–499.
[5] M. Klawe, *Semidirect product of semigroups in relation to amenability, cancellation properties, and strong Følner conditions.* Pacific J. Math. **73**(1977), no. 1, 91–106.
[6] A. L. T. Paterson, *Amenability.* Mathematical Surveys and Monographs 29, American Mathematical Society, Providence, RI, 1988.
[7] Z.-J. Ruan, *The operator amenability of $A(G)$.* Amer. J. Math. **117**(1995), no. 6, 1449–1474.
[8] V. Runde, *Lectures on Amenability.* Lecture Notes in Mathematics 1774, Springer-Verlag, Berlin, 2002.
[9] R. Stokke, *Quasi-central bounded approximate identities in group algebras of locally compact groups.* Illinois J. Math. **48**(2004), no. 1, 151–170.
[10] ———, *Approximate diagonals and Følner conditions for amenable group and semigroup algebras.* Studia Math. **164**(2004), no. 2, 139–159.
[11] Z. Yang, *Følner numbers and Følner type conditions for amenable semigroups.* Illinois J. Math. **31**(1987), no. 3, 496–517.

Department of Mathematics and Statistics
McGill University
805 Sherbrooke St. West
Montreal, QC
H3A 2K6
e-mail: janzen@math.mcgill.ca