A PROCEDURE FOR DERIVING INVERSION FORMULAE FOR INTEGRAL TRANSFORM PAIRS OF A GENERAL KIND

by IAN N. SNEDDON†

(Received 10 March, 1967)

1. The basic formulae. In recent years there have appeared solutions of several integral equations of the type

$$\int_x^1 K\left(\frac{x}{y}\right) f(y) \frac{dy}{y} = g(x) \qquad (0 \leqq x \leqq 1), \tag{1.1}$$

in which the kernel $K(x)$ contains (as a factor) one of the classical orthogonal polynomials or a hypergeometric function.

The case in which $K(x) = (1-x^2)^{-\frac{1}{2}} T_n(x^{-1})$, T_n being the Chebychev polynomial of the first kind was discussed by Ta Li [**1**] and that in which $K(x) = P_n(x^{-1})$, where P_n is the Legendre polynomial, was considered by Buschman [**2**]. Both of these integral equations are included in the case in which $K(x) = \Gamma(\lambda)(1-x^2)^{\lambda-\frac{1}{2}} C_m^\lambda(x^{-1})$ in which C_m^λ denotes the Gegenbauer polynomial; the solution in this case has been derived by Higgins [**3**]. Equations of a similar type in which the kernels involve Jacobi polynomials and associated Legendre functions have been discussed by Srivastava [**4, 5**] and Erdélyi [**6**] respectively. Integral transforms in which the kernel $K(x)$ involves a hypergeometric function have been considered by Higgins [**7**] and Wimp [**8**].

The object of the present paper is to show that the systematic use of the Mellin transform leads to a simple procedure by means of which inversion formulae for transform pairs of the type (1.1) may be discovered. The basic formulae are derived in this section and the use of the method in specific cases is illustrated in §§3–8. The adaption of the method to integral equations of a slightly different type is discussed briefly in §2.

If we make the substitutions

$$f_1(x) = f(x)H(1-x), \quad g_1(x) = g(x)\,H(1-x), \quad K_1(x) = K(x)\,H(1-x),$$

where H denotes Heaviside's unit function, we can write equation (1.1) in the form

$$\int_0^\infty K_1\left(\frac{x}{y}\right) f_1(y) \frac{dy}{y} = g_1(x) \qquad (x > 0).$$

Denoting the Mellin transform of $f_1(x)$ by

$$f_1^*(s) = \mathscr{M}[f_1(x):s] = \int_0^\infty f_1(x) x^{s-1}\, dx,$$

and applying the convolution theorem for Mellin transforms [**9**, p. 43] we find that this integral equation is equivalent to the relation

$$K_1^*(s) f_1^*(s) = g_1^*(s). \tag{1.2}$$

† Research carried out at Duke University, North Carolina in the summer of 1966 under grant AF–AFOSR 435–63 monitored by the Air Force Office of Scientific Research.

We can write this result in the equivalent form

$$f_1^*(s-l\alpha-m\beta+\gamma) = L_1^*(s-l\alpha)\alpha^l\beta^m \frac{\Gamma(s/\alpha)\Gamma((s/\beta)-l(\alpha/\beta))}{\Gamma((s/\alpha)-l)\Gamma((s/\beta)-l(\alpha/\beta)-m)} g_1^*(s-l\alpha-m\beta+\gamma),$$

where $\alpha \neq 0$, $\beta \neq 0$ and

$$L_1^*(s) = \alpha^{-l}\beta^{-m} \frac{\Gamma(s/\alpha)\Gamma((s/\beta)-m)}{\Gamma((s/\alpha)+l)\Gamma(s/\beta)K_1^*(s-m\beta+\gamma)}.$$

If the function $g(x)$ is such that

$$g^{(r)}(1) = 0 \qquad (r = 0, 1, \ldots, m-1), \tag{1.3}$$

then it is easily shown that

$$\mathcal{M}\left[\left(x^{1-\alpha}\frac{d}{dx}\right)^l \int_0^1 L_1\left(\frac{x}{y}\right)\left(y^{1-\beta}\frac{d}{dy}\right)^m \{y^\gamma g(y)\}\frac{dy}{y}; s\right]$$
$$= (-1)^{l+m}\alpha^l\beta^m L_1^*(s-l\alpha)\frac{\Gamma(s/\alpha)\Gamma((s/\beta)-l(\alpha/\beta))}{\Gamma((s/\alpha)-l)\Gamma((s/\beta)-l(\alpha/\beta)-m)} g^*(s-l\alpha-m\beta+\gamma),$$

and hence that the equation (1.2) is equivalent to the solution

$$f(x) = (-1)^{l+m}x^{l\alpha+m\beta-\gamma}\left(x^{1-\alpha}\frac{d}{dx}\right)^l \int_0^1 L_1\left(\frac{x}{y}\right)\left(y^{1-\beta}\frac{d}{dy}\right)^m \{y^\gamma g(y)\}\frac{dy}{y}, \tag{1.4}$$

provided that there are real numbers α, β, γ and positive integers l, m such that the inverse Mellin transform

$$L_1(x) = \alpha^{-l}\beta^{-m}\mathcal{M}^{-1}\left[\frac{\Gamma(s/\alpha)\Gamma((s/\beta)-m)}{\Gamma((s/\alpha)+l)\Gamma(s/\beta)K_1^*(s-m\beta+\gamma)}; s \to x\right] \tag{1.5}$$

exists. In cases in which the function $L_1(x)$ defined by this equation is of the form $L(x)H(1-x)$ we can write the solution (1.4) in the form

$$f(x) = (-1)^{l+m}x^{l\alpha+m\beta-\gamma}\left(x^{1-\alpha}\frac{d}{dx}\right)^l \int_x^1 L\left(\frac{x}{y}\right)\left(y^{1-\beta}\frac{d}{dy}\right)^m \{y^\gamma g(y)\}\frac{dy}{y}, \tag{1.6}$$

provided, of course, that the prescribed function $g(x)$ satisfies the conditions (1.3).

We have assumed that l and m are integers, but we can easily modify the solution (1.6) in cases in which it is convenient to take l, m to be positive but non-integral. The function $L_1(x)$ is defined once again by equation (1.5) and the solution (1.4) is modified only to the extent that the factor $(-1)^{l+m}$ is replaced by $(-1)^{[l]+[m]}$ and a fractional derivative of order m is defined by the relation

$$\left(\frac{d}{dx}\right)^m \psi(x) = \frac{1}{\Gamma(n-m)}\frac{d^n}{dx^n}\int_x^\infty \psi(u)(u-x)^{n-m-1}\,du,$$

where n is a positive integer, $n-1 < \operatorname{Re} m < n$.

The cases in which α or β is zero can be dealt with in a similar fashion. The solution (1.4) can be expressed in the alternative forms

$$f(x)=(-1)^{l+m}x^{m\beta-\gamma}\left(x\frac{d}{dx}\right)^{l}\int_{x}^{1}M\left(\frac{x}{y}\right)\left(y^{1-\beta}\frac{d}{dy}\right)^{m}\{y^{\gamma}g(y)\}\frac{dy}{y},\tag{1.7}$$

where

$$M(x)H(1-x)=\beta^{-m}\mathscr{M}^{-1}\left[\frac{s^{-l}\Gamma((s/\beta)-m)}{\Gamma(s/\beta)K_1^*(s-m\beta+\gamma)};x\right],\tag{1.8}$$

or

$$f(x)=(-1)^{l+m}x^{l\alpha-\gamma}\left(x^{1-\alpha}\frac{d}{dx}\right)^{l}\int_{x}^{1}N\left(\frac{x}{y}\right)\left(y\frac{d}{dy}\right)^{m}\{y^{\gamma}g(y)\}\frac{dy}{y},\tag{1.9}$$

where

$$N(x)H(1-x)=\alpha^{-l}\mathscr{M}^{-1}\left[\frac{s^{-m}\Gamma(s/\alpha)}{\Gamma((s/\alpha)+l)K_1^*(s+\gamma)};x\right],\tag{1.10}$$

or

$$f(x)=(-1)^{l+m}x^{-\gamma}\left(x\frac{d}{dx}\right)^{l}\int_{x}^{1}S\left(\frac{x}{y}\right)\left(y\frac{d}{dy}\right)^{m}\{y^{\gamma}g(y)\}\frac{dy}{y},\tag{1.11}$$

where

$$S(x)H(1-x)=\mathscr{M}^{-1}\left[\frac{s^{-l-m}}{K_1^*(s+\gamma)};x\right].\tag{1.12}$$

The case in which $l=0$ and $m=1$ arises frequently. If we put $l=0$, $m=1$ in equations (1.5) and (1.6) we find that

$$f(x)=-x^{\beta-\gamma}\int_{x}^{1}L\left(\frac{x}{y}\right)y^{-\beta}\frac{d}{dy}\{y^{\gamma}g(y)\}\,dy,$$

provided that we can find real numbers β, γ such that the inverse Mellin transform

$$L(x)H(1-x)=\mathscr{M}^{-1}\left[\frac{1}{(s-\beta)K_1^*(s-\beta+\gamma)};\ x\right]$$

exists. This solution can be written in the alternative form

$$f(x)=-\int_{x}^{1}M\left(\frac{x}{y}\right)y^{-\gamma}\frac{d}{dy}\{y^{\gamma}g(y)\}\,dy,\tag{1.13}$$

where

$$M(x)=x^{\beta-\gamma}L(x),$$

i.e. where

$$M(x)H(1-x)=\mathscr{M}^{-1}\left[\frac{1}{(s-\gamma)K_1^*(s)};x\right].\tag{1.14}$$

In certain cases it is more meaningful to express the results in terms of the operator of fractional integration K_{x^m} defined for $\alpha>0$ by the equation

$$K_{x^m}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_x^{\infty} (t^m - x^n)^{\alpha-1} f(t) m t^{m-1}\, dt, \tag{1.15}$$

and, for $\alpha < 0$, by the equation

$$K_{x^m}^{\alpha} f(x) = \left(\frac{d}{dx^m}\right)^l K_{x^m}^{\alpha+l} f(x), \tag{1.16}$$

where l is a positive integer such that $0 < \alpha + l < 1$. In either case it is easily shown that

$$\mathscr{M}[K_{x^m}^{\alpha} f(x); s] = \frac{\Gamma(s/m)}{\Gamma((s/m)+\alpha)} f^*(s+m\alpha) \tag{1.17}$$

and hence that

$$\mathscr{M}[K_{x^m}^{\alpha} x^{\gamma} K_{x^n}^{\beta} x^{\delta} f(x); s] = \frac{\Gamma(s/m)\Gamma((s+m\alpha+\gamma)/n)}{\Gamma((s/m)+\alpha)\Gamma(((s+m\alpha+\gamma)/n)+\beta)}. \tag{1.18}$$

From equations (1.2) and (1.18) we deduce immediately that, if we can find constants α, β, γ, δ and positive integers m, n such that

$$\frac{1}{K_1^*(s+m\alpha+n\beta+\gamma+\delta)} = C\frac{\Gamma(s/m)\Gamma((s+m\alpha+\gamma)/n)}{\Gamma((s/m)+\alpha)\Gamma(((s+m\alpha+\gamma)/n)+\beta)}, \tag{1.19}$$

where C does not depend on x, the solution of equation (1.1) can be written in the form

$$f(x) = Cx^{-m\alpha-n\beta-\gamma-\delta} K_{x^m}^{\alpha} x^{\gamma} K_{x^n}^{\beta} x^{\delta} g(x) H(1-x). \tag{1.20}$$

2. Integral equations of related types. The solution of an equation of the type

$$\int_x^{\infty} K\left(\frac{x}{y}\right) f(y) \frac{dy}{y} = g(x) \qquad (x > 0) \tag{2.1}$$

can be derived by an exactly similar method. If we define $K_1(x)$ as before, equation (1.2) is replaced by

$$K_1^*(s) f^*(s) = g^*(s). \tag{2.2}$$

The remainder of the analysis proceeds in exactly the same way as before but now equation (1.6) is replaced by

$$f(x) = (-1)^{l+m} x^{l\alpha+m\beta-\gamma} \left(x^{1-\alpha}\frac{d}{dx}\right)^l \int_x^{\infty} L\left(\frac{x}{y}\right)\left(y^{1-\beta}\frac{d}{dy}\right)^m \{y^{\gamma} g(y)\} \frac{dy}{y} \tag{2.3}$$

and equation (1.20) by

$$f(x) = Cx^{-m\alpha-n\beta-\gamma-\delta} K_{x^m}^{\alpha} x^{\gamma} K_{x^n}^{\beta} x^{\delta} g(x). \tag{2.4}$$

An integral equation of the type

$$\int_a^x \Lambda(x/y)\phi(y)\, dy = \psi(x) \qquad (x > a > 0) \tag{2.5}$$

can be reduced to the form (1.1) by making substitutions

$$K(x) = \Lambda(x^{-1}), \quad g(x) = \psi(ax^{-1}), \quad \phi(x) = x^{-1} f(ax^{-1}). \tag{2.6}$$

https://doi.org/10.1017/S001708950000029X Published online by Cambridge University Press

For example, the integral equation

$$\int_a^x (x^2-t^2)^{\frac{1}{2}\lambda} P_\nu^{-\lambda}(x/t)\phi(t)\,dt = \psi(x) \qquad (0<x<a), \tag{2.7}$$

considered by Erdélyi [6], is equivalent to the equation (1.1) with

$$K(x) = (1-x^2)^{\frac{1}{2}\lambda} P_\nu^{-\lambda}(x^{-1}), \quad g(x) = (x/a)^\lambda \psi(a/x), \quad \phi(x) = xf(a/x). \tag{2.8}$$

In returning to the original variables we have at our disposal the formula

$$K_{x^m}^{\alpha} f(a/x) = a^{-m\alpha} \hat{f}(a/x), \tag{2.9}$$

where $\hat{f}(x)$ is expressed in terms of the integral operator $I_{x^m}^{\alpha}$ defined by the equation

$$I_{x^m}^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x^m - t^m)^{\alpha-1} f(t) m t^{m-1}\,dt \tag{2.10}$$

through the relation

$$\hat{f}(x) = x^{1-m\alpha} I_{x^m}^{\alpha} f(x). \tag{2.11}$$

Alternatively we could treat equations of the type (2.5) *ab initio* by putting them in the form

$$\int_0^\infty K_2\left(\frac{x}{y}\right) f_2(y) \frac{dy}{y} = g_2(x) \qquad (x>0),$$

with

$$K_2(x) = K(x)H(x-1), \quad f_2(y) = f(y)H(y-a), \quad g_2(x) = g(x)H(x-a),$$

and proceeding as before.

3. Integral equations of Abel type. As a first illustration of the method outlined in §1 we consider the problem of solving an integral equation of Abel type. Suppose that

$$K(x) = (1-x)^{\lambda-1},$$

where $n-1<\lambda<n$, with n a positive integer. Then

$$K_1^*(s) = \frac{\Gamma(\lambda)\Gamma(s)}{\Gamma(s+\lambda)}.$$

Now if we take $l=0$, $m=n$, $\beta=1$, $\gamma=0$ in equation (1.5) we find that

$$L_1^*(s) = \frac{1}{\Gamma(\lambda)} \mathscr{M}^{-1}\left[\frac{\Gamma(\lambda+s-n)}{\Gamma(s)}; x\right],$$

so that $L_1(x) = L(x)\,\mathrm{H}(1-x)$ with

$$L(x) = \frac{x^{\lambda-n}(1-x)^{n-\lambda-1}}{\Gamma(\lambda)\Gamma(n-\lambda)}.$$

This shows that, if $f(x)$ satisfies the integral equation

https://doi.org/10.1017/S001708950000029X Published online by Cambridge University Press

$$\int_x^1 f(t)(t-x)^{\lambda-1}\,dt = g(x) \qquad (n-1<\lambda<n), \tag{3.1}$$

in which the prescribed function $g(x)$ satisfies the condition

$$g^{(r)}(1) = 0 \qquad (r = 0,1,2,\ldots,n-1), \tag{3.2}$$

then

$$f(x) = \frac{(-1)^n}{\Gamma(\lambda)\Gamma(n-\lambda)}\int_x^1 (t-x)^{n-\lambda-1} g^{(n)}(t)\,dt. \tag{3.3}$$

If the prescribed function does not satisfy the condition (3.2) we take $l = n$, $\alpha = 1$, $m = 0$, $\gamma = n-\lambda$, in which case equation (1.5) reduces to

$$L_1(x) = \frac{1}{\Gamma(\lambda)}\mathscr{M}^{-1}\left[\frac{\Gamma(s)}{\Gamma(s+n-\lambda)}; x\right],$$

which is of the form $L_1(x) = L(x)\,\mathrm{H}(1-x)$ with now

$$L(x) = \frac{(1-x)^{n-\lambda-1}}{\Gamma(\lambda)\Gamma(n-\lambda)}.$$

In this way we derive the solution of the equation (2.1) in the form

$$f(x) = \frac{(-1)^n}{\Gamma(\lambda)\Gamma(n-\lambda)}\cdot\frac{d^n}{dx^n}\int_x^1 (y-x)^{n-\lambda-1} g(y)\,dy. \tag{3.4}$$

4. Transforms with a Chebychev polynomial as kernel. Suppose that

$$K(x) = \frac{T_n(x^{-1})}{\sqrt{(1-x^2)}},$$

where $T_n(x)$ is the Chebychev polynomial of degree n, defined by the equation

$$T_n(x) = \cos(n\cos^{-1}x).$$

It is easily shown that

$$T_n(x) = 2^{n-1}x^n\,{}_2F_1(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n; 1-n; x^{-2})$$

and, by integrating term by term, that

$$K_1^*(s) = 2^{n-2}\sqrt{\pi}\frac{\Gamma(\frac{1}{2}s-\frac{1}{2}n)}{\Gamma(\frac{1}{2}s-\frac{1}{2}n+\frac{1}{2})}\,{}_3F_2(-\tfrac{1}{2}n, \tfrac{1}{2}-\tfrac{1}{2}n, \tfrac{1}{2}s-\tfrac{1}{2}n; \tfrac{1}{2}s-\tfrac{1}{2}n+\tfrac{1}{2}; 1).$$

The generalized hypergeometric series on the right can be summed by Saalschütz's theorem **[10**, p. 48] to give

$$K_1^*(s) = \frac{2^{s-2}\Gamma(\frac{1}{2}n+\frac{1}{2}s)\Gamma(\frac{1}{2}s-\frac{1}{2}n)}{\Gamma(s)},$$

so that

$$(s-n)K_1^*(s) = \frac{2^{s-1}\Gamma(\frac{1}{2}n+\frac{1}{2}s)\Gamma(1-\frac{1}{2}n+\frac{1}{2}s)}{\Gamma(s)}.$$

By formula (2) on p. 324 of **[11]** we see that

$$\mathscr{M}^{-1}\left[\frac{1}{(s-n)K_1^*(s)}; x\right] = \frac{2}{\pi}\cdot\frac{T_{n-1}(x)}{\sqrt{(1-x^2)}}H(1-x),$$

so that, by putting $\gamma = n$ in equations (1.13) and (1.14), we find that the integral equation

$$\int_x^1 \frac{T_n(y/x)}{\sqrt{(y^2-x^2)}}f(y)\,dy = g(x)$$

has solution

$$f(x) = -\frac{2}{\pi}\int_x^1 \frac{T_{n-1}(x/y)}{\sqrt{(y^2-x^2)}}\cdot y^{1-n}\frac{d}{dy}\{y^n g(y)\}\,dy.$$

This equation has been discussed by Ta Li **[1]**.

5. Transforms whose kernels are Legendre polynomials. If we take

$$K(x) = P_n(x^{-1}),$$

where $P_n(x)$ is the Legendre polynomial of degree n and integrate both sides of the equation

$$P_n(x^{-1}) = \frac{2^n x^{-n}\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}\,{}_2F_1(\tfrac{1}{2}-\tfrac{1}{2}n, -\tfrac{1}{2}n; \tfrac{1}{2}-n; x^{-2}),$$

we find that

$$K_1^*(s) = \frac{2^{n-1}\Gamma(n+\frac{1}{2})}{n!\,\Gamma(\frac{1}{2})}\,{}_3F_2(\tfrac{1}{2}-\tfrac{1}{2}n, -\tfrac{1}{2}n, \tfrac{1}{2}s-\tfrac{1}{2}n; \tfrac{1}{2}-n, \tfrac{1}{2}s-\tfrac{1}{2}n+1; 1).$$

Again the generalized hypergeometric series on the right terminates. Calculating its sum by Saalschütz's theorem we find that

$$K_1^*(s) = \tfrac{1}{2}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}n+\frac{1}{2})\Gamma(\frac{1}{2}s-\frac{1}{2}n)}{\Gamma(\frac{1}{2}s+\frac{1}{2})\Gamma(\frac{1}{2}s+1)}.$$

Taking $l = 0$, $m = 2$, $\beta = 2$, $\gamma = n+1$ in the formulae (1.5), (1.6), we find that the solution is given by equation (1.6) with

$$L(x)H(1-x) = \mathscr{M}^{-1}\left[\tfrac{1}{2}\frac{\Gamma(\frac{1}{2}s+\frac{1}{2}n-1)\Gamma(\frac{1}{2}s+\frac{1}{2}n-\frac{3}{2})}{\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+n-\frac{3}{2})}; x\right].$$

From formula (18) on p. 326 of **[11]** we deduce that

$$L(x) = x^{n-3}P_{n-2}(x),$$

showing that the equation

$$\int_x^1 P_n(y/x)f(y)\,dy = g(x) \qquad (0 < x < 1), \tag{5.1}$$

with $g(1) = g'(1) = 0$, has solution

$$f(x) = \int_x^1 u^{2-n}P_{n-2}\left(\frac{x}{u}\right)\left(\frac{1}{u}\frac{d}{du}\right)^2\{u^n g(u)\}\,du. \tag{5.2}$$

https://doi.org/10.1017/S001708950000029X Published online by Cambridge University Press

The solution (5.2) of the integral equation (5.1) was first derived by Buschman **[2]**.

6. Transforms whose kernels are Gegenbauer polynomials. Suppose that

$$K(x) = \tfrac{1}{2}m\Gamma(\lambda)C_m^\lambda(x^{-1})(1-x^2)^{\lambda-\frac{1}{2}},$$

where $C_m^\lambda(x)$ denotes the Gegenbauer polynomial.

Then, from the formula

$$C_m^\lambda(x) = \sum_{r=0}^{[\frac{1}{2}m]} \frac{(-1)^r(\lambda)_{m-r}}{r!(m-2r)!}(2x)^{2r-m},$$

we can easily deduce that

$$C_m^\lambda(x^{-1}) = \frac{2^m x^{-m}\Gamma(\lambda+m)}{\Gamma(\lambda)m!}\ {}_2F_1(-\tfrac{1}{2}m, \tfrac{1}{2}-\tfrac{1}{2}m; 1-\lambda-m; x^2),$$

and, integrating term by term, that

$$\mathscr{M}[C_m^\lambda(x^{-1})(1-x^2)^{\lambda-\frac{1}{2}}H(1-x); s]$$
$$= \frac{\Gamma(\lambda+\frac{1}{2})\Gamma(\frac{1}{2}s-\frac{1}{2}m)\Gamma(\lambda+m)}{2^{1-m}\Gamma(\lambda)\Gamma(\frac{1}{2}s-\frac{1}{2}m+\lambda+\frac{1}{2})m!}\,{}_3F_2(-\tfrac{1}{2}m, \tfrac{1}{2}-\tfrac{1}{2}m, \tfrac{1}{2}s-\tfrac{1}{2}m; 1-\lambda-m, \tfrac{1}{2}s-\tfrac{1}{2}m+\lambda+\tfrac{1}{2}; 1).$$

Since m is a positive integer we may use Saalschütz's theorem to evaluate the generalized hypergeometric series on the right side of this equation. In this way we obtain the result

$$\mathscr{M}[C_m^\lambda(x^{-1})(1-x^2)^{\lambda-\frac{1}{2}}H(1-x); s] = \frac{2^{s-1}\Gamma(m+2\lambda)\Gamma(\frac{1}{2}s-\frac{1}{2}m)\Gamma(\frac{1}{2}s+\lambda+\frac{1}{2}m)}{m!\,\Gamma(\lambda)\Gamma(s+2\lambda)},$$

from which it follows that

$$K_1^*(s) = \frac{2^{s-2}\Gamma(m+2\lambda)\Gamma(\frac{1}{2}s-\frac{1}{2}m)\Gamma(\frac{1}{2}s+\lambda+\frac{1}{2}m)}{(m-1)!\,\Gamma(s+2\lambda)}.$$

If we take $l=0$, $\beta=0$, $m=\lambda+\mu+1$, $\gamma=m$ in equation (1.5) we find that

$$L_1^*(s) = \frac{2^{-\lambda-\mu-1}\Gamma(\frac{1}{2}s-\lambda-\mu-1)}{\Gamma(\frac{1}{2}s)K_1^*(s-2\lambda-2\mu+m-2)},$$

from which we find that

$$L_1^*(s) = \frac{2^{\lambda+\mu-m-s-2}m!\,\Gamma(s+n-1)}{\Gamma(m+2\lambda)\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+n+\mu)}, \tag{6.1}$$

where we have written

$$\mu = \tfrac{1}{2}(m-n-1). \tag{6.2}$$

Now, from the formula

$$C_n^\mu(x) = \frac{(2\mu)_n}{n!}\,{}_2F_1(-n, n+2\mu; \mu+\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}x),$$

we can easily deduce that

$$C_n^\mu(x) = \frac{(2\mu)_n(1+x)^{\frac{1}{2}-\mu}}{n!\,2^{\frac{1}{2}-\mu}}\,{}_2F_1(n+\mu+\tfrac{1}{2}, -n-\mu+\tfrac{1}{2}; \mu+\tfrac{1}{2}; \tfrac{1}{2}-\tfrac{1}{2}x).$$

Using the formula

$$\mathscr{M}[x^{n-1}(1-x)^{\mu+r-\frac{1}{2}}H(1-x);s] = \frac{\Gamma(\mu+\frac{1}{2})\Gamma(s+n-1)(\mu+\frac{1}{2})_r}{\Gamma(s+n+\mu-\frac{1}{2})(s+n+\mu-\frac{1}{2})_r},$$

we deduce that

$$\mathscr{M}[x^{n-1}(1-x^2)^{\mu-\frac{1}{2}}C_n^{\mu}(x)H(1-x);s]$$
$$= \frac{2^{\mu-\frac{1}{2}}(2\mu)_n\Gamma(\mu+\frac{1}{2})\Gamma(s+n-1)}{n!\,\Gamma(s+n+\mu-\frac{1}{2})}\,{}_2F_1(n+\mu+\tfrac{1}{2},\,-n-\mu+\tfrac{1}{2};s+n+\mu-\tfrac{1}{2};\tfrac{1}{2}).$$

The hypergeometric series on the right-hand side of this equation can be summed by the use of Bailey's theorem [**10**, p. 32] to give

$$\mathscr{M}[x^{n-1}(1-x^2)^{\mu-\frac{1}{2}}C_n^{\mu}(x)H(1-x);s] = 2^{1-n-s}(2\mu)_n\Gamma(\tfrac{1}{2})\Gamma(\mu+\tfrac{1}{2})\cdot\frac{\Gamma(s+n-1)}{n!\,\Gamma(\frac{1}{2}s)\Gamma(\frac{1}{2}s+n+\mu)}.$$

Comparing this result with equation (1.5) we see that

$$L(x) = \frac{n\Gamma(\mu)}{A}2^{-\lambda-\mu-1}x^{n-1}(1-x^2)^{\mu-\frac{1}{2}}C_n^{\mu}(x), \tag{6.3}$$

where we have written

$$A = \frac{\pi\Gamma(m+2\lambda)\Gamma(n+2\mu)}{2^{2\lambda+2\mu+1}(m-1)!\,(n-1)!}. \tag{6.4}$$

Substituting from equation (6.3) into equation (1.6) we find that the equation

$$\tfrac{1}{2}m\Gamma(\lambda)\int_x^1 (y^2-x^2)^{\lambda-\frac{1}{2}}C_m^{\lambda}(y/x)f(y)\,dy = g(x) \qquad (0<x<1) \tag{6.5}$$

has solution

$$f(x) = (-1)^{[\lambda+\mu+1]}\frac{n\Gamma(\mu)}{A}\int_x^1 y^{2-m}(y^2-x^2)^{\mu-\frac{1}{2}}C_n^{\mu}\left(\frac{y}{x}\right)\left(\frac{1}{2y}\frac{d}{dy}\right)^{\lambda+\mu+1}\{y^m g(y)\}\,dy. \tag{6.6}$$

This is the result derived by Higgins [**3**].

7. Transforms whose kernels are associated Legendre functions. If we take

$$K(x) = (1-x^2)^{\frac{1}{2}\lambda}P_\nu^{-\lambda}(x)$$

in equation (2.1), we find that

$$K_1^*(s) = \frac{2^{-\lambda-s}\Gamma(\frac{1}{2})\Gamma(s)}{\Gamma(\frac{1}{2}\lambda-\frac{1}{2}\nu+\frac{1}{2}s+\frac{1}{2})\Gamma(\frac{1}{2}\lambda+\frac{1}{2}\nu+\frac{1}{2}s+1)},$$

and hence that

$$\frac{1}{K_1^*(s-\lambda-1)} = \frac{2^{s-1}\Gamma(\frac{1}{2}s-\frac{1}{2}\nu)\Gamma(\frac{1}{2}s+\frac{1}{2}\nu+\frac{1}{2})}{\Gamma(\frac{1}{2})\Gamma(s-\lambda-1)}.$$

Putting $m=2$, $n=1$, $\alpha=-\nu-1$, $\beta=\nu-\lambda$, $\gamma=\delta=0$ in equations (1.18) and (1.9) we find that the integral equation

$$\int_x^\infty (y^2-x^2)^{\frac{1}{2}\lambda} P_\nu^{-\lambda}(x/y) f(y)\,dy = g(x) \qquad (x>0) \tag{7.1}$$

has solution

$$f(x) = (2x)^{\nu+1} K_{x^2}^{-\nu-1} K_x^{\nu-\lambda} g(x). \tag{7.2}$$

The equation (7.1) was considered by Srivastava but his solution appears to be in error; the solution (7.2) is due to Erdélyi [6].

8. Transforms whose kernels are hypergeometric functions. For the kernel

$$K(x) = (1-x)^{c-1}\,{}_2F_1(a,b;c;1-x)$$

we find that

$$K_1^*(s) = \frac{\Gamma(c)\Gamma(s)\Gamma(c-a-b+s)}{\Gamma(c-a+s)\Gamma(c-b+s)}.$$

If we take $\beta=1$, $\gamma=l=0$ in equation (1.5) we find that

$$L_1(x) = \frac{1}{\Gamma(c)}\mathscr{M}^{-1}\left[\frac{\Gamma(c-a-m+s)\Gamma(c-b-m+s)}{\Gamma(s)\Gamma(c-m-a-b+s)};\quad s\to x\right].$$

It is readily shown that $L_1(x) = L(x)H(1-x)$, where

$$L(x) = \frac{1}{\Gamma(c)\Gamma(m-c)} x^{-m+c}(1-x)^{m-c-1}\,{}_2F_1(-a,-b;m-c;1-x^{-1}),$$

which shows that the integral equation

$$g(x) = \int_x^1 (y-x)^{c-1}\,{}_2F_1(a,b;c;1-(x/y)f(y)\,dy, \tag{8.1}$$

with $g^{(r)}(1)=0$ ($r=0,1,\ldots,m-1$), has the solution

$$f(x) = \frac{(-1)^m}{\Gamma(c)\Gamma(m-c)}\int_x^1 (y-x)^{m-c-1}\,{}_2F_1(-a,-b;m-c;1-(y/x))g^{(m)}(y)\,dy. \tag{8.2}$$

The solution in this form is due to Jet Wimp [8].

On the other hand if we take

$$K(x) = \frac{1}{\Gamma(c)} x^{-c}(1-x)^{c-1}\,{}_2F_1(a,b;c,1-x^{-1}),$$

we find that

$$K_1^*(s) = \frac{\Gamma(s+a-c)\Gamma(s+b-c)}{\Gamma(s+a+b-c)\Gamma(s)}.$$

Taking $l=0$, $\beta=1$, $\gamma=c-b$ in equation (1.5) we find that

$$L_1(x) = \mathscr{M}^{-1}\left[\frac{\Gamma(s-m-b+c)\Gamma(s-m+a)}{\Gamma(s)\Gamma(s-m+a-b)};x\right]$$

$$= \frac{H(1-x)}{\Gamma(m-c)}x^{c-b-m}(1-x)^{m-c-1}\,{}_2F_1(m-a,\,-b;m-c;1-x),$$

showing that the inversion formula for the hypergeometric function transform

$$g(x) = \frac{x^{-b}}{\Gamma(c)}\int_x^1 (y-x)^{c-1}\,{}_2F_1(a,b;c;1-(y/x))f(y)\,dy \tag{8.3}$$

is

$$f(x) = \frac{(-1)^m}{\Gamma(m-c)}\int_x^1 y^b(y-x)^{m-c-1}\,{}_2F_1(m-a,\,-b;m-c;1-(x/y))g^{(m)}(y)\,dy. \tag{8.4}$$

This is the formula derived by Higgins [7].

REFERENCES

1. Ta Li, A new class of integral transforms, *Proc. Amer. Math. Soc.* **11** (1960), 290–298.

2. R. G. Buschman, An inversion integral for a Legendre transformation, *Amer. Math. Monthly* **69** (1962), 288–289.

3. T. P. Higgins, An inversion integral for a Gegenbauer transformation, *J. Soc. Indust. Appl. Math.* **11** (1963), 886–893.

4. K. N. Srivastava, Inversion integrals involving Jacobi's polynomials, *Proc. Amer. Math. Soc.* **15** (1964), 635–638.

5. K. N. Srivastava, On some integral equations involving Jacobi polynomials, *Math. Japon.* **9** (1964), 85–88.

6. A. Erdélyi, An integral equation involving Legendre functions, *J. Soc. Indust. Appl. Math.* **12** (1964), 15–30.

7. T. P. Higgins, A hypergeometric function transform, *J. Soc. Indust. Appl. Math.* **12** (1964), 601–612.

8. Jet Wimp, Two integral transform pairs involving hypergeometric functions, *Proc. Glasgow Math. Assoc.* **7** (1965), 42–44.

9. I. N. Sneddon, *Fourier transforms*, McGraw-Hill (New York, 1951).

10. L. J. Slater, *Generalized hypergeometric functions*, Cambridge University Press (Cambridge, 1966).

11. A. Erdélyi et al., *Tables of integral transforms*, Vol. 1, McGraw-Hill (New York, 1954).

UNIVERSITY OF GLASGOW