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THE GIBBS PHENOMENON FOR GENERALIZED TAYLOR AND EULER TRANSFORMS

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1. Introduction. Let f be a real-valued function satisfying the Dirichlet conditions in a neighborhood of $x = x_0$, at which point f has a jump discontinuity. If $\{S_n(x)\}$ is the sequence of partial sums of the Fourier series of f at x, then $\{S_n(x)\}$ cannot converge uniformly at $x = x_0$. Moreover, for any number $\tau \in [-\pi, \pi]$, there exists a sequence $\{t_n\}$, where $t_n \to x_0$ and

$$S_n(t_n) = \frac{f(x_0+0) + f(x_0-0)}{2} + \frac{f(x_0+0) - f(x_0-0)}{\pi}\int_0^\tau \frac{\sin u}{u}\,du + o(1) \quad (n \to \infty).$$

This behavior of the sequence $\{S_n(x)\}$ is referred to as the Gibbs phenomenon at $x = x_0$. Let $A = (a_{nk})$ be a regular summability matrix and $\{\sigma_n(x)\}$ be the A-transform of the sequence $\{S_n(x)\}$, i.e.,

$$\sigma_n(x) = \sum_{k=0}^{\infty} a_{nk} S_k(x), \quad n = 0, 1, 2, \ldots .$$

If $\{\sigma_n(x)\}$ displays the Gibbs phenomenon at $x = x_0$ also, then A preserves the Gibbs phenomenon. Miracle [**5**] proved that the transform A preserves the Gibbs phenomenon for all functions satisfying the Dirichlet conditions over $[-\pi, \pi]$, if and only if, given $\tau \in [-\pi, \pi]$, there exists a sequence $\{t_n\}$, such that $t_n \to 0$, and

$$(1) \qquad \sigma_n(t_n) = \int_0^\tau \frac{\sin u}{u}\,du + o(1) \quad (n \to \infty),$$

where $\{\sigma_n(x)\}$ is the A-transform of the sequence of partial sums of the Fourier series for the function

$$\psi(t) = \begin{cases} -\pi/2, \text{ if } -\pi < x < 0 \\ 0, \text{ if } x \equiv 0 \pmod{\pi} \\ \pi/2, \text{ if } 0 < x < \pi, \end{cases}$$

extended to be 2π-periodic. For this particular function

$$(2) \qquad S_n(t) = \int_0^t \frac{\sin 2nu}{\sin u}\,du.$$

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Moreover, since $\sigma_n(x)$ is an odd function of x for each n, a matrix A preserves the Gibbs phenomenon if, for each nonnegative number τ, a nonnegative null sequence $\{t_n\}$ can be chosen so that (1) holds.

The preservation of the Gibbs phenomenon by a matrix transformation has been studied for the Cesáro means by Cramér [**1**] and Gronwall [**2**]; for Euler means by Szász [**9**]; for Borel means by Lorch [**4**]; for Hausdorff means by Szász [**10**]; for Riesz means by Kuttner [**3**], for Taylor means and $[F, d_n]$ means by Miracle [**5**]; and for Sonnenschein means by Sledd [**7**].

In this paper the Gibbs phenomenon is investigated for generalized Taylor methods studied by Powell [**6**; **7**] in § 2, and § 3, and for a generalized Euler method studied by Wood [**11**] in § 4.

2. The Gibbs phenomenon for the $\mathscr{T}(r_n)$ summability method. The $\mathscr{T}(r_n)$-transform [**6**] is defined as follows:

Definition 2.1. Let $\{r_n\}$ be a sequence of complex numbers, such that, for each n, $r_n \neq 1$. Let $a_{nk} = 0$ if $k < n$ and

$$\prod_{j=1}^{n+1} \frac{(1 - r_j)z}{1 - r_j z} = \sum_{k=n}^{\infty} a_{nk} z^{k+1}.$$

The matrix $A = (a_{nk})$ is the $\mathscr{T}(r_n)$ method corresponding to the sequence $\{r_n\}$.

If $r_n = r$ $(r \neq 1)$ for each n, the corresponding $\mathscr{T}(r_n)$ method is the Taylor matrix $T(r)$.

THEOREM 2.2. *Let $\{r_k\}$ be a sequence of real numbers where $0 \leqq r_k \leqq \delta$ for some $0 \leqq \delta < 1$ and all $k = 1, 2, \ldots$. Let $S_n(t)$ be given by (2), $\{\sigma_n(t)\}$ the $\mathscr{T}(r_k)$-transform of the sequence $\{S_n(t)\}$, and*

$$K_n = \sum_{j=1}^{n+1} \frac{2r_j}{1 - r_j} + 2n.$$

If $\{t_n\}$ is a sequence of positive numbers for which $\lim_{n\to\infty} t_n = 0$, while $\lim_{n\to\infty} K_n t_n = \tau$ $(0 \leqq \tau \leqq \infty)$, and $\lim_{n\to\infty} n t_n^2 = 0$, then

$$\lim_{n\to\infty} \sigma_n(t_n) = \int_0^\tau \frac{\sin u}{u} du,$$

i.e., the $\mathscr{T}(r_n)$ matrix preserves the Gibbs phenomenon.

Proof. Since $0 \leqq r_k \leqq \delta$ for some $0 \leqq \delta < 1$ and all $k = 1, 2, \ldots$ we have that the $\mathscr{T}(r_n)$-transform is regular [**6**, Theorem 3.6]. Now

$$(3) \qquad \sigma_n(t) = \sum_{k=n}^{\infty} a_{nk} \int_0^t \frac{\sin 2ku}{\sin u} du.$$

Since $|\sin 2ku| \leqq k\pi|\sin u|$ for $|u| \leqq \pi/2$ and $k = 1, 2, \ldots$ we have that

$$\left| \sum_{k=n}^{\infty} a_{nk} \frac{\sin 2ku}{\sin u} \right| \leqq \pi \sum_{k=n}^{\infty} k|a_{nk}|$$

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and

$$\sum_{k=n}^{\infty} (k+1)a_{nk} = \frac{d}{dz}\left\{\prod_{j=1}^{n+1} \frac{(1-r_j)z}{1-r_jz}\right\}\Bigg|_{z=1}$$

is absolutely convergent since

$$\prod_{j=1}^{n+1} \frac{(1-r_j)z}{1-r_jz}$$

is analytic on $\{z:|z| \leqq 1\}$. Thus $\sum_{k=n}^{\infty} a_{nk} (\sin 2ku)/(\sin u)$ is uniformly convergent for $u \in [0,t]$ $(t \leqq \pi/2)$, and (3) can be rewritten as

$$\begin{aligned}\sigma_n(t) &= \int_0^t \frac{1}{\sin u} \sum_{k=n}^{\infty} a_{nk} \sin 2ku\, du \\ &= \int_0^t \frac{1}{\sin u} \operatorname{Im}\left\{e^{-2iu} \sum_{k=n}^{\infty} a_{nk}\, e^{2(k+1)iu}\right\} du \\ &= \int_0^t \frac{1}{\sin u} \operatorname{Im}\left\{e^{-2iu} \prod_{j=1}^{n+1} \frac{(1-r_j)e^{2iu}}{1-r_je^{2iu}}\right\} du\end{aligned}$$

or

$$(4) \qquad \sigma_n(t) = \int_0^t \frac{\prod_{j=1}^{n+1} (1-r_j)}{\sin u} \operatorname{Im}\left\{\frac{e^{2inu}}{\prod_{j=1}^{n+1} (1-r_je^{2iu})}\right\} du.$$

Define the quantities ρ_j and θ_j by

$$\rho_j\, e^{-i\theta_j} = 1 - r_je^{2iu}.$$

Thus,

$$(5) \qquad \rho_j \cos\theta_j = 1 - r_j \cos 2u,$$

$$(6) \qquad \rho_j \sin\theta_j = r_j \sin 2u,$$

$$(7) \qquad \rho_j^2 = 1 - 2r_j \cos 2u + r_j^2,$$

and

$$(8) \qquad \rho_j \geqq 1 - r_j.$$

Rewriting (3) we have

$$(9) \qquad \sigma_n(t) = \int_0^t \frac{1}{\sin u} \prod_{j=1}^{n+1} \frac{1-r_j}{\rho_j} \sin\left(\sum_{j=1}^{n+1} \theta_j + 2nu\right) du.$$

Using (8),

$$0 \leqq 1 - \prod_{j=1}^{n+1} \frac{1-r_j}{\rho_j} \leqq \sum_{j=1}^{n+1} \left(1 - \frac{1-r_j}{\rho_j}\right)$$

and, by (7),

$$1 - \frac{1 - r_j}{\rho_j} \leqq \frac{2r_j(1 - \cos 2u)}{\rho_j^2}.$$

So

$$(10) \quad 1 - \frac{1 - r_j}{\rho_j} \leqq \frac{r_j}{\rho_j^2} O(u^2) \quad (u \to 0)$$

and

$$0 \leqq 1 - \prod_{j=1}^{n+1} \frac{1 - r_j}{\rho_j} \leqq \sum_{j=1}^{n+1} \frac{r_j}{\rho_j^2} O(u^2) = O(H_n u^2)$$

where

$$H_n = \sum_{j=1}^{n+1} \frac{r_j}{\rho_j^2}.$$

Since $r_k \in [0, 1)$ and $\rho_j \geqq 1 - r_j$ we have

$$\prod_{j=1}^{n+1} \frac{1 - r_j}{\rho_j} = 1 + O(nu^2)$$

and, hence, (9) becomes

$$\sigma_n(t) = \int_0^t \frac{1}{\sin u} \{1 + O(nu^2)\} \sin \left(\sum_{j=1}^{n+1} \theta_j + 2nu \right) du$$

$$= \int_0^t \frac{\sin \left(\sum_{j=1}^{n+1} \theta_j + 2nu \right)}{\sin u} du$$

$$+ O\left\{ n \int_0^t \frac{u^2}{\sin u} \sin \left(\sum_{j=1}^{n+1} \theta_j + 2nu \right) du \right\}.$$

Now

$$\left| n \int_0^t \frac{u^2}{\sin u} \sin \left(\sum_{j=1}^{n+1} \theta_j + 2nu \right) du \right| \leqq \frac{n\pi}{2} \int_0^t u \, du = O(nt^2) \quad (nt^2 \to 0).$$

Thus,

$$(11) \quad \sigma_n(t) = \int_0^t \frac{\sin \left(\sum_{j=1}^{n+1} \theta_j + 2nu \right)}{\sin u} du + O(nt^2) \quad (nt^2 \to 0).$$

Now, by (7),

$$\begin{aligned} |\rho_j \theta_j - 2r_j u| &= |\rho_j \theta_j - \rho_j \sin \theta_j + r_j \sin 2u - 2r_j u| \\ &\leqq \rho_j(\theta_j - \sin \theta_j) + r_j(2u - \sin 2u) \\ &\leqq \rho_j \theta_j^3 + 8r_j u^3, \end{aligned}$$

and, by (10),

$$\rho_j - (1 - r_j) = \frac{r_j}{\rho_j} \cdot O(u^2).$$

Thus,

$$|(1 - r_j)\theta_j - 2r_j u| \leqq |\rho_j\theta_j - 2r_j u| + |(1 - r_j)\theta_j - \rho_j\theta_j|$$

$$\leqq \rho_j\theta_j^{\,3} + 8r_j u^3 + \frac{r_j}{\rho_j}\theta_j \cdot O(u^2) \quad (u \to 0).$$

Since

$$\theta_j \leqq \frac{\pi}{2} \sin \theta_j = \frac{\pi}{2}\frac{r_j}{\rho_j} \sin 2u \leqq \frac{\pi r_j}{\rho_j} u$$

we have that

$$|(1 - r_j)\theta_j - 2r_j u| \leqq \rho_j \left(\frac{\pi r_j}{\rho_j}\right)^3 u^3 + 8r_j u^3 + \frac{\pi r_j^{\,2}}{\rho_j^{\,2}} \cdot O(u^3)$$

$$= O(u^3) \quad (u \to 0).$$

Thus,

$$\theta_j = \frac{2r_j}{1 - r_j} u + O(u^3), \quad (u \to 0)$$

and, using (11),

$$\sigma_n(t) = \int_0^t \frac{1}{\sin u} \sin \left\{ \sum_{j=1}^{n+1} \left[\frac{2r_j}{1 - r_j} u + O(u^3) \right] + 2nu \right\} du$$

$$= \int_0^t \frac{1}{\sin u} \sin \left\{ \left[\sum_{j=1}^{n+1} \frac{2r_j}{1 - r_j} + 2n \right] u + O(un^3) \right\} du$$

$$= \int_0^t \frac{1}{\sin u} \sin \left\{ \left(\sum_{j=1}^{n+1} \frac{2r_j}{1 - r_j} + 2n \right) u \right\} \cos O(nu^3)\, du$$

$$+ \int_0^t \frac{1}{\sin u} \sin O(nu^3) \cos \left\{ \left(\sum_{j=1}^{n+1} \frac{2r_j}{1 - r_j} + 2n \right) u \right\} du.$$

The procedure used to derive the error estimate in (11) shows that the second integral is $O(nt^2)$ $(nt^2 \to 0)$. Also,

$$\cos O(nu^3) = 1 + O(nu^3),$$

so

$$\sigma_n(t) = \int_0^t \frac{1}{\sin u} \sin \left\{ \left(\sum_{j=1}^{n+1} \frac{2r_j}{1 - r_j} + 2n \right) u \right\} [1 + O(nu^3)]\, du + O(nt^2)$$

$$= \int_0^t \frac{1}{\sin u} \sin K_n u \, du + O(nt^2)$$

$$= \int_0^t \frac{\sin K_n u}{u} du + O(nt^2) \quad (nt^2 \to 0).$$

If $\{t_n\}$ satisfies the hypotheses of the theorem, then

$$\sigma_n(t_n) = \int_0^{t_n} \frac{\sin K_n u}{u} du + O(nt_n^2) \quad (n \to \infty)$$

$$= \int_0^{K_n t_n} \frac{\sin u}{u} du + O(nt_n^2) \quad (n \to \infty)$$

$$= \int_0^{\tau} \frac{\sin u}{u} du + o(1) \quad (n \to \infty).$$

Letting $r_n = r \in [0, 1)$ in Theorem 2.2, we obtain the following corollary, which is essentially Theorem 4.1 due to Miracle [**5**].

COROLLARY 2.3. *Let $S_n(t)$ be given by* (2), *and let $r \in [0, 1)$. Let $\{\sigma_n(t)\}$ be the $T(r)$-transform of the sequence $\{S_n(t)\}$, and $K_n = 2(n + r)/(1 - r)$. If $\{t_n\}$ is a sequence of positive numbers for which $\lim_{n\to\infty} t_n = 0$, while $\lim_{n\to\infty} K_n t_n = \tau$ $(0 \leqq \tau \leqq \infty)$ and $\lim_{n\to\infty} nt_n^2 = 0$, then*

$$\lim_{n\to\infty} \sigma_n(t_n) = \int_0^{\tau} \frac{\sin u}{u} du.$$

3. The Gibbs phenomenon for the $L(r, t)$ summability method. The $L(r, t)$-transform is defined to be the matrix $A = (a_{nk})$, where

$$a_{nk} = \begin{cases} 0, & \text{if } k < n \\ (1 - r)^{n+1} \exp\left(\frac{tr}{1 - r}\right) L_{k-n}^{(n)}(t) r^{k-n}, & \text{if } k \geqq n; \end{cases}$$

here $L_j^{(n)}(t)$ denotes the Laguerre polynomial of degree j. It is assumed that both r and t are real, and that $0 \leqq r < 1$. The special case $L(r, 0)$ is the Taylor matrix $T(r)$. According to [**7**, Theorem 2.1], the conditions on r and t ensure that the matrix A is regular, and has real entries.

THEOREM 3.1. *Let r and t be real numbers with $0 \leqq r < 1$, let $S_n(x)$ be given by* (**2**), *and let $\{\sigma_n(x)\}$ be the $L(r, t)$-transform of $\{S_n(x)\}$. If $\{x_n\}$ is a sequence of positive numbers such that $x_n \to 0$, $[2(n + r)/(1 - r)]x_n \to \tau$ $(0 \leqq \tau \leqq \infty)$,*

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and $nx_n^2 \to 0$, *then*

$$\lim_{n\to\infty} \sigma_n(x_n) = \int_0^\tau \frac{\sin u}{u}\, du,$$

i.e., the $L(r, t)$ *matrix preserves the Gibbs phenomenon.*

Proof. We have

$$\begin{aligned}\sigma_n(x) &= \sum_{k=0}^{\infty} a_{nk} \int_0^x \frac{\sin 2ku}{\sin u}\, du \\ &= \int_0^x \frac{1}{\sin u} \operatorname{Im}\left\{\sum_{k=n}^{\infty} a_{nk} e^{2kiu}\right\} du \\ &= (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \int_0^x \frac{1}{\sin u} \operatorname{Im}\left\{\sum_{k=n}^{\infty} L_{k-n}^{(n)}(t) r^{k-n} e^{2kiu}\right\} du \\ &= (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \int_0^x \frac{1}{\sin u} \operatorname{Im}\left\{e^{2inu} \sum_{k=n}^{\infty} L_{k-n}^{(n)}(t) \right. \\ &\qquad \left. \times (re^{2iu})^{k-n}\right\} dn \\ &= (1-r)^{n+1} \exp\left(\frac{tr}{1-r}\right) \int_0^x \frac{1}{\sin u} \operatorname{Im}\left\{\frac{e^{2inu}}{(1-re^{2iu})^{n+1}} \right. \\ &\qquad \left. \times \exp\left(\frac{-tre^{2iu}}{1-re^{2iu}}\right)\right\} du.\end{aligned}$$

Thus,

$$\begin{aligned}(12) \quad \sigma_n(x) = (1-r)^{n+1} \int_0^x \frac{1}{\sin u} \operatorname{Im}&\left\{\frac{e^{2inu}}{(1-re^{2iu})^{n+1}} \right. \\ &\left. \times \exp\left(\frac{tr}{1-r} - \frac{tre^{2iu}}{1-re^{2iu}}\right)\right\} du.\end{aligned}$$

Now,

$$\begin{aligned}\left|\exp\left(\frac{tr}{1-r} - \frac{tre^{2iu}}{1-re^{2iu}}\right) - 1\right| &= \left|\sum_{k=1}^{\infty} \frac{1}{k!}\left[\frac{tr(1-e^{2iu})}{(1-r)(1-re^{2iu})}\right]^k\right| \\ &\leqq \sum_{k=1}^{\infty} \frac{1}{k!} \left|\frac{tr}{(1-r)^2}\right|^k |1-e^{2iu}|^k \\ &\leqq u \sum_{k=1}^{\infty} \frac{1}{k!} \left|\frac{tr}{(1-r)^2}\right| M^k,\end{aligned}$$

where

$$M = \sup_{0\leqslant u\leqslant \pi/2} \left|\sum_{j=1}^{\infty} \frac{(2i)^j}{j!} u^{j-1}\right| < \infty.$$

Hence,

$$(13)\quad \exp\left(\frac{tr}{1-r} - \frac{tre^{2iu}}{1-re^{2iu}}\right) = 1 + O(u) \quad (u \to 0).$$

From (12) and (13) it follows that

$$\sigma_n(x) = (1-r)^{n+1} \int_0^x \frac{1}{\sin u} \operatorname{Im}\left\{\frac{e^{2inu}}{(1-re^{2iu})^{n+1}}\right\} du + O\left\{\int_0^x \frac{u}{\sin u} \operatorname{Im}\left[e^{2inu}\left(\frac{1-r}{1-re^{2iu}}\right)^{n+1}\right] du\right\},$$

i.e.,

$$(14)\quad \sigma_n(x) = (1-r)^{n+1} \int_0^x \frac{1}{\sin u} \operatorname{Im}\left\{\frac{e^{2inu}}{(1-re^{2iu})^{n+1}}\right\} du + O(x) \quad (x \to 0).$$

But the integral appearing in (14) is precisely the integral which arises when the Taylor transform $T(r)$ is applied to the series (2). According to [**5**, Theorem 4.1], we, therefore, have that

$$\lim_{n\to\infty} \sigma_n(x_n) = \int_0^\tau \frac{\sin u}{u} du,$$

provided $\{x_n\}$ satisfies the hypotheses of the theorem.

4. The Gibbs phenomenon for the $J(g_m, z)$ summability method. The $J(g_m, z)$-transform [**11**] is defined as follows:

Definition 4.1. Let $\{g_m\}$ be a convergent sequence of functions, each function analytic on $\{z: |z| < \rho\}$ for $\rho > 1$, and let

$$g(z) = \lim_{m\to\infty} g_m(z)$$

where the convergence is uniform on compact subsets of $\{z: |z| < \rho\}$ and $g(u) \neq 0$ for $-1 \leqq u \leqq 0$. Let $\{\zeta_n^{(m)}(x)\}$ be a sequence of polynomials defined by

$$g_m(u)(1-u)^x = \sum_{n=0}^{\infty} \zeta_n^{(m)}(x) u^n$$

and define the linear operator L_m by

$$L_m(f, x) = \frac{1}{g(x-1)} \sum_{n=0}^{m} (-1)^{m-n} \zeta_{m-n}^{(m)}(-n-1) x^n (1-x)^{m-n} f\left(\frac{n}{m}\right)$$

where f is defined on [0, 1] and $x \in [0, 1]$. The matrix $B = (b_{mn})$ given by

$$L_m(f, z) = \sum_{n=0}^{m} b_{mn} f\left(\frac{n}{m}\right) \quad \text{for } |1-z| < \rho$$

is the $J(g_m, z)$ summability method corresponding to the sequence $\{g_m\}$ and the parameter z.

If $g_m(z) \equiv 1$ for all m then the $J(g_m, z)$ transform is merely the Euler transform $E(z)$.

In the following let z be real and $0 < z \leqq 1$ (write $z = x$). Thus, the $J(g_m, x)$ method is regular [**11**, Theorem 2.1]. Also, denote

$$g_m(x) = \sum_{n=0}^{\infty} (-1)^n a_{mn} x^n,$$

$$h_m(x) = \sum_{n=m+1}^{\infty} (-1)^k a_{mn} x^n,$$

$$g(x) = \sum_{n=0}^{\infty} (-1)^n a_n x^n.$$

The following lemma is the result of a straightforward calculation.

LEMMA 4.2. *The $J(g_m, x)$ summability transform $B = (b_{mn})$ is given by*

$$b_{mn} = \begin{cases} \dfrac{1}{g(x-1)} x^n (1-x)^{m-n} \displaystyle\sum_{k=0}^{m-n} \binom{m-k}{n} a_{mk}, & \text{if } 0 \leqq n \leqq m \\ 0, & \text{if } n > m. \end{cases}$$

THEOREM 4.3. *Let a_{mn} be real for all $m, n = 1, \ldots,$ and let $x \in (0, 1]$. Let $S_n(t)$ be given by* (2) *and $\{\sigma_n(t)\}$ be the $J(g_m, x)$-transform of $\{S_n(t)\}$. If $\{t_m\}$ is a sequence of positive numbers such that $t_m \to 0$, $mt_m \to \tau/x$ $(0 \leqq \tau \leqq \infty)$ and $mt_m^2 \to 0$ then*

$$\lim_{m\to\infty} \sigma_m(t_m) = \int_0^\tau \frac{\sin u}{u} du,$$

i.e., the $J(g_m, x)$ matrix preserves the Gibbs phenomenon.

Proof. By Lemma 4.2,

$$\sigma_m(t) = \frac{1}{g(x-1)} \sum_{n=0}^{m} x^n (1-x)^{m-n} \sum_{k=0}^{m-n} \binom{m-k}{n} a_{mk} \int_0^t \frac{\sin 2nu}{\sin u} du$$

$$= \frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u}$$

$$\times \operatorname{Im} \left\{ \sum_{k=0}^{m} a_{mk} (1-x)^k \sum_{n=0}^{m-k} \binom{m-k}{n} (xe^{2iu})^n (1-x)^{m-k-n} \right\} du,$$

so

$$\sigma_m(t) = \frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u} \times \operatorname{Im} \left\{ \sum_{k=0}^{m} \frac{g_m^{(k)}(0)}{k!} (x-1)^k (1-x+xe^{2iu})^{m-k} \right\} du. \tag{15}$$

Write $\xi = 1 - x + xe^{2iu}$ and

$$P_m(t) = \int_0^t \frac{1}{\sin u} \operatorname{Im} (\xi^m)\, du.$$

By (15)

$$\sigma_m(t) = \frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u} \operatorname{Im} \left\{ \xi^m \sum_{k=0}^{m} (-1)^k a_{mk} \left(\frac{x-1}{1-x+xe^{2iu}} \right)^k \right\} du$$

$$= \frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u}$$

$$\times \operatorname{Im} \left\{ \xi^m \left[g_m \left(\frac{x-1}{1-x+xe^{2iu}} \right) - h_m \left(\frac{x-1}{1-x+xe^{2iu}} \right) \right] \right\} du.$$

So

$$(16) \quad \sigma_m(t) = \frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u} \operatorname{Im} \left\{ \xi^m \left[g_m \left(\frac{x-1}{1-x+xe^{2iu}} \right) - g_m(x-1) \right] \right\} du$$

$$+ \frac{g_m(x-1)}{g(x-1)} P_m(t) + \frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u}$$

$$\times \operatorname{Im} \left\{ \xi^m \left[h_m(x-1) - h_m \left(\frac{x-1}{1-x+xe^{2iu}} \right) \right] \right\} du$$

$$- \frac{h_m(x-1)}{g(x-1)} P_m(t).$$

In order to estimate the first and third terms in (16), we use the following: Since, for each m, $g_m(z) = \sum_{k=0}^{\infty} (-1)^k a_{mk} z^k$ is analytic on $D = \{z: |z| < \rho\}$, $\rho > 1$, and $\{g_m\}$ converges uniformly to g on compact subsets of D, given ρ_1 $(1 < \rho_1 < \rho)$ there exists $A > 0$ such that

$$(17) \quad \sum_{k=0}^{\infty} |a_{mk}|^2 \rho_1^{2k} \leqq A \quad \text{for all } m.$$

Also, if $|z_1| \leqq 1$ and $|z_2| \leqq 1$, then

$$|g_m(z_1) - g_m(z_2)| \leqq |z_1 - z_2| \sup_{z \in [z_1, z_2]} |g_m'(z)|,$$

and, using Hölder's inequality,

$$|g_m'(z)| \leqq \sum_{k=1}^{\infty} k |a_{mk}|$$

$$\leqq \left[\sum_{k=1}^{\infty} |a_{mk}|^2 \rho_1^{2k} \right]^{1/2} \left[\sum_{k=1}^{\infty} \frac{k^2}{\rho_1^{2k}} \right]^{1/2}$$

$$\leqq A \left[\sum_{k=1}^{\infty} \frac{k^2}{\rho_1^{2k}} \right]^{1/2}$$

$$= A_1.$$

Hence, if $|z_1| \leqq 1$ and $|z_2| \leqq 1$, then

$$|g_m(z_1) - g_m(z_2)| \leqq A_1|z_1 - z_2|.$$

If $u \in [0, \pi/4]$ and $x \in (0, 1]$, then

$$|x - 1| \leqq 1 \quad \text{and} \quad \left|\frac{x-1}{1 - x + xe^{2iu}}\right| \leqq 1.$$

Thus,

$$\left|\frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u} \operatorname{Im} \left\{\xi^m \left[g_m\left(\frac{x-1}{1-x+xe^{2iu}}\right) - g_m(x-1)\right]\right\} du\right|$$
$$\leqq \frac{1}{|g(x-1)|} \int_0^t \frac{1}{\sin u} |\xi|^m \left|g_m\left(\frac{x-1}{1-x+xe^{2iu}}\right) - g_m(x-1)\right| du$$
$$\leqq \frac{A_1}{|g(x-1)|} \int_0^t \frac{1}{\sin u} \left|\frac{x-1}{1-x+xe^{2iu}} - (x-1)\right| du$$
$$\leqq \frac{A_2}{|g(x-1)|} \int_0^t \frac{u}{\sin u} du$$
$$= O(t) \quad (t \to 0).$$

Similarly,

$$\left|\frac{1}{g(x-1)} \int_0^t \frac{1}{\sin u} \operatorname{Im} \left\{\xi^m \left[h_m(x-1) - h_m\left(\frac{x-1}{1-x+xe^{2iu}}\right)\right]\right\} du\right|$$
$$= O(t) \quad (t \to 0).$$

Also, by (17),

$$|h_m(x-1)| \leqq \sum_{k=m+1}^{\infty} |a_{mk}|$$
$$\leqq \left[\sum_{k=m+1}^{\infty} |a_{mk}|^2 \rho_1^{2k}\right]^{1/2} \left[\sum_{k=m+1}^{\infty} \frac{1}{\rho_1^{2k}}\right]^{1/2}$$
$$\leqq A\left[\sum_{k=m+1}^{\infty} \frac{1}{\rho_1^{2k}}\right]^{1/2}.$$

Thus,

$$\frac{h_m(x-1)}{g(x-1)} P_m(t) = o(1) \quad (m \to \infty).$$

By (16),

$$\sigma_m(t) = \frac{g_m(x-1)}{g(x-1)} P_m(t) + o(1) \quad (t \to 0, m \to \infty),$$

i.e.,

$$\sigma_m(t) = P_m(t) + o(1) \quad (t \to 0,\ m \to \infty).$$

If $\{t_m\}$ satisfies the hypotheses of the theorem, then

$$\sigma_m(t_m) = \int_0^{t_m} \frac{1}{\sin u} \operatorname{Im}\{(1 - x + xe^{2iu})^m\}\, du + o(1) \quad (m \to \infty).$$

Using the result

$$\lim_{m\to\infty} \int_0^{t_m} \frac{1}{\sin u} \operatorname{Im}\{(1 - x + xe^{2iu})^m\}\, du = \int_0^{\tau} \frac{\sin u}{u}\, du$$

proved by Szász [**9**] we have

$$\lim_{m\to\infty} \sigma_m(t_m) = \int_0^{\tau} \frac{\sin u}{u}\, du.$$

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