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Varieties of topological groups III

Sidney A. Morris

This paper continues the investigation of varieties of topological groups. It is shown that the family of all varieties of topological groups with any given underlying algebraic variety is a class and not a set. In fact the family of all β-varieties with any given underlying algebraic variety is a class and not a set. A variety generated by a family of topological groups of bounded cardinal is not a full variety.

The varieties $\underline{\underline{V}}(R)$ and $\underline{\underline{V}}(T)$ generated by the additive group of reals and the circle group respectively each with its usual topology are examined. In particular it is shown that a locally compact Hausdorff abelian group is in $\underline{\underline{V}}(T)$ if and only if it is compact. Thus $\underline{\underline{V}}(R)$ properly contains $\underline{\underline{V}}(T)$.

It is proved that any free topological group of a non-indiscrete variety is disconnected. Finally, some comments are made on topologies on free groups.

1. Subgroup topologies

We will use the notation and terminology developed in [6] and [7].

DEFINITION. Let G be a group and M a family of subgroups of G . If M is an open basis at the identity for the topology τ of G , then τ is said to be a subgroup topology and G an S-group. If all the subgroups in M have index in G strictly less than an infinite cardinal m , then τ is said to be an $S(m)$-topology and G an $S(m)$-group.

The proof of the following lemma is elementary and therefore omitted.

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LEMMA 1.1. *If* A *and* B *are subgroups of* G *of index strictly less than an infinite cardinal* m *, then the subgroup* $A \cap B$ *has index in* G *strictly less than* m *.*

THEOREM 1.2. *Let* G *be any group and* M *the family of all normal subgroups of* G *of index strictly less than some infinite cardinal* m *. Then* M *is an open basis at the identity for a group topology* τ *on* G *. Further,* τ *is discrete if and only if* G *has cardinal strictly less than* m *.*

Proof. We check 4.5 (i) - (v) of [3]. Clearly (i) - (iv) are satisfied and (v) follows from Lemma 1.1. The last statement in the theorem is obvious.

COROLLARY 1.3. *Let* $\bar{G}$ *be any residually finite abstract group* [9]. *Then* $\bar{G}$ *admits a Hausdorff* $S(\aleph_0)$*-topology.*

THEOREM 1.4. *Let* $\bar{\underline{V}}$ *be an algebraic variety which is generated by its finite abstract groups. Then all free abstract groups of* $\bar{\underline{V}}$ *admit Hausdorff* $S(\aleph_0)$*-topologies. In particular this is the case when* $\bar{\underline{V}}$ *is the algebraic variety of all groups or of all abelian groups, or any locally finite algebraic variety.*

Proof. This is an immediate consequence of Theorem 17.81 of [9] and Corollary 1.3.

THEOREM 1.5. *Let* $\bar{F}$ *be any algebraically relatively free abstract group. Then for any cardinal* $m > \aleph_0$ *,* $\bar{F}$ *admits a Hausdorff* $S(m)$*-topology.*

Proof. This result follows from Theorem 1.2 and the fact, implied by Theorem 17.81 of [9], that $\bar{F}$ is residually countable.

THEOREM 1.6. *If* G *is a Hausdorff* S*-group, then it is totally disconnected.*

Proof. Let C be the component of the identity e . If $C \neq \{e\}$ then it has a proper subset A containing e which is open in the induced topology τ on C . Clearly A contains a subgroup B of C which is open (and therefore closed) in τ . This implies C is not connected, which is a contradiction. Thus $C = \{e\}$, and the proof is complete.

2. S(m)-varieties

THEOREM 2.1. *Let m be any infinite cardinal and $\{G_\alpha : \alpha \in I\}$ be a family of $S(m)$-groups. Then the variety generated by this family contains only $S(m)$-groups.*

Proof. Clearly it is sufficient to show that subgroups, quotient groups and cartesian products of $S(m)$-groups are $S(m)$-groups.

Let G be a group with an open basis at the identity e consisting of a family M of subgroups of index in G strictly less than m. For any subgroup H of G, the family $N = \{N_i : N_i = M_i \cap H , M_i \in M\}$ is an open basis at e for the induced topology on H. Since the index of each N_i in H is strictly less than m, the indeced topology on H is an $S(m)$-topology. Thus subgroups of $S(m)$-groups are $S(m)$-groups.

Now consider the quotient group G/K for K any normal subgroup of G. Let Φ be the natural homomorphism of G onto G/K. Then $\{\Phi(M_i) : M_i \in M\}$ is an open basis at the identity of G/K for the quotient topology. Furthermore, the index of each $\Phi(M_i)$ in G/K is strictly less than m. Thus G/K has an $S(m)$-topology. Hence quotient groups of $S(m)$-groups are $S(m)$-groups.

Let G_0 be the cartesian product of a set $\{G_j : j \in J\}$ of $S(m)$-groups. Let the family $\{N_{jk} : k \in K_j\}$ of subgroups of index strictly less than m in G_j be an open basis at the identity of G_j for its topology. The family T of all subgroups of G_0 of the form $\prod_{j \in J} H_j$, where $H_j = G_j$ for all but a finite number of j in J and $H_j \neq G_j$ implies $H_j = N_{jk}$ for some $k \in K_j$, is an open basis at the identity for the product topology on G_0. Since each group in T is of index in G_0 strictly less than m, G_0 has an $S(m)$-topology. Thus cartesian products of $S(m)$-groups are $S(m)$-groups. The proof of the theorem is complete.

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COROLLARY 2.2. *Let* $\{G_\alpha : \alpha \in I\}$ *be a family of* S*-groups. Then the variety generated by this family contains only* S*-groups.*

Proof. This result is a consequence of the proof of the above theorem.

DEFINITION. A variety which contains only S-groups ($S(m)$-groups) is said to be an S-variety ($S(m)$-variety).

It is obvious that every $S(m)$-variety is an S-variety whilst there exist S-varieties which are not $S(m)$-varieties for any m .

THEOREM 2.3. *Let* $\bar{\underline{\underline{V}}}$ *be any algebraic variety. For each infinite cardinal* m *let* $\underline{\underline{V}}_m$ *be the class of all* $S(m)$*-groups* G *such that* $\bar{G} \in \bar{\underline{\underline{V}}}$. *Then* $\underline{\underline{V}}_m$ *is a variety such that* $\bar{\underline{\underline{V}}}_m = \bar{\underline{\underline{V}}}$ *and for any infinite cardinal* $n \neq m$, $\underline{\underline{V}}_n \neq \underline{\underline{V}}_m$.

Proof. By Theorems 1.2 and 2.1, $\underline{\underline{V}}_m$ is a variety and $\bar{\underline{\underline{V}}}_m = \bar{\underline{\underline{V}}}$. Without loss of generality, assume $n > m$. Let $\bar{G}$ be any abstract group in $\bar{\underline{\underline{V}}}$ of order m . Then by Theorem 1.2, $\bar{G}$ with the discrete topology is in $\underline{\underline{V}}_n$ but not $\underline{\underline{V}}_m$. Thus $\underline{\underline{V}}_n \neq \underline{\underline{V}}_m$.

REMARK 2.4. One outstanding unsolved problem in the theory of algebraic varieties of abstract groups is:- How many algebraic varieties are there? (See [9])[1]. Theorem 2.3 provides the answer to the corresponding question for varieties of (topological) groups. The family of all varieties of groups is a class and not a set. In fact for each algebraic variety $\bar{\underline{\underline{V}}}$, the family of all varieties with underlying algebraic variety $\bar{\underline{\underline{V}}}$ is a class and not a set.

THEOREM 2.5. *If* $\underline{\underline{V}}$ *is an* S*-variety then it is not a* β*-variety and therefore not a full variety.*

Proof. This follows immediately from Theorem 1.6, and Theorems 2.3 and 2.1 of [7].

THEOREM 2.6. *Let* F *be algebraically relatively free and the*

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[1] This problem has very recently been solved by A.Ju. Ol'šanskiĭ (unpublished), who has shown that there are $2^{\aleph_0}$ such varieties. - Editor

family of all normal subgroups of index in F *strictly less than some infinite cardinal* m *be an open basis at the identity for the topology of* F *. Further, let* X *be any subspace of* F *which is a free algebraic basis of* F *. Then* F *is* $F(X, \underline{\underline{V}}(F))$ *(cf. Theorem 3.3 of [6]). If* m *is strictly greater than* $\aleph_0$ *, then* X *has the discrete topology.*

Proof. The fact that F is $F(X, \underline{\underline{V}}(F))$ follows from the proof of Theorem 2.1. To show that X is discrete if $m > \aleph_0$, let x be any element of X and Y be $X - x$. Let G be the smallest normal subgroup of F containing Y . Clearly G is of countable index and is therefore open. Thus $xG \cap X$ is an open subset of X . Since $xG \cap X = \{x\}$, X has the discrete topology.

3. Some basic questions answered

In the theory of varieties of groups the following questions naturally arise: Let A be any group in a (non-indiscrete) variety $\underline{\underline{V}}$. If B is a group algebraically isomorphic to A which has

(i) a strictly finer topology than A or

(ii) a strictly coarser topology than A ,

is B necessarily in $\underline{\underline{V}}$?

In the light of the results of §1 and §2 these questions can now be answered.

EXAMPLE 3.1. Let $\bar{\underline{\underline{V}}}$ be any algebraic variety and m any cardinal $> c$. Let $\underline{\underline{V}}$ be the $\underline{\underline{V}}_m$ of Theorem 2.3. Then by Theorem 1.2, if $\bar{F}$ is any free abstract group of $\bar{\underline{\underline{V}}}$ of cardinal c , $\bar{F}$ with the discrete topology τ is in $\underline{\underline{V}}$. Let $\underline{\underline{W}}$ be the full variety with the property that $\bar{\underline{\underline{W}}} = \bar{\underline{\underline{V}}}$, and $F_1 = F(X, \underline{\underline{W}})$, where X is the closed interval $[0, 1]$ of reals. By Theorems 1.1 and 1.2 of [7], F_1 is Hausdorff. Clearly then by Theorem 1.6, $F_1 \notin \underline{\underline{V}}$ whilst F_1 is algebraically isomorphic to $\bar{F}$ and has a strictly coarser topology than τ . Thus question (i) above is answered in the negative.

Let $\bar{Y}$ be a set of cardinal m and $\bar{F}_2$ be the free abstract group of $\bar{\underline{\underline{V}}}$ on $\bar{Y}$. By Theorem 1.5, $\bar{F}_2$ appears in $\underline{\underline{V}}$ with a Hausdorff topology. However, Theorem 1.2 shows that $\bar{F}_2$ with the discrete topology does not appear in $\underline{\underline{V}}$. Thus question (ii) above is answered in the negative.

In [5] it was shown that if $\underline{\underline{V}}$ is the variety of all groups or all abelian groups and X is any Tychonoff space, then $F(X, \underline{\underline{V}})$ is Hausdorff. This prompted the author to ask the question: If $\underline{\underline{V}}$ is any variety and X is any Hausdorff space such that $F(X, \underline{\underline{V}})$ exists, is $F(X, \underline{\underline{V}})$ necessarily Hausdorff? The example below provided by John Looker shows that the answer is in the negative.

EXAMPLE 3.2. Let $\underline{\underline{V}}$ be the class of all groups G having the property that the intersection of all neighbourhoods of the identity in G contains the commutator subgroup of G . It can be readily verified that $\underline{\underline{V}}$ is a variety such that $\bar{\underline{\underline{V}}}$ is the algebraic variety of all abstract groups. Clearly the additive group of reals with the usual topology is in $\underline{\underline{V}}$. Thus, by Corollary 2.10 of [6], $F(X, \underline{\underline{V}})$ exists for any Tychonoff space X . However, it is not Hausdorff.

It was shown in [7] that moderately free groups on Hausdorff spaces are Hausdorff. This, together with the above example, led I.D. Macdonald to ask the question:- If the Hausdorff group F is a free group of the variety it generates is it necessarily moderately free? Example 3.3 shows that this is not so.

EXAMPLE 3.3. Let Z be the additive group of integers with the finest $S(\aleph_0)$-topology. By Theorem 1.4, Z is Hausdorff. Further, by Theorem 1.2, Z is not discrete whilst by Theorem 2.6 Z is free in the variety it generates.

4. T(m)-topologies

DEFINITION. The group G is said to be a $T(m)$-group if there exists an $S(m)$-group which is algebraically isomorphic to G and has a finer topology than G .

Clearly if G is any group of cardinal n , then for any infinite cardinal $m > n$, G is a $T(m)$-group. (Simply compare G and $\bar{G}$ with the discrete topology.)

LEMMA 4.1. *Let* $\{G_\alpha : \alpha \in I\}$ *and* $\{H_\alpha : \alpha \in I\}$ *be families of* *groups such that for each* $\alpha \in I$, H_α *is algebraically isomorphic to* G_α *and has a finer topology than* G_α. *Then for any group* G *in the variety* $\underline{\underline{V}}$ *generated by the family* $\{G_\alpha : \alpha \in I\}$ *there is a group* H *in the variety* $\underline{\underline{W}}$ *generated by* $\{H_\alpha : \alpha \in I\}$ *which is algebraically isomorphic to* G *and has a finer topology than* G.

Proof. Clearly G is in $\underline{\underline{V}}$ if and only if it can be obtained from $\{G_\alpha : \alpha \in I\}$ by a finite number of applications of the operations of taking cartesian products, subgroups and quotient groups. If we apply the same operations (as those used to obtain G) to the family $\{H_\alpha : \alpha \in I\}$ we obtain a group H which is algebraically isomorphic to G and has a finer topology than G. Further H is in $\underline{\underline{W}}$.

THEOREM 4.2. *A variety generated by a family of* $T(m)$*-groups contains only* $T(m)$*-groups.*

Proof. This result is an immediate consequence of Theorem 2.1 and Lemma 4.1.

DEFINITION. A variety which contains only $T(m)$-groups is said to be a $T(m)$-variety.

COROLLARY 4.3. *A* $T(m)$*-variety is not a full variety.*

Proof. This follows from Theorem 1.2.

THEOREM 4.4. *Let* $\{G_\alpha : \alpha \in I\}$ *be a family of groups each of cardinal strictly less than some infinite cardinal* m. *Then the variety generated by* $\{G_\alpha : \alpha \in I\}$ *is a* $T(m)$*-variety and consequently not a full variety.*

Proof. This is immediate from Theorem 4.2 and Corollary 4.3.

REMARK 4.5. The above result contrasts with the algebraic result that every abstract variety is generated by its finitely generated abstract groups.

Theorem 4.4 can be strengthened in the case that $m = \aleph_0$.

THEOREM 4.6. *A variety generated by finite groups is an* $S(\aleph_0)$*-variety and consequently not a* β*-variety.*

Proof. It is easily proved [8] that finite groups are $S(\aleph_0)$-groups. The result then follows from Theorems 2.1 and 2.5.

THEOREM 4.7. *Let* R *be the additive group of reals and* T *the circle group each having its usual topology. Then* $\underline{\underline{V}}(R)$ *and* $\underline{\underline{V}}(T)$ *are* β*-varieties but not full varieties.*

Proof. The fact that $\underline{\underline{V}}(R)$ and $\underline{\underline{V}}(T)$ are not full varieties follows from Theorem 4.4. That $\underline{\underline{V}}(R)$ and $\underline{\underline{V}}(T)$ are β-varieties can be deduced from the proof of Theorem 4.5 of [6].

The next lemma is obvious.

LEMMA 4.8. *If* $\underline{\underline{V}}$ *is a* β*-variety and* $\underline{\underline{W}}$ *is a variety such that* $\bar{\underline{\underline{W}}} = \bar{\underline{\underline{V}}}$ *and* $\underline{\underline{W}} \supset \underline{\underline{V}}$ *, then* $\underline{\underline{W}}$ *is a* β*-variety.*

LEMMA 4.9. *If* $\underline{\underline{W}}$ *is any full variety,* $X = [0, 1]$ *and* $F = F(X, \underline{\underline{W}})$ *, then* $\underline{\underline{V}}(F)$ *is a* β*-variety.*

Proof. This result follows from Theorems 1.1, 1.2 and 2.3 of [7] and Theorem 3.3 of [6].

THEOREM 4.10. *Let* $\bar{\underline{\underline{V}}}$ *be any algebraic variety. Then the family of all* β*-varieties* $\underline{\underline{W}}$ *such that* $\bar{\underline{\underline{W}}} = \bar{\underline{\underline{V}}}$ *is a class and not a set.*

Proof. For each cardinal m strictly greater than c , let $\underline{\underline{W}}_m$ be the family of all $T(m)$-groups G such that $\bar{G} \in \bar{\underline{\underline{V}}}$. By Theorem 4.2 this is a variety and clearly $\underline{\underline{W}}_m \supseteq \underline{\underline{V}}(F)$ (in the notation of Lemma 4.9). Thus by Lemmas 4.8 and 4.9, $\underline{\underline{W}}_m$ is a β-variety. Further for n any cardinal strictly greater than m , it is clear that $\underline{\underline{W}}_m \neq \underline{\underline{W}}_n$. The proof is complete.

5. $\underline{\underline{V}}(R)$ and $\underline{\underline{V}}(T)$

In this section R and T will denote the additive group of reals and the circle group respectively each with its usual topology.

THEOREM 5.1. *Let* M *be a family of groups each having the property that it can be imbedded in a compact abelian group. Then every group in the variety generated by this family also has this property.*

Proof. It is obvious that this property is preserved by the

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operations of taking subgroups, quotient groups and cartesian products. From this the result immediately follows.

LEMMA 5.2. *If* G *is a locally compact Hausdorff subgroup of a compact group* H , *then* G *is compact.*

Proof. Let N be the closure of the identity in H . Then H/N is a compact Hausdorff group. If p is the natural homomorphism of H onto H/N , then $p(G)$ is topologically isomorphic with G . Therefore by Theorem 5.11 of [3], $p(G)$ (and therefore G) is compact.

LEMMA 5.3. *If* G *is a compact Hausdorff abelian group then it can be imbedded in a suitable cartesian product of copies of* T .

Proof. Just as Theorem 2.2.6 of [10] follows from Theorem 2.1.2 so too does this result.

THEOREM 5.4. *Let* G *be a locally compact Hausdorff abelian group. Then* G *is in the variety* $\underline{\underline{V}}(T)$ *generated by* T *if and only if* G *is compact.*

Proof. If G is in $\underline{\underline{V}}(T)$, then by Theorem 5.1 and Lemma 5.2, G is compact. Conversely if G is compact then by Lemma 5.3 G is in $\underline{\underline{V}}(T)$.

COROLLARY 5.5. *The variety* $\underline{\underline{V}}(R)$ *properly contains* $\underline{\underline{V}}(T)$.

THEOREM 5.6. *If* G *is a locally compact Hausdorff compactly generated abelian group, then* G *is in* $\underline{\underline{V}}(R)$.

Proof. This result follows from Theorem 9.8 of [3] and Theorem 5.4.

COROLLARY 5.7. *A locally compact Hausdorff compactly generated abelian group is a* $T(m)$*-group, for any* $m > c$.

Proof. This is an immediate consequence of Theorems 4.2 and 5.6.

THEOREM 5.8. *Let* G *be a connected Hausdorff locally compact non-compact abelian group. Then* $\underline{\underline{V}}(G) = \underline{\underline{V}}(R)$.

Proof. This follows from Theorem 9.14 of [3] and Theorem 5.4.

COROLLARY 5.9. *A connected Hausdorff locally compact abelian group is a* $T(m)$*-group for any* $m > c$.

Proof. This follows from Theorems 5.8, 5.4 and 4.2.

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LEMMA 5.10. *The abelian group* G *is a* $T(m)$*-group if and only if every open neighbourhood of the identity contains a subgroup of index strictly less than* m .

LEMMA 5.11. *The group* T *is not a* $T(c)$*-group.*

Proof. This is obvious from Lemma 5.10 and the fact that subgroups of T are finite or dense.

THEOREM 5.12. *The varieties* $\underline{\underline{V}}(T)$ *and* $\underline{\underline{V}}(R)$ *are* $T(m)$*-varieties if and only if* $m > c$.

6. Miscellaneous results

Theorem 6.1 generalizes Theorem 1.13 of [7].

THEOREM 6.1. *If* $\underline{\underline{V}}$ *is any non-indiscrete variety and* $F(X, \underline{\underline{V}})$ *is the free group of* $\underline{\underline{V}}$ *on any space* X *, then* $F(X, \underline{\underline{V}})$ *is disconnected.*

Proof. Let G be any non-indiscrete group in $\underline{\underline{V}}$. Then G has a proper open subset O . Let g be any element of G not in O . Then the subgroup H of G generated by g is a non-indiscrete countable group. Thus, by p. 32 of [1], H is a non-indiscrete countable completely regular space. Therefore H is disconnected.

Define the continuous mapping ϕ of X into H by $\phi(X) = g$. Then there exists a continuous homomorphism Φ of $F(X, \underline{\underline{V}})$ onto H such that $\Phi|X = \phi$. Consequently $F(X, \underline{\underline{V}})$ is disconnected.

COROLLARY 6.2. *Any non-indiscrete relatively free group is disconnected.*

Proof. This is a corollary of the proof of Theorem 6.1.

The author is indebted to Professor John L. Kelley for Theorem 6.3 and Corollary 6.4.

THEOREM 6.3. *Let* X *be any completely regular space and* Y *be the quotient space of* X *obtained by identifying any pair of points of* X *which are limit points of each other and giving* Y *the quotient topology. If* Y *can be imbedded in a Hausdorff group* G *, then* X *can be imbedded in the product group* $G \times K$ *, for* K *any sufficiently large indiscrete group.*

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Proof. Let f be the natural mapping of X into Y , p the projection of $G \times K$ onto K and ϕ the imbedding of Y into G . Consider the diagram:

$$\begin{tikzcd} X \arrow[r, "\theta"] \arrow[d, "f"'] & G\times K \arrow[d, "p"] \\ Y \arrow[r, "\phi"'] & G \end{tikzcd}$$

Clearly any $1-1$ map θ which makes the above diagram commute is an imbedding. Further it is obvious that if the cardinal of K is greater than the cardinal of G then such maps do exist and the result is proved.

COROLLARY 6.4. *Any completely regular space can be imbedded in a topological group.*

Proof. This is an immediate consequence of Theorem 7 Chapter IV of [4] and Theorem 6.3.

THEOREM 6.5. *Let* $\underline{\underline{V}}$ *be any variety for which* $F(X, \underline{\underline{V}})$ *exists for all Tychonoff spaces* X . *Then for any completely regular space* Y , $F(Y, \underline{\underline{V}})$ *exists. In particular this is the case when* $\underline{\underline{V}}$ *is a* β*-variety or more particularly a full variety.*

Proof. The existence of $F(Y, \underline{\underline{V}})$ can be deduced from Theorem 6.3 above and Lemma 2.7 and Theorem 2.6 of [6]. The last statement of the theorem is obvious.

REMARK 6.6. We point out that Theorem 6.5 is applicable to varieties other than β-varieties. (See Example 3.2.)

Notation. If G is a group and N the closure of the identity, then the quotient group G/N will be denoted by G^* .

THEOREM 6.7. *Let* $\underline{\underline{V}}$ *be a variety and* G *a group such that* $\bar{G} \in \bar{\underline{\underline{V}}}$. *If* $G^* \in \underline{\underline{V}}$ *then* $G \in \underline{\underline{V}}$.

Proof. Let H be the indiscrete group algebraically isomorphic to G . Since $\bar{G} \in \underline{\underline{V}}$, by Lemma 2.7 of [6], $H \in \underline{\underline{V}}$. Let i be the natural homomorphism of G onto H and p be the natural projection of G onto

G^*. Define the homomorphism ϕ of G into $G^* \times H$ by $\phi(g) = (p(g), i(g))$ for all g in G. Then ϕ is an imbedding. Since $G^* \times H$ is in $\underline{\underline{V}}$, G is in $\underline{\underline{V}}$.

THEOREM 6.8. *Let M be a family of groups and $N = \{M_i^* : M_i \in M\}$. Then any Hausdorff group in the variety generated by the family M is in the variety generated by the family N.*

Proof. It is sufficient to show that for each positive integer n and any group H obtainable from M by n operations (of taking cartesian products, subgroups or quotient groups) the group H^* is obtainable from N by n operations.

This proposition is proved by induction. Firstly consider $n = 1$. If H is a subgroup of $M_i \in M$ then clearly H^* is a subgroup of M_i^*. If $H = M_j/A$, $M_j \in M$ then, by Theorem 5.36 of [3], H^* is a quotient group of M_j^*. Finally if $H = \prod_{\alpha \in I} M_\alpha$, $M_\alpha \in M$ then $H^* = \prod_{\alpha \in I} M_\alpha^*$.

The remainder of the induction proof is obvious.

THEOREM 6.9. *Let G be a compact Hausdorff abelian torsion group. Then there exists a finite cyclic discrete group C such that $G \in \underline{\underline{V}}(C)$. Consequently G is an $S(\aleph_0)$-group.*

Proof. This is immediate from Theorem 25.9 of [3].

7. Topologies on free groups

In this section we will examine Theorem 6 of [2]. At the bottom of p. 741 of [2] the following appears: "Finally, any group topology on $F_0(X)$ relative to which X is embedded topologically is one for which all functions $\tilde{f}$ are continuous". This statement appeared not to be obviously true to us. In correspondence Professor B.R. Gelbaum has agreed that it is not obvious. In fact Example 7.1 shows that it is not true. Thus, in Gelbaum's notation, T_{AP} is not necessarily the infimum of F; that is the last statement in Theorem 6 of [2] is false.

EXAMPLE 7.1. Let $F_0(a)$ be the infinite cyclic group with generator $\{a\}$ which has as an open basis at the identity for its topology τ, the family of subgroups generated by a^{3n}, $n = 1, 2, \ldots$. Clearly τ is Hausdorff. Consider the map f of $\{a\}$ into the quaternions given by $f(a) = i$. There exists a homomorphism $\tilde{f}$ of $F_0(a)$ into the quaternions such that $\tilde{f}(a) = f(a) = i$. Then $\tilde{f}(F_0(a)) = \{i, -1, -i, 1\}$ and $\tilde{f}^{-1}\{1\}$ is the subgroup of $F_0(a)$ generated by a^4, which is not an open subset of τ. Therefore τ is not continuous.

The question might be asked: In every Hausdorff group topology on $F_0(X)$ is X a closed subset? This would have been a consequence of Theorem 6 of [2], but is in fact not so as Example 7.2 shows.

EXAMPLE 7.2. Let X be any infinite set and $F(x)$ the free abstract group on X. Let the family A of the normal closures of all subgroups of $F(X)$ which are generated by cofinite subsets of X be an open basis at the identity for a group topology τ on $F(X)$. Clearly τ is Hausdorff. Suppose X is closed. Then the complement $C(X)$ of X in $F(X)$ is an open neighbourhood of the identity. Then $C(X)$ contains an element of A. Thus $C(X)$ contains an element of X, which is a contradiction. Hence X is not closed in τ.

References

[1] R.E. Edwards, *Functional analysis* (Holt, Rinehart and Winston, New York, 1955).

[2] B.R. Gelbaum, "Free topological groups", *Proc. Amer. Math. Soc.* 12 (1961), 737-743.

[3] Edwin Hewitt and Kenneth A. Ross, *Abstract harmonic analysis* (Academic Press, New York, 1963).

[4] John L. Kelley, *General topology* (Van Nostrand, New York, 1955).

[5] A.A. Markov, "On free topological groups", *C.R. (Doklady) Acad. Sci. URSS, (N.S.)* 31 (1941), 299-301. *Bull. Acad. Sci. URSS. Sér. Math.* [*Izvestiya Akad. Nauk SSSR*] 9 (1945), 3-64. (Russian.

English summary). English transl., *Amer. Math. Soc. Translation no.* 30 (1950), 11-88; reprint *Amer. Math. Soc. Transl.* (1) 8 (1962), 195-273.

[6] Sidney A. Morris, "Varieties of topological groups", *Bull. Austral. Math. Soc.* 1 (1969), 145-160.

[7] Sidney A. Morris, "On varieties of topological groups", *Bull. Austral. Math. Soc.* 2 (1970), 1-13.

[8] Sidney A. Morris and H.B. Thompson, "Topologies on finite groups", *Bull. Austral. Math. Soc.* 1 (1969), 315-317.

[9] Hanna Neumann, *Varieties of groups* (Ergebnisse der Mathematik und ihrer Grenzgebiete, Band 37, Springer-Verlag, Berlin, Heidelberg, New York, 1967).

[10] Walter Rudin, *Fourier analysis on groups* (Interscience, New York, London, 1962).

The Flinders University of South Australia,
Bedford Park, South Australia,
and
University of Adelaide,
Adelaide, South Australia.