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STOCHASTIC MEASURE DIFFUSION PROCESSES

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1. **Introduction.** The purpose of this article is to give an introduction to the study of a class of stochastic partial differential equations and to give a brief review of some of the recent developments in this field. This study has evolved naturally out of the theory of stochastic differential equations initiated in a pioneering paper of K. Itô [13]. In order to set this review in its appropriate setting we begin by considering a simple scalar stochastic differential equation. Such an equation is written in differential form as

$$(1.1)\qquad \begin{aligned} dZ(t) &= a(t, Z(t))\,dt + \gamma(t, Z(t))\,db(t), \qquad t \geq 0,\\ Z(0) &= z_0 \end{aligned}$$

where $b(t)$, $t \geq 0$, refers to a standard Wiener process and appropriate regularity conditions are imposed on the coefficients $a(.,.)$ and $\gamma(.,.)$. Under the appropriate conditions the solution, $Z(t)$, is a stochastic diffusion process, that is, a Markov process with almost surely continuous sample paths. Such a process is specified by a family of probability measures $\{P_{z_0} : z_0 \in R\}$ defined on a canonical measure space $(\Omega, \mathbf{F})$ where Ω denotes the space of continuous functions from $[0, \infty)$ into R and $\mathbf{F}$ denotes the σ-algebra of Borel subsets (with respect to an appropriate topology on Ω). Then $Z(.,.)$ is described as the canonical mapping $Z : [0, \infty) \times \Omega \to R$ defined by $Z(t, \omega) \equiv \omega(t)$ for $t \geq 0$, $\omega \in \Omega$. The probability measure P_{z_0} describes the behavior of the system when the initial condition is $Z(0) = z_0$. This family of probability measures can be uniquely specified by a semigroup of contraction operators $\{T_t : t \geq 0\}$, defined on $C_0(R)$ the Banach space of bounded real-valued continuous functions vanishing at as furnished with the supremum norm. The semigroup is defined by

$$T_t f(z) \equiv E_z(f(Z(t))), \qquad f \in C_0(R),\ t \geq 0,\ z \in R,$$

and E_z denotes the expectation operator with respect to the probability measure P_z. In turn the semigroup is identified by means of its infinitesimal

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generator, L, which is defined by

$$Lf(z) \equiv \lim_{t \downarrow 0} [(T_t f(z) - f(z))/t] \quad \text{(when the limit belongs to } C_0(R)).$$

In general L is an unbounded operator with domain $D(L)$. Thus the diffusion process, $Z(t)$, can be specified by the semigroup $\{T_t : t \geq 0\}$, the infinitesimal generator $(L, D(L))$ or the stochastic differential equation (1.1). Returning to equation (1.1) it should be noted that it is not a differential equation in the usual sense since the Wiener process $\{b(t) : t \geq 0\}$ is with probability one nowhere differentiable. It was for this reason that K. Itô introduced his stochastic calculus which made possible the study of (1.1) as a well defined mathematical object. Stochastic differential equations have become an important tool in applied mathematics. One reason for this is that such equations arise naturally as approximations to many complex and mathematically untractable stochastic systems. Typical examples arise in queueing theory, transport theory [16] and genetics [10].

In recent years considerable interest has developed in the study of spatially distributed stochastic systems and it natural to extend the methods of stochastic calculus in this direction. Examples of spatially distributed systems arise in chemical kinetics, statistical physics and population biology. Translated into the language of stochastic calculus these problems involve stochastic partial differential equations. Of these an important family consists of the parabolic equations with non-negative solutions and it is this family that we discuss below. Such an equation can be written formally as

$$\partial u / \partial t = Gu + F(u) + W(u) \tag{1.2}$$

where $u(.,.) : [0, \infty) \times R^d \to R^+$ (the nonnegative reals), G is a linear elliptic operator, $F(u)$ is a nonlinear operator and $W(u)$ is state dependent fluctuation or noise term. Heuristically, $u(t, .)$ describes the distribution of the population in R^d at time t, G describes the spatial dispersion of the population, $F(u)$ describes the interaction effects such as competition and $W(.)$ describes fluctuations due to demographic or environmental stochasticity. However there are more serious obstacles to the rigorous mathematical development of (1.2) than in the case of (1.1). To obtain a mathematically well-posed version of (1.2) it is necessary to abandon the search for function-valued solutions and rather look for generalized function or measure-valued solutions. Secondly rather than try to give a rigorous sense to the expression $W(u)$, we must reformulate (1.2) as a weak equation known as a martingale problem. This approach which is described in the next section is based on the generalization of an extremely useful reformulation of the stochastic differential equation (1.1) due to Stroock and Varadhan [17].

2. **Measure-valued martingale problems.** To simplify the exposition of this section we assume that the system is confined to a compact subset, D, of R^d. Then the state space for the process $\{X(t): t \geq 0\}$ to be constructed is $M(D)$, the space of nonegative Borel measures on D. Let $C(D)$ denote the space of continuous functions on D. For $\mu \in M(D)$, $f \in C(D)$, let $\langle \mu, f \rangle$ denote the integral $\int f(x)\mu(dx)$. Let $C_b(M(D))$ denote the Banach space of bounded continuous functions on $M(D)$ furnished with the supremum norm.

Let Ω denote the space of continuous functions from $[0, \infty)$ into $M(D)$ furnished with the topology of uniform convergence on bounded intervals. Let $\mathbf{F}$ denote the σ-algebra of Borel subsets of Ω and let $X(.,.):[0,\infty)\times\Omega \to M(D)$ be defined by

$$X(t, \omega) \equiv \omega(t) \quad \text{for} \quad t \in [0, \infty) \quad \text{and} \quad \omega \in \Omega.$$

Finally, let $\Pi(\Omega)$ denote the family of probability measures on Ω.

A martingale problem is specified by a pair $(L, D(L))$ where L is an unbounded linear operator defined on a linear subspace, $D(L)$, of $C_b(M(D))$. A solution to the martingale problem associated with the pair $(L, D(L))$ is a mapping $\mu \to P_\mu$ from $M(D)$ to $\Pi(\Omega)$ such that

(2.1a) $$P_\mu(X(0) = \mu) = 1,$$

and

(2.1b) for every $\Psi \in D(L)$ and $\mu \in M(D)$,

$$Y_\Psi(t) \equiv \Psi(X(t)) - \int_0^t L\Psi(X(s))\, ds$$

is a P_μ-martingale with respect to the increasing family of σ-algebras $\mathbf{F}_t \equiv \sigma\{X(s, B): 0 \leq s \leq t,\ B \text{ Borel in } D\}$. The martingale condition means that if $t > s$, then

$$E_\mu(Y_\Psi(t) \mid \mathbf{F}_s) = Y_\Psi(s), \qquad P_\mu\text{-a.s.}$$

where $E_\mu(. \mid \mathbf{F}_s)$ denotes conditional expectation with respect to the σ-algebra $\mathbf{F}_s$.

The significance of the martingale problem is that if it has a *unique* solution satisfying a mild regularity condition, then the family of measures $\{P_\mu : \mu \in M(D)\}$ is induced by a Markov process and hence can be identified with a semigroup of contraction operators $\{T_t: t \geq 0\}$ on $C_b(M(D))$. Furthermore, for $\Psi \in C_b(M(D))$,

$$T_t\Psi(\mu) = E_\mu(\Psi(X(t)).$$

The Markov property implies that for $0 < s < t$, and any Borel subset B of $M(D)$,

$$P_\mu(X(t) \in B \mid \mathbf{F}_s) = P_{X(s)}(X(t-s) \in B), \quad P_\mu\text{-a.s.}$$

where $P_\mu(.\,|\,\mathbf{F}_s)$ denotes the conditional probability with respect to the σ-algebra $\mathbf{F}_s$. However note that for fixed Borel subset B of D, the real-valued process $X(t, B)$ may not be a Markov process. finally, the infinitesimal generator of the semigroup $\{T_t : t \geq 0\}$ is given by the closure of the operator L.

Thus to reformulate equation (1.2) as a martingale problem the operator L must be specified. The appropriate choice is

$$L\Psi(\mu) = \langle \Psi'(\mu), G^*\mu + F(\mu)\rangle + \tfrac{1}{2}\langle Q(\mu), \Psi''(\mu)\rangle \tag{2.2}$$

where G^* denotes the adjoint of the infinitesimal operator of a diffusion semigroup on $C(D)$, $F : M(D) \to M(D)$ is a continuous mapping, Ψ', Ψ'' denote the first and second Fréchet derivatives, and Q is a continuous mapping from $M(D)$ into $M^*(D \times D)$. $M^*(D \times D)$ denotes the collection of symmetric signed measures on $D \times D$ which are positive definite, that is, for every $f \in C(D)$,

$$\iint f(x)f(y)Q(\mu : dx \times dy) \geq 0.$$

Furthermore, we assume that if $\mu(B) = 0$, then $Q(\mu : B \times B) = 0$. The martingale problem is then specified by the (2.1), (2.2) and the triple (G, F, Q). A unique solution to the martingale problem is called a *stochastic measure diffusion process* with spatial diffusion generator G, interaction F and fluctuation quadratic form Q. Heuristically, $Q(\mu : dx \times dy)$ represents the infinitesimal variance-covariance structure of the population at locations x and y when the current population distribution is given by μ.

Thus the problem is now well-posed, namely to find conditions under which the martingale problem associated with the triple (G, F, Q) has a unique solution and then to study the structure and behavior of this solution. In the next section we describe a martingale problem of this type for which an explicit solution can be obtained.

3. **The branching measure diffusion process.** The model to be described in this section arises as a diffusion approximation to a stochastic population model of a population subject to reproduction and spatial dispersion. We assume that D is a Green domain, for example, a sphere, and that G is the d-dimensional Laplacian, Δ, with absorbing boundary conditions on ∂D. We assume that $F(.) = c$, a constant and

$$Q(\mu ; dx \times dy) = \gamma\delta_{x-y}\mu\ (dx) \tag{3.1}$$

where $\gamma > 0$ and δ denotes the Dirac delta function.

This martingale problem has been solved explicitly (see for example [5], [7]) in terms of its characteristic functional, that is, the Fourier transform of the finite dimensional distributions of the probability measures P_μ. The transition characteristic functional of the measure-valued process $X(t)$, $t \geq 0$, is defined

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by

$$L_{t,\mu}(\phi) \equiv E_\mu(\exp(i\langle X(t), \phi\rangle)) \tag{3.2}$$

for $\phi \in C(D)$ and $\mu \in M(D)$. The following theorem is established in [7].

THEOREM 3.1. *The martingale problem associated with the triple* (Δ, c, Q) *where* Q *is given by* (3.1) *has a unique solution with transition characteristic functional given by*

$$L_{t,\mu}(\phi) = \exp(i \int u(t, x)\mu\,(dx)), \qquad \phi \in C(D), \tag{3.3}$$

where $u(.,.)$ *is the solution of the nonlinear initial value problem*

$$\partial u(t, x)/\partial t = \Delta u(t, x) + i\gamma u^2(t, x) + cu(t, x), \qquad t > 0, \tag{3.4}$$

$x \in D - \partial D$, *and*

$$u(0, x) = \phi(x)$$

$$u(t, x) = 0, t > 0, \qquad x \in \partial D.$$

To indicate the significance of this process we next indicate how it arises as the diffusion approximation to a branching random field. Let $Y_\varepsilon(t)$, $t \geq 0$, denote the measure-valued Markov process which lives on D and has the following evolution. At time $t = 0$, $Y_\varepsilon(0)$ is given by a Poisson random field on D with intensity λ_ε. Each particle has a random lifetime which is exponentially distributed with mean V_ε^{-1}. At the end of its lifetime each particle branches into n identical particles with probability p_n, $n = 0, 1, 2, 3, \ldots$ We assume that the mean offspring size $m_\varepsilon = \sum np_n$, and that $m_{2,\varepsilon} = \sum n(n-1)p_n < \infty$. In addition, during its lifetime each particle moves in D according to a Brownian motion and is killed if it reaches the boundary ∂D. For $\varepsilon > 0$, let $\lambda_\varepsilon = \lambda/\varepsilon$, $m_\varepsilon = c\varepsilon$, $V_\varepsilon = V/\varepsilon$ and $m_{2,\varepsilon} \to m_2$ as $\varepsilon \to 0$. Under these conditions, the following theorem is established in [4], [6].

THEOREM 3.2. *The branching random fields,* $Y_\varepsilon(t)$, $t \geq 0$, *converge in the sense of weak convergence of probability measures as* $\varepsilon \to 0$ *to the branching measure diffusion process* $X(t)$, $t \geq 0$, *that is, the solution of the martingale problem* (3.1) *with* $\gamma = \frac{1}{2}Vm_2$.

Recall that for each $t \geq 0$, the branching measure diffusion process is described by a random measure on D. The following result which is proved in [9] provides insight into the structure of these random measures.

THEOREM 3.3. *Let* $X(t)$, $t \geq 0$, *denote the branching measure diffusion process in* $D \subset R^d$, $d \geq 2$. *Then for fixed* $t > 0$, *there exists a random set* $B \subset D$, *such that*

$$X(t, \omega, D \cap B(\omega)) = X(t, \omega, D)$$

for P_μ-almost every $\omega \in \Omega$, and

$$\dim B(\omega) \leq 2,$$

for P_μ-almost every $\omega \in \Omega$. (The dimension refers to Hausdorff dimension.)

A simple corollary to Theorem 3.3 is that for $d \geq 3$, and $t > 0$, the random measure $X(t)$ is singular-measure-valued with probability one. This explains why equation (1.2) has no function-valued solution in $d \geq 3$. In fact as written, it is essentially the equation for the "density" of $X(t)$ which of course does not exist if $X(t)$ is singular.

4. **Some further results on the martingale problem.** In this section we briefly review some recent results on martingale problems of the type (2.1). Although general existence and uniqueness results for the most general martingale problem have not yet been obtained, considerable progress has been made on identifying and solving some important classes of martingale problems.

Although we have described a martingale problem above which does not have a function-valued solution, there is an important class of problems which do have function-valued solutions. These have been studied by M. Viot [18] and are examples of measure-valued martingale problems which have absolutely continuous measure-valued solutions whose densities belong to an appropriate Hilbert space of functions. Let λ denote Lebesgue measure and for an absolutely continuous measure, μ, let $\tilde{\mu}$ denote its Radon–Nikodym derivative with respect to λ. The martingale problems studied by Viot are given by triples (G, F, Q) where there are fairly weak hypothesis placed on G and F but a strong hypothesis is placed on Q. $Q(.)$ is defined for an absolutely continuous measure μ, by

$$Q(\mu; dx \times dy) = h(\tilde{\mu}(x))h(\tilde{\mu}(y))q(x, y)\lambda\,(dx)\lambda\,(dy)$$

where $h(.)$ is a continuous function with $h(x) \geq 0$ if $x \geq 0$ and $h(0) = 0$. Furthermore the essential assumption is that $q(x, y)$ is a positive definite function which is the kernel of a nuclear operator on the Hilbert space $L^2(D)$. Note that this implicitly requires that the infinitesimal fluctuations be spatially correlated in higher dimensions thus excluding those $Q(.)$'s which are concentrated on the diagonal in $D \times D$.

P. L. Chow [2] and S. Mizuno [15] have obtained existence results for the martingale problem with $G = \Delta$, $F = 0$, and $Q(\mu : dx \times dy) = q(x, y)\mu\,(dx)\mu\,(dy)$ where $q(x, y)$ is a symmetric positive definite function on $D \times D$. This represents a model of a population growing in a random environment and can also be viewed as a random linear evolution equation.

Fleming and Viot [11] have introduced another martingale problem whose solution is a model for the evolution of a continuous genetic characteristic. For this model the triple is given by $G = \Delta$ with reflecting boundary conditions on

∂D, $F=0$ and

$$Q(\mu:dx\times dy)=\delta_{x-y}\mu\,(dx)-\mu\,(\mathrm{dx})\mu\,(dy).$$

In this case the process is measure-preserving and the state space is $M_1(D)$, the space of probability measures on D. They prove existence by obtaining a solution as a diffusion limit of a sequence of discrete models. They prove uniqueness by showing that the conditional moments are uniquely determined by the martingale problem.

The martingale problem of Section 3 has the property that the measure $Q(\mu:.)$ is concentrated on the diagonal in $D\times D$. This corresponds to the fact that the infinitesimal fluctuations in disjoint regions are independent conditioned on the current distribution. A. Bose [1] has studied the family of martingale problems with $G=F=0$ and in which $Q(.)$ possesses this property. In this case the solution $X(t)$ is for every $t>0$, an atomic random measure even if $X(0)$ is non-atomic. In this case each atom evolves according to a one dimensional diffusion and Bose establishes a connection between the behavior of the measure-valued process and the one dimensional diffusion by exploiting some deep results of Feller and McKean concerning the latter.

Up to this point the martingale problems which were discussed were interaction free, that is, $F(.)$ was either linear or zero. This limitation can be removed by the extension to measure-valued processes of the formula of Cameron–Martin–Girsanov for the function space Radon–Nikodym derivative of one probability measure with respect to another. To describe this method consider a martingale problem given by the triple (G, F, Q) and compare it with the martingale problem given by the triple $(G, 0, Q)$. Let $\{P_\mu:\mu\in M(D)\}$ and $\{P^F_\mu:\mu\in M(D)\}$ denote the solutions of the two martingale problems. Then the following result is established in [7].

Theorem 4.1. *Assume that*

$$F(\mu;dx)=\int f(\mu;y)Q(\mu:dx\times dy).$$

Then for every $\mu\in M(D)$ and $t<\infty$, the probability measure $P^F_{\mu\mathbf{F}_t}$ is absolutely continuous with respect to the probability measure $P_{u\mathbf{F}t}$ and the Radon–Nikodym $dP^F_{\mu\,\mathbf{F}_t}/dP_{\mu\,\mathbf{F}_t}\equiv R(t)$ is given by

$$R(t)=\exp\biggl(\int_0^t\langle f(X(s)),dX_G(s)\rangle-\frac{1}{2}\int_0^t\int\int f(X(s);x)f(X(s);y)Q(X(s):dx\times y)\,ds)$$

where

$$X_G(t)=X(t)-\int_0^t G^*X(s)\,ds,$$

and

$$\int_0^t \langle f(X(s)), dX_G(s)\rangle$$

denotes an appropriate Itô stochastic integral which is analogous to the stochastic integral introduced by Itô for the finite dimensional case.

Applications of this formula have been made to the branching measure diffusion process and the Fleming–Viot model (refer to [7]).

The family of quadratic functionals which have been discussed above together with the Cameron–Martin–Girsanov formula makes possible the study of many distributed population models which arise in chemical kinetics, genetics and ecology.

5. **Global behavior of stochastic measure diffusion processes.** In addition to questions of existence and uniqueness, the major open problems lie in the study of the global behavior of solutions of measure-valued martingale problems. Of greatest importance are questions of stability, singularity, critical behavior and bifurcation of solutions. These may form the basis for the study of nonequilibrium phase transitions arising in chemical kinetics and population biology. Some first steps in this direction have only recently been realized (see for example [2], and [8] and the references therein). The detailed description of these problems and results are beyond the scope of this review. However in order to give an introduction to this viewpoint we now describe one aspect of global behavior which can be described in the context of the branching measure diffusion process.

The first thing to note is that the branching measure diffusion can be extended to all of R^d. In this setting assume that the system is spatially homogeneous, that is, consider the case in which $X(0)=\lambda$, a constant multiple of Lebesgue measure. We also assume that $c=0$ so that $E_\lambda(X(t,B))$ is finite and constant in t for every relatively compact subset B of R^d. The process is then a fluctuation dissipation process with fluctuations produced locally and then dissipated via spatial diffusion. The question of interest is whether the fluctuations grow indefinitely or reach a steady state behavior. This question is answered by the following theorem which is proved in [5].

THEOREM 5.1. *Let $\{X(t), t\geq 0\}$, denote the branching measure diffusion process in R^d with the initial condition $X(0)=\lambda$.*

(a) *For $d>2$, $X(t)$ converges (in the topology of weak convergence of probability measures) to a steady state random measure, denoted by X_∞.*

(b) *For $d\leq 2$, and K a compact subset of R^d, $X(t,K)\to 0$ in probability as $t\to\infty$.*

The study of the structure of the steady state X_∞ is also of interest. The *covariance operator*

$$\Gamma(\psi, \phi) \equiv E(\langle X_\infty, \phi\rangle \cdot \langle X_\infty, \psi\rangle) - E(\langle X_\infty, \phi\rangle)E(\langle X_\infty, \psi\rangle).$$

It has been proved in [5] that

$$\Gamma(\psi, \phi) = \int [\phi(x)\psi(y)/|x-y|^{d-2}]\, dx\, dy.$$

This means that the random measure has long-tailed correlations, a case not traditionally studied in probability theory but one which also arises in the study of critical point behavior in statistical physics. In particular the usual type of central limit theorem argument which depends on some type of "asymptotic independence" fails in this case. Nevertheless both central limit theorems and invariance principles have recently been proved in this setting ([5], [8] and Holley and Stroock [12]). To describe one of these results we introduce the following group of scaling transformations: for $K>0$,

$$X^K(t, A) \equiv K^{-(d+2)/2}(X(K^2t, A_K) - E(X(K^2t, A_K)))$$

where $A_K \equiv \{x : x/K \in A\}$.

THEOREM 5.2 ([12], [8]). *Let $X(t)$, $t \geq 0$, denote the branching measure diffusion with initial condition $X(0) = a$ multiple of Lebesgue measure in R^d, $d>2$, and assume that $c=0$. Then as $K \to \infty$, $X^K(.,.)$ converges weakly to a Gaussian stochastic process known as the infinite dimensional Ornstein–Uhlenbeck process, $Y(t)$, $t \geq 0$.*

The Ornstein–Uhlenbeck process is of course not a measure-valued process (because of the centering) but is a process whose state space is $\mathbf{S}'(R^d)$, the space of tempered distributions in R^d. The Ornstein–Uhlenbeck process is also the solution of a martingale problem, namely, the martingale problem associated with the triple $(\Delta, 0, Q)$ where $Q(\mu; dx \times dy) = a\delta_{x-y}\, dx$ where a is a positive constant. It should also be pointed out that weak convergence results of this type are most easily proved within the martingale problem context because of the stability of the martingale property under weak limits (see [17] and [15]). The previous result is typical of those involving weak convergence to fixed points of the group of scaling transformations.

REFERENCES

1. A. Bose, *Brownian measure processes.* Ph.D. Thesis, Carleton University, 1977.

2. P. L. Chow, *Stochastic partial equations in turbulence related problems.* In Probabilistic Analysis and Related Topics, Vol. 1, Academic Press, 1978.

3. D. A. Dawson, *Generalized stochastic integrals and equations.* Trans. Amer. Math. Soc. **147** (1970), 473–506.

4. ——, *Stochastic evolution equations and related measure processes*, J. Mult. Anal. **5** (1975), 1–52.

5. ——, *The critical measure diffusion.* Zeit. Wahr. verw Geb. **40** (1977), 125–145.

6. ——, *Limit theorems for interaction free geostochastic systems.* To appear in Seria Colloquia Mathematica Societatis Janos Bolyai.

7. ——, *Geostochastic calculus.* Can. J. Stat., **6** (1979), 143–168.

8. ——, *An infinite geostochastic system.* To appear in Multivariate Analysis **5,** North Holland.

9. D. A. Dawson and K. J. Hochberg, *The carrying dimension of a stochastic measure diffusion*, Ann. Prob., to appear.

10. W. Feller, *Diffusion processes in genetics.* In Proc. Second Berkeley Symp., Univ. of Calif. Press, 1951, 227–246.

11. W. Fleming and M. Viot, *Some measure-valued population processes.* Proc. Intl. Conf. Stoch. Anal., Northwestern Univ., Academic Press, 1978.

12. R. A. Holley and D. W. Stroock, *Generalized Ornstein–Uhlenbeck processes and infinite particle branching Brownian motion*, to appear.

13. K. Itô, *On stochastic differential equations.* Mem. Amer. Math. Soc. No. **4,** 1951.

14. H. P. McKean, *Stochastic integrals.* Academic Press, 1969.

15. S. Mizuno, *On some infinite dimensional martingale problems and related stochastic evolution equations.* Ph.D. thesis, Carleton University, 1978.

16. G. C. Papanicolaou, *Asymptotic analysis of transport processes.* Bull. Amer. Math. Soc. **81** (1975), 330–392.

17. D. W. Stroock and S. R. S. Varadhan, *Diffusion processes with continuous coefficients*, **I, II**. Comm. Pure Appl. Math. **22** (1969), 345–400, 479–530.

18. M. Viot, *Méthodes de compacité et de monotonie compacité pour les équations aux dérivées partielles stochastiques.* Thèse, Univ. de Paris, 1975.

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