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Duality for hypergeometric functions and invariant Gauss-Manin systems

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Abstract. We present some basic identities for hypergeometric functions associated with the integrals of Euler type. We give a geometrical proof for formulae such as the identity between the single and double integrals expressing Appell's hypergeometric series $F_1(a,b,b',c;x,y)$.

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0. Introduction

It is well-known that the Appell hypergeometric series $F_1(a,b_1,b_2,c;z_1,z_2)$ admits two integral representations of Euler type, one of which is a single integral and the other a double integral

$$
\begin{aligned}
&F_1(a,b_1,b_2,c;z_1,z_2)\\
&= \sum_{m_1,m_2=0}^{\infty} \frac{(a;m_1+m_2)(b_1;m_1)(b_2;m_2)}{(c;m_1+m_2)(1;m_1)(1;m_2)} z_1^{m_1} z_2^{m_2}\\
&= C_1(a,c)\int_0^1 s^a(1-s)^{c-a}(1-z_1s)^{-b_1}(1-z_2s)^{-b_2}\frac{\mathrm{d}s}{s(1-s)} =\\
&= C_2(b,c)\iint_{\substack{s_1,s_2>0\\ 1-s_1-s_2>0}} (1-z_1s_1-z_2s_2)^{-a}\\
&\qquad \times {s_1}^{b_1}{s_2}^{b_2}(1-s_1-s_2)^{c-b_1-b_2}\frac{\mathrm{d}s_1\wedge \mathrm{d}s_2}{s_1s_2(1-s_1-s_2)},
\end{aligned}
$$

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where $|z_1| < 1, |z_2| < 1, (\alpha, m) = \alpha(\alpha+1)\cdots(\alpha+m-1)$, and

$$C_1(a,c) = \frac{\Gamma(c)}{\Gamma(a)\Gamma(c-a)}, \qquad C_2(b,c) = \frac{\Gamma(c)}{\Gamma(b_1)\Gamma(b_2)\Gamma(c-b_1-b_2)}.$$

In order to look into the feature of this identity between these two integrals, we express the identity as

$$\int_{\Delta_{01}(x)} U^{\alpha}(x)\varphi_{01}(x) = C_{01,01}\int_{\Delta_{234}(y)} U^{-\alpha}(y)\varphi_{234}(y),$$

where

$$\alpha = (\alpha_0, \ldots, \alpha_4) = (a, c-a, -b_1, -b_2, b_1+b_2-c),$$

$$x = \begin{pmatrix} 1 & -1 & -z_1 & -z_2 & 0 \\ 0 & 1 & 1 & 1 & 1 \end{pmatrix}, \qquad y = \begin{pmatrix} 1 & 1 & 0 & 0 & 1 \\ -z_1 & 0 & 1 & 0 & -1 \\ -z_2 & 0 & 0 & 1 & -1 \end{pmatrix},$$

$$(L_0(x), \ldots, L_4(x)) = (s,1)x, \qquad (L_0(y), \ldots, L_4(y)) = (1, s_1, s_2)y,$$

$$U^{\alpha}(x) = \prod_{j=0}^{4} L_j(x)^{\alpha_j}, \qquad U^{-\alpha}(y) = \prod_{j=0}^{4} L_j(y)^{-\alpha_j},$$

$$\varphi_{01}(x) = \mathrm{d}\log\frac{L_1(x)}{L_0(x)}, \qquad \varphi_{234}(y) = \mathrm{d}\log\frac{L_3(y)}{L_2(y)} \wedge \mathrm{d}\log\frac{L_4(y)}{L_3(y)},$$

and, furthermore, $\Delta_{01}(x)$ is the 1-dimensional simplex bounded by $L_0(x) = 0$ and $L_1(x) = 0$; $\Delta_{234}(y)$ is the 2-dimensional simplex bounded by $L_2(y) = 0, L_3(y) = 0$ and $L_4(y) = 0$; and finally $C_{01,01}$ is a constant expressed in terms of α_j. We observe that

(i) there exists a regular diagonal 5×5 matrix H such that $x\ H\ {}^t y = 0$, in other words, the configuration $[y]$ of the 5 hyperplanes $L_j(y) = 0$ in the 2-dimensional projective space $\mathbb{P}^2$, which is the equivalence class of the set of ordered 5 hyperplanes $L_j(y) = 0$ in $\mathbb{P}^2$ modulo the the projective transformations, is dual to the configuration $[x]$ of the 5 points $L_j(x) = 0$ in $\mathbb{P}^1$,

(ii) for each j, the exponent of $L_j(x)$ in $U^{\alpha}(x)$ and $L_j(y)$ in $U^{-\alpha}(y)$ differ only by the sign,

(iii) the multi-index of 1-form $\varphi_{01}(x)$ (resp. 1-cycle $\Delta_{01}(x)$) and that of $\varphi_{234}(y)$ (resp. $\Delta_{234}(y)$) are complementary.

By the above observation, we can expect that the identity

$$\int_{\Delta_J(x)} U^{\alpha}(x)\varphi_I(x) = C_{IJ} \int_{\Delta_{J^\perp}(y)} U^{-\alpha}(y)\varphi_{I^\perp}(y) \tag{0.1}$$

holds for any multi-indices $I = \{i_0, i_1\}$ and $J = \{j_0, j_1\}$ (put $I^\perp = \{i_2, i_3, i_4\} = \{0, \ldots, 4\} \setminus I$ and $J^\perp = \{j_2, j_3, j_4\} = \{0, \ldots, 4\} \setminus J$), where

$$\varphi_I(x) = \mathrm{d}\log\frac{L_{i_1}(x)}{L_{i_0}(x)}, \qquad \varphi_{I^\perp}(y) = \mathrm{d}\log\frac{L_{i_3}(y)}{L_{i_2}(y)} \wedge \mathrm{d}\log\frac{L_{i_4}(y)}{L_{i_3}(y)},$$

and $\Delta_J(x)$ is the 1-dimensional simplex bounded by $L_{j_0}(x) = 0$ and $L_{j_1}(x) = 0$, and $\Delta_{J^\perp}(y)$ is the 2-dimensional simplex bounded by $L_{j_2}(y) = 0$, $L_{j_3}(y) = 0$ and $L_{j_4}(y) = 0$. Note that we need to assign a suitable branch of $U^{\alpha}(x)$ on $\Delta_J(x)$, and that of $U^{-\alpha}(y)$ on $\Delta_{J^\perp}(y)$ in order to state (0.1) precisely. For the case $J = \{0, 1\}$, there are the standard assignment of branch of $U^{\alpha}(x)$ on $\Delta_J(x)$ and that of $U^{-\alpha}(y)$ on $\Delta_{J^\perp}(y)$, for $|z_1| < 1$ and $|z_2| < 1$. These yield the identities between the hypergeometric series and the integrals. For a general multi-index J, we have neither standard assignments of branches nor expressions by series for the integrals. To show (0.1), we must find systematical assignments of branches and determine the constant C_{IJ} depending on the assignments of branches.

More generally, it is shown in [GGr1] and [Kit1] that the hypergeometric series of $k \times l$ variables with parameters $(a_1, \ldots, a_k, b_1, \ldots, b_l, c)$ admits k-fold and l-fold integrals both of Euler type. We can see that the feature of the identity between these integrals is similar to (i) $\sim$ (iii) as follows. Put

$$n = k + l,$$

$$(\alpha_0, \ldots, \alpha_{n+1}) = \left(a_1, \ldots, a_k, c - \sum_{j=1}^{k} a_j, -b_1, \ldots, -b_l, -c + \sum_{j=1}^{l} b_j\right),$$

and define a $(k+1) \times (n+2)$-matrix x and $(l+1) \times (n+2)$-matrix y from the linear forms $L_j(x)$ and $L_j(y)$ in the k-fold and the l-fold integrals, respectively. Then the configuration $[y]$ of $L_j(y) = 0$ in $\mathbb{P}^l$ is dual to the configuration $[x]$ of $L_j(x) = 0$ in $\mathbb{P}^k$, i.e., there exists a regular diagonal $(n+2) \times (n+2)$ matrix H such that $x\ H\ {}^t y = 0$, and the identity is expressed as (0.1) for $I = J = \{0, 1, \ldots, k\}$ and $I^\perp = J^\perp = \{k+1, \ldots, n, n+1\}$.

In this paper, we show the identity (0.1) for general multi-indices I and J of cardinality $k + 1$. Since the correspondence of the variables in (0.1) is the duality of the configurations as we saw, it is convenient to define functions of the configuration $[x]$ of hyperplanes in $\mathbb{P}^k$ with parameter $\alpha = (\alpha_0, \ldots, \alpha_{n+1}) \in (\mathbb{C} \setminus \mathbb{Z})^{n+2}$ satisfying $\sum_{j=0}^{n+1} \alpha_j = 0$ by modifying the left-hand side in (0.1), where $x = (x_{ij})_{0 \leqslant i \leqslant k, 0 \leqslant j \leqslant n+1}$ is a $(k+1) \times (n+2)$ complex matrix such that no

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$(k+1)$-minor vanishes. We prepare two kinds of such functions $F^+_{IJ}(\alpha,[x])$ and $F^-_{IJ}(\alpha,[x])$ by assigning combinatorially two branches $U^\alpha_{\Delta^+_J}$ and $U^\alpha_{\Delta^-_J}$ of U^α on $\Delta_J(x)$. Our main theorem stated strictly is the identity

$$F^+_{I_0J_0}(\alpha,[x]) = c(I_0,J_0)F^-_{I_0^\perp J_0^\perp}(-\alpha,[x]^\perp),$$

where $[x]^\perp$ is the dual configuration of $[x]$, i.e., the configuration $[x]^\perp$ is represented by an $(l+1)\times(n+2)$ matrix y of rank $(l+1)$ such that $x\ H\ {}^t y = 0$, and

$$I_0 = \{0,i_1,\ldots,i_k\}, J_0 = \{0,j_1,\ldots,j_k\}, \quad i_k, j_k \leqslant n;$$

$I_0^\perp$ and $J_0^\perp$ are the complements of I_0 and J_0, respectively. The constant $c(I_0,J_0)$ is expressed combinatorially in terms of α_j. For $I_0 = J_0 = \{0,\ldots,k\}$, this identity reduces to the identity obtained from the hypergeometric series, which will be seen in section 5.2.

We construct the $\binom{n}{k}\times\binom{n}{k}$ matrices $\Pi^\pm_0(\alpha,[x])$ (resp. $\Pi^\pm_{n+1}(\alpha,[x])$) by arranging the functions $F^\pm_{IJ}(\alpha,[x])$ lexicographically for the set of multi-indices I and J satisfying $i_0 = j_0 = 0$ and $i_k, j_k \leqslant n$ (resp. $1 \leqslant i_0, j_0$ and $i_k = j_k = n+1$). We call them *the hypergeometric period matrices of type* (k,n). We present our main theorem as the identity between $\Pi^+_0(\alpha,[x])$ and $\Pi^-_{n+1}(-\alpha,[x]^\perp)$.

In our proof of the main theorem, – it is essential to consider the hypergeometric period matrices – there are three keys: the wedge formulae for hypergeometric period matrices studied in [Ter] and [Var], twisted Riemann's period relations presented in [CM], and the invariant Gauss–Manin system on the configuration space, essentially obtained in [Aom], or [AK, Ch 3.8]. Our proof enables us to present constant $c(I_0,J_0)$ in terms of geometrical quantities, which are both intersection numbers of forms and those of cycles.

1. The hypergeometric period matrices

1.1. Let $M = M(k+1,n+2)$ be the set of $(k+1)\times(n+2)$ complex matrices such that no $(k+1)$-minor vanishes; for an element $x = (x_{ij})_{0\leqslant i\leqslant k, 0\leqslant j\leqslant n+1} \in M(k+1,n+2)$, put

$$x\langle J\rangle = \det(x_{ij_\lambda})_{0\leqslant i,\lambda\leqslant k},$$

where $J = \{j_0,j_1,\ldots,j_k\}, 0 \leqslant j_0 < j_1 < \cdots < j_k \leqslant n+1$, denotes a multi-index. We define $\mathcal{M} = \mathcal{M}(k+1,n+2)$ as

$$\mathcal{M}(k+1,n+2) = (\mathbb{P}^k \times M(k+1,n+2))\Big\backslash \bigcup_{j=0}^{n+1}\{L_j = 0\},$$

$$L_j = L_j(t,x) = \sum_{i=0}^{k} t_i x_{ij},$$

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where $t = (t_0, t_1, \dots, t_k)$ is a homogeneous coordinate system of the complex projective space $\mathbb{P}^k$. Let μ be the projection from $\mathcal{M}$ to M; the triple $(\mathcal{M}, M, \mu)$ is a C^∞ fiber bundle. We denote the fiber $\mu^{-1}(x)$ over x by $T(x)$ and the inclusion map of $T(x)$ into $\mathcal{M}$ by $\tau_x\colon T(x) \to \mathcal{M}$. The space $T(x)$ is given by

$$T(x) = \mathbb{P}^k \Big\backslash \bigcup_{j=0}^{n+1} \{t \in \mathbb{P}^k \mid L_j(t,x) = 0\}.$$

We define the holomorphic 1-from ω^α on $\mathcal{M}$ by

$$\omega^\alpha = \omega^\alpha(t,x) = \sum_{j=0}^{n+1} \alpha_j \,\mathrm{d}\log L_j(t,x) - \frac{1}{\binom{n}{k}} \sum_J \alpha_J \,\mathrm{d}\log x\langle J\rangle,$$

where

$$\alpha = (\alpha_0, \alpha_1 \dots, \alpha_{n+1}), \quad \alpha_j \in \mathbb{C} \setminus \mathbb{Z}, \qquad \sum_{j=0}^{n+1} \alpha_j = 0, \tag{1.1.1}$$

$$\alpha_J = \alpha_{j_0} + \cdots + \alpha_{j_k},$$

and J runs over all subsets of $\{0, 1, \dots, n+1\}$ with cardinality $k+1$; note that $\omega^{-\alpha} = -\omega^\alpha$. Let $\mathcal{L}^\alpha$ be the kernel of the connection $\nabla^\alpha = d + \omega^\alpha \wedge$ and $\mathcal{L}^\alpha(x)$ the restriction of $\mathcal{L}^\alpha$ on $T(x)$; $\mathcal{L}^\alpha$ is a locally constant subsheaf of $\mathcal{O}_\mathcal{M}$ of rank 1. Since each local branch of the multi-valued function

$$U^\alpha = U^\alpha(t,x) = \prod_{j=0}^{n+1} L_j(t,x)^{\alpha_j} \Big/ D(x),$$

$$D(x) = \prod_J x\langle J\rangle^{\alpha_J/\binom{n}{k}},$$

on a simply connected open set $\mathcal{V}$ of $\mathcal{M}$ is a solution of $\nabla^{-\alpha} u = 0$, it is a section of $\mathcal{L}^{-\alpha}$ on $\mathcal{V}$ and its restriction on $T(x)$ is that of $\mathcal{L}^{-\alpha}(x)$ on $\mathcal{V} \cap T(x)$ for $x \in \mu(\mathcal{V})$.

1.2. For $0 \leqslant i \leqslant n+1$, put

$$\psi_i = \psi_i(t,x) = \mathrm{d}\log L_i(t,x) - \frac{1}{\binom{n}{k}} \sum_{J_i} \mathrm{d}\log x\langle J_i\rangle,$$

where J_i runs over the multi-indices of cardinality $k+1$ including the index i; note that

$$\omega^\alpha(t,x) = \sum_{i=0}^{n+1} \alpha_i \psi_i(t,x).$$

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We define holomorphic k-forms $\varphi_I = \varphi_I(t,x)$ on $\mathcal{M}$ by

$$\varphi_I(t,x) = (\psi_{i_0} - \psi_{i_1}) \wedge \cdots \wedge (\psi_{i_{k-1}} - \psi_{i_k}),$$

where $I = \{i_0, i_1, \ldots, i_k\}, 0 \leqslant i_0 < i_1 < \cdots < i_k \leqslant n+1$. Let Φ_k be the $\mathbb{C}$-vector space spanned by the φ_I's, where we regard Φ_0 as $\mathbb{C}$. We can easily show that the quotient space $\Phi_k/(\omega^\alpha \wedge \Phi_{k-1})$ is $\binom{n}{k}$-dimensional and that the equivalence classes of φ_{I_0}'s and those of $\varphi_{I_{n+1}}$'s form different bases of the space, where I_0's and I_{n+1}'s are multi-indices of the following type

$$I_0 = \{0, i_1, \ldots, i_k\}, \qquad I_{n+1} = \{i_1, \ldots, i_k, n+1\},$$

$$1 \leqslant i_1 < \cdots < i_k \leqslant n.$$

For a fixed $x \in M$, it is known that the twisted cohomology groups with coefficients in $\mathcal{L}^\alpha(x)$ survive only at the kth degree and that $H^k(T(x), \mathcal{L}^\alpha(x))$ is canonically isomorphic to the pull back of $\Phi_k/(\omega^\alpha \wedge \Phi_{k-1})$ by $\tau_x\colon T(x) \to \mathcal{M}$; especially, its rank is $\binom{n}{k}$; refer to [AK] and [KN]. Note that the pull-back $\tau_x^*(\varphi_I)$ of φ_I by τ_x is given by

$$\tau_x^*(\varphi_I) = \mathrm{d}_t \log \frac{L_{i_0}(t,x)}{L_{i_1}(t,x)} \wedge \cdots \wedge \mathrm{d}_t \log \frac{L_{i_{k-1}}(t,x)}{L_{i_k}(t,x)}. \tag{1.2.1}$$

1.3. Since the direct image sheaf $\mu_*(\mathcal{L}^{-\alpha})$ of $\mathcal{L}^{-\alpha}$ by the smooth map μ is locally constant, the sheaf $\mathcal{H}_p(M, \mu_*(\mathcal{L}^{-\alpha}))$ over M associated to the presheaf $V \mapsto H_p(V, \mu_*(\mathcal{L}^{-\alpha}))$ whose stalk on x is the pth twisted homology group $H_p(T(x), \mathcal{L}^{-\alpha}(x))$ with coefficients in $\mathcal{L}^{-\alpha}(x)$, is also locally constant. For any $x \in M$, it is known that the twisted homology groups with coefficients in $\mathcal{L}^{-\alpha}(x)$ survive only at the kth degree and that the rank of $H_k(T(x), \mathcal{L}^{-\alpha}(x))$ is $\binom{n}{k}$; see [IK1], [IK2] and [KN]. Let ξ be a fixed element of M given by real numbers $0 \leqslant \zeta_0 < \zeta_1 < \cdots < \zeta_n$ as

$$\xi = \xi_k = \begin{pmatrix} 1 & 1 & \ldots & 1 & 0 \\ \zeta_0 & \zeta_1 & \ldots & \zeta_n & 0 \\ \zeta_0^2 & \zeta_1^2 & \ldots & \zeta_n^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \zeta_0^k & \zeta_1^k & \ldots & \zeta_n^k & 1 \end{pmatrix} \in M(k+1, n+2).$$

For each multi-index $J = \{j_0, j_1, \ldots, j_k\}, 0 \leqslant j_0 < j_1 < \cdots < j_k \leqslant n+1$, we will define, in section 3.2, an element $\gamma_J^+(\alpha)$ (resp. $\gamma_J^-(\alpha)$) of $H_k(T(\xi), \mathcal{L}^{-\alpha}(\xi))$ as a pair $(\Delta_J^+, U^\alpha_{\Delta_J^+})$ (resp. $(\Delta_J^-, U^\alpha_{\Delta_J^-})$) of the real k-dimensional surface Δ_J^+ in

$T(\xi)$ and the branch $U^{\alpha}_{\Delta^+_J}$ of U^{α} on Δ^+_J (resp. Δ^-_J and $U^{\alpha}_{\Delta^-_J}$). The $\gamma^+_{J_0}(\alpha)$'s as well as $\gamma^-_{J_{n+1}}(\alpha)$'s form a basis, where the multi-indices J_0's and J_{n+1}'s are of type

$$J_0 = \{0, j_1, \dots, j_k\}, \qquad J_{n+1} = \{j_1, \dots, j_k, n+1\},$$

$$1 \leqslant j_1 < \dots < j_k \leqslant n.$$

The local triviality of $\mathcal{H}_k(M, \mu_*(\mathcal{L}^{-\alpha}))$ enables us to define elements $\gamma^{\pm}_J(\alpha, x)$ in $H_k(T(x), \mathcal{L}^{-\alpha}(x))$ on a general $x \in M$ as the continuation of $\gamma^{\pm}_J(\alpha)$ along a path $x(s)$ from ξ to x in M

$$x(s)\colon\ [0,1] \to M, \qquad x(0) = \xi, \qquad x(1) = x;$$

note that they depend on the choice of $x(s)$.

1.4. The duality of the spaces $H^k(T(x), \mathcal{L}^{\alpha}(x))$ and $H_k(T(x), \mathcal{L}^{-\alpha}(x))$ induces the natural pairing between $\tau^*_x(\varphi_I)$ and $\gamma^{\pm}_J(\alpha, x)$, which defines the hypergeometric functions $F^+_{IJ}(\alpha, x)$ and $F^-_{IJ}(\alpha, x)$ on $M(k+1, n+2)$

$$\begin{aligned} F^{\pm}_{IJ}(\alpha, x) &= F^{\pm}_{IJ}(\alpha, x(s)) = \langle \tau^*_x(\varphi_I), \gamma^{\pm}_J(\alpha, x)\rangle \\ &= \int_{\Delta^{\pm}_J(x)} U^{\alpha}_{\Delta^{\pm}_J(x)} \tau^*_x(\varphi_I), \end{aligned} \tag{1.4.1}$$

where $\gamma^{\pm}_J(\alpha, x)$'s are defined by the path $x(s)$ from ξ to x in M and are represented by $(\Delta^{\pm}_J(x), U^{\alpha}_{\Delta^{\pm}_J(x)})$. Since $\gamma^{\pm}_J(\alpha, x)$ depend on the choice of $x(s)$, $F^{\pm}_{IJ}(\alpha, x)$ are multi-valued holomorphic functions on M; more precisely, they are holomorphic functions on the universal covering $\tilde{M}(k+1, n+1) = \tilde{M}$ of M with the base point ξ.

DEFINITION 1.4.1. *The* $\binom{n}{k} \times \binom{n}{k}$ *matrices*

$$\Pi^+_0(\alpha, x) = (F^+_{I_0 J_0}(\alpha, x))_{I_0, J_0} \quad \textit{and}$$

$$\Pi^-_{n+1}(\alpha, x) = (F^-_{I_{n+1} J_{n+1}}(\alpha, x))_{I_{n+1}, J_{n+1}}$$

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are called the hypergeometric period matrices of type (k,n) *with parameter* α, *where the multi-indices*

$$I_0=\{0,i_1,\dots,i_k\},\qquad I_{n+1}=\{i_1,\dots,i_k,n+1\},$$
$$1\leqslant i_1<\dots<i_k\leqslant n,$$
$$J_0=\{0,j_1,\dots,j_k\},\qquad J_{n+1}=\{j_1,\dots,j_k,n+1\},$$
$$1\leqslant j_1<\dots<j_k\leqslant n,$$

are arranged lexicographically.

1.5. We define actions of the group $G=\mathrm{GL}_{k+1}(\mathbb{C})\times(\mathbb{C}^*)^{n+2}$ on $M(k+1,n+2)$ and its universal covering $\tilde{G}$ on $\tilde{M}(k+1,n+2)$ as follows

$$(g,r)\colon\ x\mapsto g\cdot x\cdot \mathrm{diag}(r_0,\dots,r_{n+1}),$$
$$(g(s),r(s))\colon\ x(s)\mapsto gxr(s)=g(s)\cdot x(s)\cdot\mathrm{diag}(r_0(s),\dots,r_{n+1}(s)),$$

where

$$x(s)\colon\ [0,1]\to M,\quad x(0)=\xi,\ \ x(1)=x,$$
$$g(s)\colon\ [0,1]\to \mathrm{GL}_{k+1}(\mathbb{C}),\quad g(0)=1_{k+1},\quad g(1)=g,$$
$$r(s)\colon\ [0,1]\to(\mathbb{C}^*)^{n+2},\quad r(0)=(1,\dots,1),\quad r(1)=r=(r_0,\dots,r_{n+1}),$$

are paths from ξ to x in M, from 1_{k+1} to g in $\mathrm{GL}_{k+1}(\mathbb{C})$, and from $(1,\dots,1)$ to r in $(\mathbb{C}^*)^{n+1}$, respectively. We call the space

$$X=X(k,l)=M(k+1,n+2)/G,\quad l=n-k$$

the configuration space of ordered $k+l+2$ hyperplanes on $\mathbb{P}^k$ in general position and denote by $[x]$ the element of X represented by $x\in M$. By the action of G, any element $x\in M$ can be transformed into the following form

$$\begin{pmatrix} (-1)^k & & & -1 & & & & 0\\ & \ddots & & \vdots & & -z[x] & & \vdots\\ & & (-1)^1 & -1 & & & & 0\\ & & & 1 & 1 & \cdots & 1 & 1\end{pmatrix}, \tag{1.5.1}$$

$z[x]$ is a $(k\times l)$-matrix of which $(p+1,q-k)$ component of $z[x]$ is

$$\frac{x\langle J^{q\backslash j_p}\rangle x\langle J^{k\backslash j_k}\rangle}{x\langle J^{k\backslash j_p}\rangle x\langle J^{q\backslash j_k}\rangle}\quad(1\leqslant p+1\leqslant k,1\leqslant q-k\leqslant l),$$

where

$$J = \{j_0, \ldots, j_k\} = \{0, 1, \ldots, k-1, n+1\}, \qquad J^{q\setminus j_p} = J \cup \{q\} \setminus \{j_p\}.$$

Indeed, Cramer's formula implies that the (p, q) component $x'_{p,q}$ of $x' = x\langle J\rangle^{-1} \cdot x$ is

$$x'_{p,q} = \begin{cases} \delta_{j_p,q}, & q \in J, \\ (-1)^{k-p-1} \frac{x\langle J^{q\setminus j_p}\rangle}{x\langle J\rangle}, & q \notin J, \quad p < k, \\ \frac{x\langle J^{q\setminus j_p}\rangle}{x\langle J\rangle}, & q \notin J, \quad p = k; \end{cases}$$

by acting

$$\mathrm{diag}\left(\frac{-1}{x'_{0,k}}, \ldots, \frac{-1}{x'_{k-1,k}}, \frac{1}{x'_{k,k}}\right)$$
$$\times \left((-1)^{k-1} x'_{0,k}, \ldots, (-1)^0 x'_{k-1,k}, 1, \frac{x'_{k,k}}{x'_{k,k+1}}, \ldots, \frac{x'_{k,k}}{x'_{k,n}}, x'_{k,k}\right)$$

on x', we have (1.5.1). Note that each component of $z[x]$ is invariant under the action of G.

The normal form (1.5.1) implies that X is a $(k \times l)$-dimensional affine manifold. Since the subgroup $G' = \{(g, r) \in G \mid r_{n+1} = 1\}$ acts freely on M and M is included in the G'-orbit of the set of normal forms (1.5.1), we have

$$M(k+1, n+2) \simeq X(k, l) \times G'.$$

Noting that the universal covering $\tilde{M}$ is isomorphic to $\tilde{X} \times \tilde{G}'$, we have

$$\tilde{M}/\tilde{G}' \simeq \tilde{X}. \tag{1.5.2}$$

LEMMA 1.5.1. *The functions $F^{\pm}_{IJ}(\alpha, x)$ are invariant under the action of $\tilde{G}$*

$$F^{\pm}_{IJ}(\alpha, gxr(s)) = F^{\pm}_{IJ}(\alpha, x(s)).$$

Proof. It is sufficient to prove

$$F^{\pm}_{IJ}(\alpha, g \cdot x \cdot \mathrm{diag}(r)) = F^{\pm}_{IJ}(\alpha, x),$$

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for $(g,r) \in G$ near to the unity. We have

$$\begin{aligned} &D(g \cdot x \cdot \mathrm{diag}(r)) \\ &\quad = \prod_J (\det(g) \cdot x\langle J\rangle \cdot (r_{j_0} \dots r_{j_k}))^{\alpha_J/\binom{n}{k}} \\ &\quad = \det(g)^{\Sigma_J \alpha_J/\binom{n}{k}} D(x) \prod_{i=0}^{n+1} r_i^{\Sigma_{J_i} \alpha_{J_i}/\binom{n}{k}} = D(x) \prod_{j=0}^{n+1} r_j^{\alpha_j}. \end{aligned}$$

Since $L_j(t, gx) = L_j(tg, x)$, the action of g induces the map $g\colon T(g\cdot x) \ni t \to tg \in T(x)$. We have

$$\begin{aligned} &g(\Delta_J^{\pm}(g \cdot x)) = \Delta_J^{\pm}(x), \\ &U^{\alpha}_{\Delta_J^{\pm}(g\cdot x)}(t, g\cdot x) = g^*(U^{\alpha}_{\Delta_J^{\pm}(x)}(t,x)), \\ &\tau^*_{g\cdot x}(\varphi_I(t, g\cdot x)) = g^*(\tau^*_x(\varphi_I(t,x))), \end{aligned}$$

which imply

$$\begin{aligned} &\int_{\Delta_J^{\pm}(g\cdot x)} U^{\alpha}_{\Delta_J^{\pm}(g\cdot x)}(t, g\cdot x)\tau^*_{g\cdot x}(\varphi_I(t, g\cdot x)) \\ &\quad = \int_{\Delta_J^{\pm}(x)} U^{\alpha}_{\Delta_J^{\pm}(x)}(t,x)\tau^*_x(\varphi_I(t,x)), \\ &F^{\pm}_{IJ}(\alpha, g\cdot x) = F^{\pm}_{IJ}(\alpha, x). \end{aligned}$$

Since $T(x) = T(x \cdot \mathrm{diag}(r)), \Delta_J^{\pm}(x)$ is invariant under the action of r. We have

$$\begin{aligned} &U^{\alpha}_{\Delta_J^{\pm}(x)}(t, x\cdot \mathrm{diag}(r)) = U^{\alpha}_{\Delta_J^{\pm}(x)}(t,x), \\ &\tau^*_x(\varphi_I(t, x\cdot\mathrm{diag}(r))) = \tau^*_x(\varphi_I(t,x)), \end{aligned}$$

which imply

$$F^{\pm}_{IJ}(\alpha, x\cdot \mathrm{diag}(r)) = F^{\pm}_{IJ}(\alpha, x). \qquad \square$$

This lemma together with (1.5.2) shows that the functions $F^{\pm}_{IJ}(\alpha, x)$ are defined on $\tilde{X}$. When we regard them as multi-valued functions on the configuration space X, we denote them by $F^{\pm}_{IJ}(\alpha, [x])$ and the hypergeometric period matrices by $\Pi_0^+(\alpha, [x])$ and $\Pi^-_{n+1}(\alpha, [x])$. Refer to [MSTY] for the monodromy behavior of the hypergeometric period matrices defined on X.

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2. The duality of the configuration spaces

2.1. For every $x \in M(k+1, n+2)$ there exists a unique $x^* \in M(l+1, n+2)$ modulo $\mathrm{GL}_{l+1}(\mathbb{C})$ such that $x\,{}^t x^* = O$. Moreover, we have

$$(x \cdot \mathrm{diag}(r))\,{}^t(x^* \cdot \mathrm{diag}(r)^{-1}) = x\,{}^t x^* = O,$$

$$r = (r_0, \dots, r_{n+1}) \in (\mathbb{C}^*)^{n+2}.$$

We give a bijective map $\perp$ as follows.

DEFINITION 2.1.1. *The map* $\perp\colon X(k,l) \to X(l,k)$ *is defined by*

$$\perp\colon X(k,l) \ni [x] \mapsto [x]^{\perp} = [x^*] \in X(l,k),$$

where

$$x\,{}^t x^* = O, \quad x \in M(k+1, n+2), \quad x^* \in M(l+1, n+2).$$

Note that such x^* is given by

$$x^* = ({}^t x_{I^{\perp}}\,{}^t x_I^{-1}, -1_{l+1})$$

for

$$x = (x_I, x_{I^{\perp}}), \quad x_I \in \mathrm{GL}_{k+1}(\mathbb{C}), \quad I = \{0, \dots, k\}.$$

The straightforward calculation shows the following lemma.

LEMMA 2.1.2. *For any* $(k \times l)$*-matrix* z*, we have*

$$x_z\,\mathrm{diag}((-1)^n, (-1)^{n-1}, \dots, (-1)^0, (-1)^{-1})\,{}^t y_z = O,$$

$$x_z\langle J\rangle = (-1)^{n(k-l+1)/2}\, y_z\langle J^{\perp}\rangle,$$

where

$$x_z = \begin{pmatrix} (-1)^k & & & -1 & & & & 0 \\ & \ddots & & \vdots & & -z & & \vdots \\ & & (-1)^1 & -1 & & & & 0 \\ & & & 1 & 1 & \cdots & 1 & 1 \end{pmatrix},$$

$$y_z = \begin{pmatrix} 1 & \cdots & 1 & 1 & & & & (-1)^l \\ & & & 0 & (-1)^0 & & & (-1)^{l-1} \\ & -{}^t z & & \vdots & & \ddots & & \vdots \\ & & & 0 & & & (-1)^{l-1} & (-1)^{l-1} \end{pmatrix}. \tag{2.1.1}$$

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Recall that the base point $\xi_k \in M(k+1, n+2)$ is given by

$$\xi_k = \begin{pmatrix} 1 & 1 & \cdots & 1 & 0 \\ \zeta_0 & \zeta_1 & \cdots & \zeta_n & 0 \\ \zeta_0^2 & \zeta_1^2 & \cdots & \zeta_n^2 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \zeta_0^k & \zeta_1^k & \cdots & \zeta_n^k & 1 \end{pmatrix} \in M(k+1, n+2).$$

We define a $(k \times l)$-matrix as

$$z[\xi_k] = \begin{pmatrix} \frac{\zeta_k-\zeta_0}{\zeta_{k+1}-\zeta_0} & \cdots & \frac{\zeta_k-\zeta_0}{\zeta_n-\zeta_0} \\ \vdots & \frac{\zeta_k-\zeta_i}{\zeta_j-\zeta_i} & \vdots \\ \frac{\zeta_k-\zeta_{k-1}}{\zeta_{k+1}-\zeta_{k-1}} & \cdots & \frac{\zeta_k-\zeta_{k-1}}{\zeta_n-\zeta_{k-1}} \end{pmatrix},$$

$$0 \leqslant i < k < j \leqslant n.$$

LEMMA 2.1.3. *The element* $\xi_k \in M(k+1, n+2)$ *is transformed into* $x_{z[\xi_k]}$ *in* (2.1.1) *by the actions of* $\mathrm{GL}_{k+1}(\mathbb{R})$ *and* $(\mathbb{R}_{>0})^{n+2}$*; the element* $\xi_l \in M(l+1, n+2)$ *is transformed into* $y_{z[\xi_k]}$ *in* (2.1.1) *by the actions of* $\mathrm{GL}_{l+1}(\mathbb{R})$ *and* $(\mathbb{R}_{>0})^{n+2}$*. Hence we have*

$$[\xi_k]^\perp = [\xi_l]. \tag{2.1.2}$$

Proof. Use the Vandermonde determinant formula and the argument leading the normal form (1.5.1). □

3. The main theorem

3.1. We introduce some notations in order to state our main theorem. Let E_{lk} be the $\binom{n}{l} \times \binom{n}{k}$ matrix

$$E_{lk} = ((-1)^{(l(l+1)/2)+p_1+\cdots+p_l}\delta_{P,J^\perp})_{P,J},$$

where multi-indices $P = \{p_1, \ldots, p_l\}, 1 \leqslant p_1 < \cdots < p_l \leqslant n$ and $J = \{j_1, \ldots, j_k\}, 1 \leqslant j_1 < \cdots < j_k \leqslant n$ are arranged lexicographically, $J^\perp = \{1, \ldots, n\} \setminus J$ and $\delta_{P,J^\perp}$ is Kronecker's symbol. Note that E_{lk} is anti-diagonal. For an element $g = (g_{pq}) \in \mathrm{GL}_n(\mathbb{C})$, put

$$\wedge^l g = (\det(g_{pq})_{p\in P, q\in Q})_{PQ} \in \mathrm{GL}_{\binom{n}{l}}(\mathbb{C}),$$

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where the multi-indices P and Q of cardinality l are arranged lexicographically. Note that

$$\wedge^l(g_1g_2) = (\wedge^l g_1)(\wedge^l g_2), \qquad \wedge^l g_1^{-1} = (\wedge^l g_1)^{-1}$$

for $g_1, g_2 \in \mathrm{GL}_n(\mathbb{C})$.

The following is our main theorem.

THEOREM 3.1.1. *(Duality for hypergeometric period matrices)*

$$\begin{aligned}\Pi_0^+(\alpha,[x]) \;&=\; V(\alpha)\,{}^tE_{lk}(\wedge^l I_{ch}(\alpha)^{-1}) \\ &\quad\times \Pi_{n+1}^-(-\alpha,[x]^\perp)(\wedge^l I_h(\alpha)^{-1})E_{lk},\end{aligned} \tag{3.1.1}$$

where

$$V(\alpha) = \mathrm{e}^{n\pi\sqrt{-1}\alpha_0}\,\mathrm{e}^{(n-1)\pi\sqrt{-1}\alpha_1}\ldots\mathrm{e}^{\pi\sqrt{-1}\alpha_{n-1}}\frac{\Gamma(\alpha_0)\Gamma(\alpha_1)\ldots\Gamma(\alpha_n)}{\Gamma(-\alpha_{n+1})},$$

$$I_{ch}(\alpha) = \mathrm{diag}\left(\frac{2\pi\sqrt{-1}}{-\alpha_1}, \frac{2\pi\sqrt{-1}}{-\alpha_2}, \ldots, \frac{2\pi\sqrt{-1}}{-\alpha_n}\right),$$

$$I_h(\alpha) = \mathrm{diag}\left(\frac{\mathrm{e}^{2\pi\sqrt{-1}(\alpha_0+\alpha_1)}}{\mathrm{e}^{2\pi\sqrt{-1}\alpha_1}-1}, \frac{\mathrm{e}^{2\pi\sqrt{-1}(\alpha_0+\alpha_1+\alpha_2)}}{\mathrm{e}^{2\pi\sqrt{-1}\alpha_2}-1}, \ldots, \frac{\mathrm{e}^{2\pi\sqrt{-1}(\alpha_0+\cdots+\alpha_n)}}{\mathrm{e}^{2\pi\sqrt{-1}\alpha_n}-1}\right),$$

and the path from $[\xi(l)]$ to $[x]^\perp$ defining $\Pi_{n+1}^-(-\alpha,[x]^\perp)$ is the $\perp$-image of the path defining $\Pi_0^+(\alpha,[x])$.

Remark 3.1.2. Each component of (3.1.1) says

$$F^+_{I_0J_0}(\alpha,[x]) = c(I_0,J_0)F^-_{I_0^\perp J_0^\perp}(-\alpha,[x]^\perp)$$

for a constant $c(I_0,J_0) \in \mathbb{C}^*$.

3.2. We construct $\gamma_J^\pm(\alpha) \in H_k(T(\xi_k), \mathcal{L}^{-\alpha}(\xi_k))$ for $J = \{j_0,\ldots,j_k\}$. Since

$$\xi_k\langle I\rangle = \prod_{0\leqslant\kappa<\lambda\leqslant k}\xi_1\langle i_\kappa, i_\lambda\rangle > 0, \tag{3.2.1}$$

we assign $\arg(\xi_k\langle I\rangle) = 0$ for every I. Let Δ_J be the simplex in $\mathbb{P}^k$ defined by

$$0 < -L_{j_{\lambda-1}}(t,\xi_k)/L_{j_\lambda}(t,\xi_k) < \infty, \quad 1 \leqslant \lambda \leqslant k;$$

it will turn out in the next section that Δ_J and the hyperplane $L_j(t,\xi_k)=0$ intersect for $j_{\lambda-1}<j<j_\lambda$. We assign arguments of L_j/L_{n+1}'s on $\Delta_J\cap T(\xi_k)$ as follows

$$\arg\tfrac{L_j(t,\xi_k)}{L_{n+1}(t,\xi_k)}=\begin{cases}k\pi, & j<j_0,\\ 0, & j>j_k,\\ (k-\lambda)\pi, & j=j_\lambda,\end{cases}$$

$$\arg\tfrac{L_j(t,\xi_k)}{L_{n+1}(t,\xi_k)} \tag{3.2.2}$$

$$=\begin{cases}(k-\lambda+1)\pi, & \text{for points } (-1)^{k-\lambda}(L_j/L_{n+1})<0,\\ (k-\lambda)\pi, & \text{for points } (-1)^{k-\lambda}(L_j/L_{n+1})>0,\end{cases}$$

$$j_{\lambda-1}<j<j_\lambda,$$

which fix the choice of branch $U^\alpha_{\Delta_J^+}(t,\xi_k)$ on $\Delta_J\cap T(\xi_k)$. We define $\gamma_J^+(\alpha)$ as the pair of $\Delta_J\cap T(\xi_k)$ and the branch $U^\alpha_{\Delta_J^+}(t,\xi_k)$ of U^α by the above assignment. Similarly, we define $\gamma_J^-(\alpha)$ as the pair of $\Delta_J\cap T(\xi_k)$ and the branch $U^\alpha_{\Delta_J^-}(t,\xi_k)$ of U^α by the assignment of the argument of $L_j(t,\xi_k)/L_{n+1}(t,\xi_k)$ with the minus sign of (3.2.2).

3.3. It is not so easy to see the structure of $\gamma_J^\pm(\alpha)\in H_k(T(\xi_k),\mathcal{L}^{-\alpha}(\xi_k))$ for $k\neq 1$. Here we give the another description of $\gamma_J^\pm(\alpha)$ for a general k. Let ι_k be the map

$$\mathbb{C}^k\ni(s_{(1)},\ldots,s_{(k)})\mapsto(\sigma_k,\ldots,\sigma_1)\in\mathbb{C}^k,$$

where σ_j is the jth fundamental symmetric polynomial of $s_{(i)}$'s, i.e.,

$$\sigma_j=\sum_{1\leqslant i_1<\cdots<i_j\leqslant k}s_{(i_1)}\ldots s_{(i_j)}.$$

We can regard ι_k as a map from $(\mathbb{P}^1)^k=\mathbb{P}^1_{(1)}\times\cdots\times\mathbb{P}^1_{(k)}$ to $\mathbb{P}^k$. By using affine coordinates $s_{(i)}=(t_0/t_1)_{(i)}$ on $\mathbb{P}^1_{(i)}$ and $(t_0/t_k,\ldots,t_{k-1}/t_k)$ on $\mathbb{P}^k$, the pull-back of $L_j(t,\xi_k)/L_{n+1}(t,\xi_k)$ by ι_k is easily obtained as follows

$$\iota_k^*\left(\frac{L_j((t_0,\ldots,t_k),\xi_k)}{L_{n+1}((t_0,\ldots,t_k),\xi_k)}\right)$$

$$=\iota_k^*\left(\sum_{i=0}^k(t_i/t_k)\zeta_j^i\right)=\prod_{i=1}^k(s_{(i)}+\zeta_j)$$

$$=\prod_{i=1}^k\frac{L_j((t_0,t_1)_{(i)},\xi_1)}{L_{n+1}((t_0,t_1)_{(i)},\xi_1)}. \tag{3.3.1}$$

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Then ι_k induces the map from $T(\xi_1)_{(1)} \times \ldots \times T(\xi_1)_{(k)}$ to $T(\xi_k)$. Though the map ι_k is of $k! : 1$, the restriction of ι_k on $\Delta_{\{j_0,j_1\}} \times \ldots \times \Delta_{\{j_{\kappa-1},j_\kappa\}}$ is bijective for $J = \{j_0, \ldots, j_k\}$ since each intersection $\Delta_{\{j_{\kappa-1},j_\kappa\}} \cap \Delta_{\{j_{\lambda-1},j_\lambda\}} (1 \leqslant \kappa < \lambda \leqslant k)$ is empty. Let us show

$$\gamma_J^+(\alpha) = \iota_k(\gamma^+_{\{j_0,j_1\}}(\alpha)_{(1)} \times \cdots \times \gamma^+_{\{j_{k-1},j_k\}}(\alpha)_{(\kappa)}),$$

i.e.,

$$\Delta_J \cap T(\xi_k) = \iota_k(\Delta_{\{j_0,j_1\}} \times \cdots \times \Delta_{\{j_{\kappa-1},j_\kappa\}}), \tag{3.3.2}$$

$$U^\alpha_{\Delta_J^+}((t_0, \ldots, t_k), \xi_k) = \iota_{k*}\left(\prod_{\lambda=1}^k U^\alpha_{\Delta^+_{\{j_{\lambda-1},j_\lambda\}}}((t_0,t_1)_{(\lambda)}, \xi_1)\right). \tag{3.3.3}$$

Because of (3.2.1), we have

$$D(\xi_k) = \prod_J \xi_k\langle J\rangle^{\alpha_J/\binom{n}{k}} = \left(\prod_{0\leqslant i<j\leqslant n+1} \xi_1\langle i,j\rangle^{(\alpha_i+\alpha_j)/n}\right)^k = D(\xi_1)^k.$$

Recall that the argument of $L_j(t,\xi_1)/L_{n+1}(t,\xi_1)$ on the simplex $\Delta_{\{j_{\lambda-1},j_\lambda\}}$ defining the branch $U^\alpha_{\Delta^+_{\{j_{\lambda-1},j_\lambda\}}}$ is given by

$$\arg\frac{L_j(t,\xi_1)}{L_{n+1}(t,\xi_1)} = \begin{cases} \pi, & j \leqslant j_{\lambda-1}, \\ 0, & j \geqslant j_\lambda, \\ \pi \text{ or } 0, & j_{\lambda-1} < j < j_\lambda. \end{cases}$$

By summing up the values for $1 \leqslant \lambda \leqslant k$, we have the argument of

$$\iota_{k*}\left(\prod_{\lambda=1}^k \frac{L_j((t_0,t_1)_{(\lambda)},\xi_1)}{L_{n+1}((t_0,t_1)_{(i)},\xi_1)}\right) = \frac{L_j((t_0,\ldots,t_k),\xi_k)}{L_{n+1}((t_0,\ldots,t_k),\xi_k)}$$

on $\iota_k(\Delta_{\{j_0,j_1\}} \times \cdots \times \Delta_{\{j_{k-1},j_k\}})$, which coincides with that of $L_j(t,\xi_k)/L_{n+1}(t,\xi_k)$ on Δ_J in (3.2.2). This implies (3.3.2) and (3.3.3). Now that the assignment in (3.2.2) is justified, we can see that Δ_J and the hyperplane $L_j(t,\xi_k) = 0$ intersect for $j_{\lambda-1} < j < j_\lambda$.

4. Proof of the main theorem

4.1. The following proposition was essentially proved in [Ter]; since the choice of our forms and cycles is distinct from that in [Ter], a proof shall be attached.

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PROPOSITION 4.1.1. *(Wedge formulae for period matrices)*

$$\wedge^k \Pi_0^+(\alpha, \xi_1) = \Pi_0^+(\alpha, \xi_k),$$

$$\wedge^k \Pi_{n+1}^-(\alpha, \xi_1) = \Pi_{n+1}^-(\alpha, \xi_k),$$

in particular,

$$\wedge^n \Pi_0^+(\alpha, \xi_1) = \Pi_0^+(\alpha, \xi_n) = V(\alpha).$$

Proof. By (3.3.1), we have

$$\begin{aligned}
&\iota_k^*(\varphi_I((t_0, \ldots, t_k), \xi_k)) \\
&= \left(\sum_{\lambda=1}^{k} \varphi_1((t_0, t_1)_{(\lambda)}, \xi_1)\right) \wedge \cdots \wedge \left(\sum_{\lambda=1}^{k} \varphi_k((t_0, t_1)_{(\lambda)}, \xi_1)\right) \\
&= \sum_{\varsigma \in \mathfrak{S}_k} \operatorname{sign}(\varsigma)(\varphi_{\varsigma(1)}((t_0, t_1)_{(1)}, \xi_1) \wedge \cdots \wedge \varphi_{\varsigma(k)}((t_0, t_1)_{(k)}, \xi_1)),
\end{aligned}$$

where $\mathfrak{S}_k$ is the symmetric group of degree k. This identity and (3.3.2), (3.3.3) yield

$$\wedge^k \Pi_0^+(\alpha, \xi_1) = \Pi_0^+(\alpha, \xi_k).$$

By Lemma 2.1.3, and (3.2.2), we can easily obtain

$$\Pi_0^+(\alpha, \xi_n) = V(\alpha). \qquad \square$$

This proposition is deeply related to the isomorphisms between $\wedge^k H^1(T(\xi_1), \mathcal{L}^\alpha(\xi_1))$ and $H^k(T(\xi_k), \mathcal{L}^\alpha(\xi_k))$ and between $\wedge^k H_1(T(\xi_1), \mathcal{L}^{-\alpha}(\xi_1))$ and $H_k(T(\xi_k), \mathcal{L}^{-\alpha}(\xi_k))$ studied in [IK1] and [IK2].

4.2. The integral (1.4.1) does not converge in the usual sense for a general parameter α. In order that it makes sense, we introduce the notion of the regularization of $\gamma_J^\pm(\alpha)$. In this section we explicitly give the regularization $\tilde{\gamma}_J^\pm(\alpha)$ of $\gamma_J^\pm(\alpha)$ for $k = 1$. Deform $\Delta_J, J = \{j_0, j_1\}$, into Δ_J^+ and Δ_J^- as shown in the following figure.

The assignments of $\arg(L_j(t, \xi_1)/L_{n+1}(t, \xi_1))$ on $\Delta_J^\pm$ are naturally defined by the deformations, these induce the branches $U^\alpha_{\Delta_J^\pm}$ on $\Delta_J^\pm$; note that the assignment on Δ_J^+ is determined by (3.2.2) and that on Δ_J^- is determined by the minus sign of (3.2.2). For a sufficiently small positive number ε, let $C_J(j_\lambda), \lambda = 0, 1$, be the

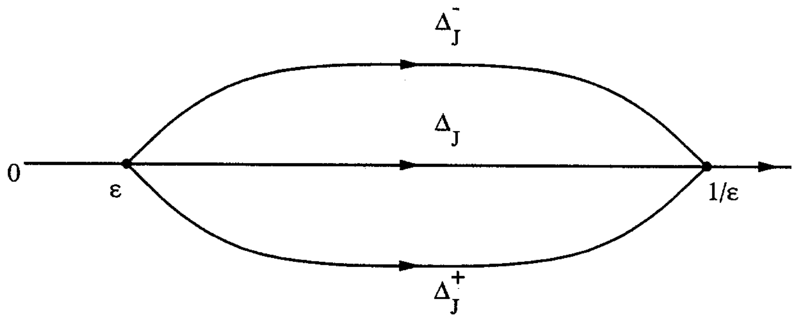

Figure 4.2.1.

circles

$$C_J(j_0)\colon\ -L_{j_0}/L_{j_1} = \varepsilon\, \mathrm{e}^{\sqrt{-1}s}, \quad 0 \leqslant s < 2\pi,$$

$$C_J(j_1)\colon\ -L_{j_0}/L_{j_1} = \varepsilon^{-1}\, \mathrm{e}^{-\sqrt{-1}s}, \quad 0 \leqslant s < 2\pi.$$

We define branches $U^\alpha_{C_J^\pm(j_\lambda)}$ on $C_J(j_\lambda)$ by the continuations of the branches $U^\alpha_{\Delta_J^\pm}$. We define the regularizations $\tilde{\gamma}_J^\pm(\alpha)$ of $\gamma_J^\pm(\alpha)$ by the following formal summations

$$\tilde{\gamma}_J^\pm(\alpha) = \frac{1}{c_{j_0}-1}(C_J(j_0), U^\alpha_{C_J^\pm(j_0)}) + (\Delta_J^\pm, U^\alpha_{\Delta_J^\pm}) - \frac{1}{c_{j_1}-1}(C_J(j_1), U^\alpha_{C_J^\pm(j_1)}), \tag{4.2.2}$$

where $c_j = \exp(2\pi\sqrt{-1}\alpha_j)$.

For a general k, we define $\tilde{\gamma}_J^\pm(\alpha)$ by

$$\tilde{\gamma}_J^\pm(\alpha) = \iota_k(\tilde{\gamma}^\pm_{\{j_0,j_1\}}(\alpha)_{(1)} \times \cdots \times \tilde{\gamma}^\pm_{\{j_{k-1},j_k\}}(\alpha)_{(k)}).$$

The values $\langle \tau^*_{\xi_k}(\phi_I), \tilde{\gamma}_J^\pm(\alpha)\rangle$ are well-defined under the condition (1.1.1), moreover the Cauchy integral theorem implies that they are independent of the choice of the small positive number ε and that they coincide with $\langle \tau^*_{\xi_k}(\phi_I), \gamma_J^\pm(\alpha)\rangle$ when it exists in the classical sense.

4.3. We compute the intersection number of $\tilde{\gamma}^+_{I_0}(\alpha) \in H_1(T(\xi_1), \mathcal{L}^{-\alpha}(\xi_1))$ and $\tilde{\gamma}^-_{J_{n+1}}(-\alpha) \in H_1(T(\xi_1), \mathcal{L}^{\alpha}(\xi_1))(I_0 = \{0,i\}, J_{n+1} = \{j, n+1\})$, which is defined as the summation of the products of the topological intersection number

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of chains and the branches U^{α} and $U^{-\alpha}$ at every intersection point, refer to [KY1] for details. Note that, if $i \neq j$, then the topological chains defining $\tilde{\gamma}^{+}_{I_0}(\alpha)$ and $\tilde{\gamma}^{-}_{J_{n+1}}(-\alpha)$ do not intersect and that if $i = j$ then there is one intersection point on the chain $C_{I_0}(i)$, where we assume that the small positive number ε for $\tilde{\gamma}^{-}_{J_{n+1}}(-\alpha)$ is much smaller than that for $\tilde{\gamma}^{+}_{I_0}(\alpha)$; the following figure helps us to understand the situation.

$-L_0/L_{n+1}$ space

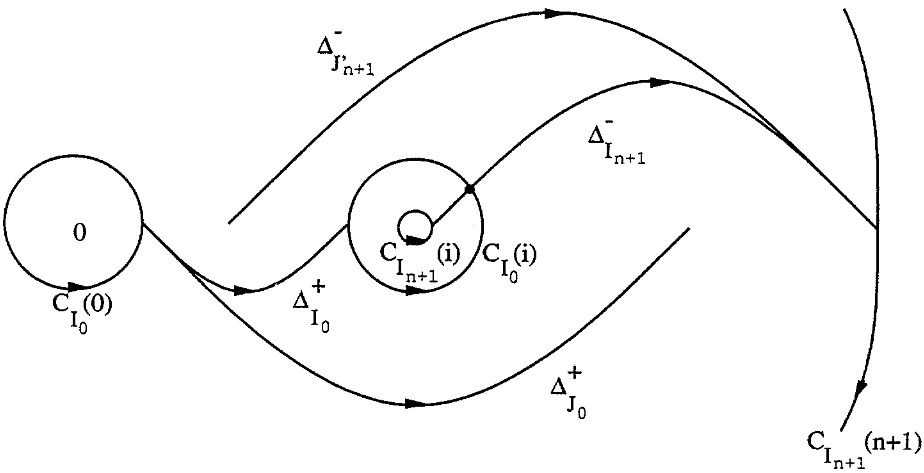

Figure 4.3.1.

By considering the coefficient of $C_{I_0}(i)$ and the signature and branches $U^{\alpha}_{C^{+}_{I_0}(i)}$ and $U^{-\alpha}_{\Delta^{-}_{J_{n+1}}}$ at the intersection point, the intersection number is given by

$$\langle \tilde{\gamma}^{+}_{I_0}(\alpha), \tilde{\gamma}^{-}_{J_{n+1}}(-\alpha) \rangle = \delta_{ij} \frac{e^{2\pi\sqrt{-1}(\alpha_0+\cdots+\alpha_j)}}{e^{2\pi\sqrt{-1}\alpha_j} - 1},$$

$$I_0 = \{0, i\}, J_{n+1} = \{j, n+1\},$$

where δ_{ij} is Kronecker's symbol. The regularity of the intersection matrix

$$I_h(\alpha) = (\langle \tilde{\gamma}^{+}_{I_0}(\alpha), \tilde{\gamma}^{-}_{J_{n+1}}(-\alpha) \rangle)_{1 \leqslant i,j \leqslant n}$$

shows that the cycles $\tilde{\gamma}^{+}_{I_0}(\alpha)$'s and $\tilde{\gamma}^{-}_{J_{n+1}}(-\alpha)$'s form bases of $H_1(T(\xi_1), \mathcal{L}^{-\alpha}(\xi_1))$ and $H_1(T(\xi_1), \mathcal{L}^{\alpha}(\xi_1))$, respectively.

The first theorem in [CM] implies that the intersection number of the 1-forms $\tau_{\xi_1}^*(\varphi_{I_0}) \in \tau_{\xi_1}^*(\Phi_1/\mathbb{C}\omega^{\alpha})$ and $\tau_{\xi_1}^*(\varphi_{J_{n+1}}) \in \tau_{\xi_1}^*(\Phi_1/\mathbb{C}\omega^{-\alpha})$ is

$$\langle \tau_{\xi_1}^*(\varphi_{I_0}), \tau_{\xi_1}^*(\varphi_{J_{n+1}})\rangle = \delta_{ij}\frac{2\pi\sqrt{-1}}{-\alpha_j},$$

$$I_0 = \{0, i\}, \quad J_{n+1} = \{j, n+1\}.$$

The regularity of the intersection matrix

$$I_{ch}(\alpha) = (\langle \tau_{\xi_1}^*(\varphi_{I_0}), \tau_{\xi_1}^*(\varphi_{J_{n+1}})\rangle)_{1\leqslant i,j\leqslant n}$$

shows that the 1-forms $\tau_{\xi_1}^*(\varphi_{I_0})$'s and $\tau_{\xi_1}^*(\varphi_{J_{n+1}})$'s form bases of $H^1(T(\xi_1), \mathcal{L}^{\alpha}(\xi_1))$ and $H^1(T(\xi_1), \mathcal{L}^{-\alpha}(\xi_1))$, respectively.

The second theorem in [CM] implies the following proposition.

PROPOSITION 4.3.1. *(The twisted Riemann's period relation for $k = 1$)*

$$\Pi_{n+1}^-(-\alpha, [\xi_1]) I_h(\alpha)^{-1}\, {}^t\Pi_0^+(\alpha, [\xi_1]) = I_{ch}(\alpha). \tag{4.3.1}$$

We give a key lemma to prove our main theorem.

LEMMA 4.3.2. *The identity* (3.1.1) *holds for the point* $[x] = [\xi_k]$.

Proof. We have proved $[\xi_k]^{\perp} = [\xi_l]$ in Lemma 2.1.3. By taking the l-fold wedge product of (4.3.1), we have

$$\begin{aligned}&(\wedge^l I_{ch}(\alpha)^{-1})(\wedge^l \Pi_{n+1}^-(-\alpha; [\xi_1]))(\wedge^l I_h(\alpha)^{-1})\\ &\quad = {}^t(\wedge^l \Pi_0^+(\alpha; [\xi_1]))^{-1}.\end{aligned}$$

The Laplace expansion formula yields that

$$\begin{aligned}&{}^t(\wedge^l \Pi_0^+(\alpha; [\xi_1]))^{-1}\\ &\quad = \frac{1}{\det(\Pi_0^+(\alpha; [\xi_1]))} E_{lk}(\wedge^k \Pi_0^+(\alpha; [\xi_1]))\, {}^tE_{lk},\end{aligned}$$

Proposition 4.1.1 implies our claim. □

4.4. We present the differential equation associated to $\Pi_0^+(\alpha; x)$.

PROPOSITION 4.4.1. *(The invariant Gauss-Manin system). The hypergeometric period matrix $\Pi_0^+(\alpha; x)$ satisfies the following differential equation*

$$\mathrm{d}\Pi_0^+(\alpha; x) = \Theta_0^{\alpha}[x]\Pi_0^+(\alpha; x). \tag{4.4.1}$$

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The connection form $\Theta_0^\alpha[x] = (\theta^\alpha_{I_0 J_0})_{I_0 J_0}$ *is given by*

$$\theta^\alpha_{J_0 J_0} = \sum_{j_\kappa \in J_0} \alpha_{j_\kappa} \, \mathrm{d}\log \frac{x\langle J_0^{n+1\backslash 0}\rangle}{x\langle J_0^{n+1\backslash j_\kappa}\rangle}$$

$$+ \sum_{j_\lambda \in J_0^\perp} \alpha_{j_\lambda} \, \mathrm{d}\log \frac{x\langle J_0^{j_\lambda\backslash 0}\rangle}{x\langle J_0\rangle} - \frac{1}{\binom{n}{k}} \sum_J \alpha_J \, \mathrm{d}\log x\langle J\rangle,$$

$$\theta^\alpha_{J_0 J_0^{j_\lambda\backslash j_\kappa}} = \frac{j_\lambda - j_\kappa}{|j_\lambda - j_\kappa|}(-1)^{j_\lambda - \kappa - \lambda + k}\alpha_{j_\lambda} \, \mathrm{d}\log \frac{x\langle J_0^{j_\lambda\backslash j_\kappa}\rangle x\langle J_0^{n+1\backslash 0}\rangle}{x\langle J_0^{j_\lambda\backslash 0}\rangle x\langle J_0^{n+1\backslash j_\kappa}\rangle},$$

$$\theta^\alpha_{I_0 J_0} = 0 \quad \text{otherwise,}$$

where J *runs over the multi-indices of cardinality* $k+1$ *and* $J_0^{j_\lambda\backslash j_\kappa}$ *is the multi-index corresponding to the set* $(J_0 \setminus \{j_\kappa\}) \cup \{j_\lambda\}$, $j_\kappa \in J_0 = \{0, j_1, \ldots, j_k\}, j_\lambda \in J_0^\perp = \{j_{k+1}, \ldots, j_n, n+1\}$. *The connection form* $\Theta_0^\alpha[x]$ *is invariant under the action of* $\mathrm{GL}_{k+1}(\mathbb{C}) \times (\mathbb{C}^*)^{n+2}$ *on* $x \in M(k+1, n+2)$.

Proof. By using the results in [Aom] or [AK] Ch. 3.8, we can show that the hypergeometric period matrix satisfies the system of differential equation stated in the proposition. We have only to show that the invariance of $\Pi_0^+(\alpha; x)$, the invariance is clear for $\theta^\alpha_{J_0 J_0^{j_\lambda\backslash j_\kappa}}$. In order to see the invariance of $\theta^\alpha_{J_0 J_0}$, we eliminate α_0 from $\theta^\alpha_{J_0 J_0}$ by $\sum_{j=0}^{n+1} \alpha_j = 0$ and see the coefficient of α_j in the expression of $\theta^\alpha_{J_0 J_0}$. Then we have

$$\theta^\alpha_{J_0 J_0} = \frac{1}{\binom{n}{k}} \sum_{j \in J_0} \alpha_j \, \mathrm{d}\log \frac{x\langle J_0^{n+1\backslash 0}\rangle^{\binom{n}{k}} \prod_{I_0} x\langle I_0\rangle}{x\langle J_0^{n+1\backslash j}\rangle^{\binom{n}{k}} \prod_{I_j} x\langle I_j\rangle}$$

$$+ \frac{1}{\binom{n}{k}} \sum_{j \in J_0^\perp} \alpha_j \, \mathrm{d}\log \frac{x\langle J_0^{j\backslash 0}\rangle^{\binom{n}{k}} \prod_{I_0} x\langle I_0\rangle}{x\langle J_0\rangle^{\binom{n}{k}} \prod_{I_j} x\langle I_j\rangle}.$$

The homogeneity of the terms in the logarithmic differentials above shows the desired invariance. □

The connection form $\Theta_0^\alpha[x]$ is called the Gauss-Manin connection on the configuration space $X(k,l)$ for the basis φ_{I_0}'s.

EXAMPLE 1. Type $(1,2), k = l = 1, n = 2, \{0,1,2,3\}$.

$$\theta_0(01;01;\alpha) = \frac{\alpha_1}{2} \, \mathrm{d}\log \frac{x\langle 13\rangle x\langle 02\rangle}{x\langle 03\rangle x\langle 12\rangle}$$

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$$+\frac{\alpha_2}{2}\,\mathrm{d}\log\frac{x\langle 12\rangle x\langle 03\rangle}{x\langle 01\rangle x\langle 23\rangle}+\frac{\alpha_3}{2}\,\mathrm{d}\log\frac{x\langle 13\rangle x\langle 02\rangle}{x\langle 01\rangle x\langle 23\rangle},$$

$$\theta_0(02;02;\alpha)=\frac{\alpha_1}{2}\,\mathrm{d}\log\frac{x\langle 12\rangle x\langle 03\rangle}{x\langle 02\rangle x\langle 13\rangle}$$

$$+\frac{\alpha_2}{2}\,\mathrm{d}\log\frac{x\langle 23\rangle x\langle 01\rangle}{x\langle 03\rangle x\langle 12\rangle}+\frac{\alpha_3}{2}\,\mathrm{d}\log\frac{x\langle 23\rangle x\langle 01\rangle}{x\langle 02\rangle x\langle 13\rangle},$$

$$\theta_0(01;02;\alpha)=\alpha_2\,\mathrm{d}\log\frac{x\langle 02\rangle x\langle 13\rangle}{x\langle 12\rangle x\langle 03\rangle},$$

$$\theta_0(02;01;\alpha)=\alpha_1\,\mathrm{d}\log\frac{x\langle 01\rangle x\langle 23\rangle}{x\langle 12\rangle x\langle 03\rangle}.$$

EXAMPLE 2. Type $(1,3), k=1, l=2, n=3, \{0,1,2,3,4\}$. We give only $\theta_0(01;01;\alpha)$

$$\theta_0(01;01;\alpha)=\frac{\alpha_1}{3}\,\mathrm{d}\log\frac{x\langle 14\rangle^2 x\langle 02\rangle x\langle 03\rangle}{x\langle 04\rangle^2 x\langle 12\rangle x\langle 13\rangle}$$

$$+\frac{\alpha_2}{3}\,\mathrm{d}\log\frac{x\langle 12\rangle^2 x\langle 03\rangle x\langle 04\rangle}{x\langle 01\rangle^2 x\langle 23\rangle x\langle 24\rangle}$$

$$+\frac{\alpha_3}{3}\,\mathrm{d}\log\frac{x\langle 13\rangle^2 x\langle 02\rangle x\langle 04\rangle}{x\langle 01\rangle^2 x\langle 23\rangle x\langle 34\rangle}$$

$$+\frac{\alpha_4}{3}\,\mathrm{d}\log\frac{x\langle 14\rangle^2 x\langle 02\rangle x\langle 03\rangle}{x\langle 01\rangle^2 x\langle 24\rangle x\langle 34\rangle}.$$

4.5. A similar calculation as in the proof of Proposition 4.4.1 leads to a system of differential equations for $\Pi^-_{n+1}(-\alpha,y), y\in M(l+1,n+2)$.

LEMMA 4.5.1. *We have*

$$\mathrm{d}\Pi^-_{n+1}(-\alpha,y)=\Theta^{-\alpha}_{n+1}[y]\Pi^-_{n+1}(-\alpha,y).$$

The connection form $\Theta^{-\alpha}_{n+1}[y]=(\theta^{-\alpha}_{P_{n+1}Q_{n+1}})_{P_{n+1}Q_{n+1}}$ *is given by*

$$\theta^{-\alpha}_{P_{n+1}P_{n+1}}=\sum_{p_\nu\in P_{n+1}}\alpha_{p_\nu}\ \mathrm{d}\log\frac{y\langle P^{0\backslash p_\nu}_{n+1}\rangle}{y\langle P^{0\backslash n+1}_{n+1}\rangle}$$

$$+\sum_{p_\upsilon\in P^\perp_{n+1}}\alpha_{p_\upsilon}\ \mathrm{d}\log\frac{y\langle P_{n+1}\rangle}{y\langle P^{p_\upsilon\backslash n+1}_{n+1}\rangle}$$

$$+\frac{1}{\binom{n}{l}}\sum_{P}(\sum_{p\in P}\alpha_p)\ \mathrm{d}\log y\langle P\rangle$$

$$=\frac{1}{\binom{n}{l}}\sum_{p\in P_{n+1}}\alpha_p\,\mathrm{d}\log\frac{y\langle P_{n+1}^{0\backslash p}\rangle^{\binom{n}{l}}\prod_{Q_p}y\langle Q_p\rangle}{y\langle P_{n+1}^{0\backslash n+1}\rangle^{\binom{n}{l}}\prod_{Q_{n+1}}y\langle Q_{n+1}\rangle}$$

$$+\frac{1}{\binom{n}{l}}\sum_{p\in P_{n+1}^{\perp}}\alpha_p\ \mathrm{d}\log\frac{y\langle P_{n+1}\rangle^{\binom{n}{l}}\prod_{Q_p}y\langle Q_p\rangle}{y\langle P_{n+1}^{p\backslash n+1}\rangle^{\binom{n}{l}}\prod_{Q_{n+1}}y\langle Q_{n+1}\rangle},$$

$$\theta^{-\alpha}_{P_{n+1}P_{n+1}^{p_\upsilon\backslash p_\nu}}=-\frac{p_\upsilon-p_\nu}{|p_\upsilon-p_\nu|}(-1)^{p_\upsilon-\nu-\upsilon+l}\alpha_{p_\upsilon}\,\mathrm{d}\log\frac{y\langle P_{n+1}^{0\backslash n+1}\rangle y\langle P_{n+1}^{p_\upsilon\backslash p_\nu}\rangle}{y\langle P_{n+1}^{p_\upsilon\backslash n+1}\rangle y\langle P_{n+1}^{0\backslash p_\nu}\rangle},$$

$$\theta^{-\alpha}_{P_{n+1}Q_{n+1}}=0\quad\text{otherwise},$$

where P_{n+1} and Q_{n+1} are multi-indices of type

$$P_{n+1}=\{p_0,\ldots,p_l,n+1\}\quad 0<p_0\leqslant\cdots\leqslant p_l<n+1,$$

$$Q_{n+1}=\{q_0,\ldots,q_l,n+1\}\quad 0<q_0\leqslant\cdots\leqslant q_l<n+1,$$

and $P_{n+1}^{p_\upsilon\backslash p_\nu}$ is the multi-index corresponding to the set $(P_{n+1}\setminus\{p_\nu\})\cup\{p_\upsilon\}, p_\nu\in P_{n+1}=\{p_1,\ldots,p_l,n+1\}, p_\upsilon\in P_{n+1}^{\perp}=\{0,p_{l+1},\ldots,p_n\}$. The $\theta_{n+1}^{-\alpha}[y]$ is invariant under the action of $\mathrm{GL}_{l+1}(\mathbb{C})\times(\mathbb{C}^*)^{n+2}$; hence it induces a system differential equations on the configuration space $X(l,k)$.

REMARK 4.5.2. The connection form $\Theta_{n+1}^{-\alpha}[y]$ is obtained from $\Theta_0^{\alpha}[x]$ just by replacing

$$\alpha\to-\alpha,\quad k\to l\quad x\to y,\quad J\to P,$$

$$\text{the index } 0\to\ \text{the index } n+1.$$

4.6. The following proposition concludes our proof of the main theorem.

PROPOSITION 4.6.1. *The right-hand side of* (3.1.1) *satisfies the system* (4.4.1).

Proof. Since we have

$$\begin{aligned}\mathrm{d}(g_1\Pi^-_{n+1}(-\alpha,y)g_2)&=g_1\,\mathrm{d}\Pi^-_{n+1}(-\alpha,y)g_2\\&=g_1\Theta^{-\alpha}_{n+1}[y]\Pi^-_{n+1}(-\alpha,y)g_2\\&=(g_1\Theta^{-\alpha}_{n+1}[y]g_1^{-1})(g_1\Pi^-_{n+1}(-\alpha,y)g_2),\end{aligned}$$

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for $g_1, g_2 \in \mathrm{GL}_{\binom{n}{l}}(\mathbb{C})$, the connection form associated to the right hand side of (3.1.1) is the pull-back of

$$^{t}E_{lk}(\wedge^l I_{ch}(\alpha)^{-1})\Theta^{-\alpha}_{n+1}[y](\wedge^l I_{ch}(\alpha))^{t}E_{lk}^{-1}$$

by the map $\perp$: $[x] \mapsto [x]^{\perp} = [y]$. By virtue of Lemma 2.1.2, we have only to substitute $y\langle P\rangle$ into $x\langle P^{\perp}\rangle$ in order to get the pull-back $\perp^* \, (\Theta^{-\alpha}_{n+1}[y])$ of $\Theta^{-\alpha}_{n+1}[y]$ by $\perp$. We put $P = J^{\perp}$ and

$$\begin{aligned} P_{n+1} &= \{p_1, \ldots, p_\nu, \ldots, p_l, n+1\} \\ &= \{j_{k+1}, \ldots, j_\lambda, \ldots, j_{k+l}, n+1\} = J_0^{\perp}, \quad p_\nu = j_\lambda, \\ P_{n+1}^{\perp} &= \{0, p_{l+1}, \ldots, p_\upsilon, \ldots, p_{l+k}\} \\ &= \{0, j_1, \ldots, j_\kappa, \ldots, j_k\} = J_0, \quad p_\upsilon = j_\kappa; \end{aligned}$$

note that

$$p_\nu \in P_{n+1} \Leftrightarrow j_\lambda \in J_0^{\perp}, \quad p_\upsilon \in P_{n+1}^{\perp} \Leftrightarrow j_\kappa \in J_0,$$

$$\nu = \lambda - k, \quad \upsilon = \kappa + l,$$

$$(P_{n+1})^{\perp} = J_0, \qquad (P_{n+1}^{0\backslash n+1})^{\perp} = J_0^{n+1\backslash 0}, \qquad (P_{n+1}^{0\backslash p_\nu})^{\perp} = J_0^{j_\lambda\backslash 0},$$

$$(P_{n+1}^{p_\upsilon\backslash n+1})^{\perp} = J_0^{n+1\backslash j_\kappa}, (P_{n+1}^{p_\upsilon\backslash p_\nu})^{\perp} = J_0^{j_\lambda\backslash j_\kappa},$$

and that

$$\alpha_0 \,\mathrm{d}\log \frac{x\langle J_0\rangle}{x\langle J_0^{n+1\backslash j_\kappa}\rangle} = -\left(\sum_{j=1}^{n+1}\alpha_j\right)\,\mathrm{d}\log \frac{x\langle J_0\rangle}{x\langle J_0^{n+1\backslash j_\kappa}\rangle}.$$

We have

$$\begin{aligned} \perp^* \, (\theta^{-\alpha}_{P_{n+1}P_{n+1}}) = &\sum_{j_\lambda \in J_0^{\perp}} \alpha_{j_\lambda} \,\mathrm{d}\log \frac{x\langle J_0^{j_\lambda\backslash 0}\rangle}{x\langle J_0^{n+1\backslash 0}\rangle} \\ &+ \sum_{j_\kappa \in J_0} \alpha_{j_\kappa} \,\mathrm{d}\log \frac{x\langle J_0\rangle}{x\langle J_0^{n+1\backslash j_\kappa}\rangle} \\ &+ \frac{1}{\binom{n}{l}} \sum_P \Big(\sum_{j \in P^{\perp}} -\alpha_j\Big) \,\mathrm{d}\log x\langle P^{\perp}\rangle \end{aligned}$$

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$$= \sum_{j_\kappa \in J_0} \alpha_{j_\kappa} \, \mathrm{d}\log \frac{x\langle J_0^{n+1\setminus 0}\rangle}{x\langle J_0^{n+1\setminus j_\kappa}\rangle}$$

$$+ \sum_{j_\lambda \in J_0^\perp} \alpha_{j_\lambda} \, \mathrm{d}\log \frac{x\langle J_0^{j_\lambda\setminus 0}\rangle}{x\langle J_0\rangle}$$

$$-\frac{1}{\binom{n}{k}} \sum_J \left(\sum_{j \in J} \alpha_j \right) \mathrm{d}\log x\langle J\rangle,$$

$$\perp^* (\theta^{-\alpha}_{P_{n+1}P_{n+1}^{p_\upsilon\setminus p_\nu}}) = -\frac{j_\kappa - j_\lambda}{|j_\kappa - j_\lambda|}(-1)^{j_\kappa-\kappa-l-\lambda+k+l}$$

$$\times \alpha_{j_\kappa} \, \mathrm{d}\log \frac{x\langle J_0^{n+1\setminus 0}\rangle x\langle J_0^{j_\lambda\setminus j_\kappa}\rangle}{x\langle J_0^{n+1\setminus j_\kappa}\rangle x\langle J_0^{j_\lambda\setminus 0}\rangle},$$

$$= \frac{j_\lambda - j_\kappa}{|j_\lambda - j_\kappa|}(-1)^{j_\kappa-\kappa-\lambda+k} \alpha_{j_\kappa} \, d\log \frac{x\langle J_0^{n+1\setminus 0}\rangle x\langle J_0^{j_\lambda\setminus j_\kappa}\rangle}{x\langle J_0^{n+1\setminus j_\kappa}\rangle x\langle J_0^{j_\lambda\setminus 0}\rangle}.$$

By taking the conjugate, we see

$$^tE_{lk}(\wedge^l I_{ch}(\alpha)^{-1}) \perp^* (\Theta^{-\alpha}_{n+1}[y])(\wedge^l I_{ch}(\alpha))E_{lk}^{-1},$$

$\perp^* (\theta^{-\alpha}_{P_{n+1}P_{n+1}})$ is the (J_0, J_0)-component and $\perp^* (\theta^{-\alpha}_{P_{n+1}P_{n+1}^{p_\upsilon\setminus p_\nu}})$ is multiplied

$$\left(\prod_{p \in P_{n+1}} (-1)^p \alpha_p \right) \left(\prod_{p \in P_{n+1}^{p_\upsilon\setminus p_\nu}} (-1)^p \alpha_p^{-1} \right)$$

$$= (-1)^{p_\nu - p_\upsilon} \alpha_{p_\nu}/\alpha_{p_\upsilon} = (-1)^{j_\lambda - j_\kappa} \alpha_{j_\lambda}/\alpha_{j_\kappa}$$

and the $(J_0, J_0^{j_\lambda\setminus j_\kappa})$-component, which are equal to those of $\Theta_0^\alpha[x]$. □

The Cauchy fundamental theorem together with Lemma 4.3.2 and Proposition 4.6.1 proves our main theorem.

5. Examples

5.1. We recall the definition of the hypergeometric series $F(a, b, c; z)$ of type (k, l)

$$F(a, b, c; z) = \sum_m \frac{\prod_{i=1}^k (a_i; |m_i|) \prod_{j=1}^l (b_j; |^t m_j|)}{(c; |m|)m!} z^m,$$

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where $m = (m_{ij})$ runs over the set $\mathbb{Z}^{kl}_{\geqslant 0}$ and

$$|m_i| = \sum_{j=1}^{l} m_{ij}, \qquad |{}^t m_j| = \sum_{i=1}^{k} m_{ij},$$

$$|m| = \sum_{1\leqslant i\leqslant k, 1\leqslant j\leqslant l} m_{ij}, \qquad m! = \prod_{1\leqslant i\leqslant k, 1\leqslant j\leqslant l} m_{ij}!,$$

$$a = (a_1, \ldots, a_k) \in \mathbb{C}^k, \qquad b = (b_1, \ldots, b_l) \in \mathbb{C}^l, \quad c \in \mathbb{C} - \mathbb{Z}_{<0},$$

$$(c; |m|) = c(c+1)\ldots(c+|m|-1);$$

$z = (z_{ij})$ is an element of $\mathbb{C}^{kl}$ near to 0 and

$$z^m = \prod_{1\leqslant i\leqslant k, 1\leqslant j\leqslant l} z_{ij}^{m_{ij}}.$$

Note that $F(a, b, c; z)$ is the Gauss hypergeometric series

$$\sum_{m=0}^{\infty} \frac{(a; m)(b; m)}{(c; m)m!} z^m,$$

when $(k, l) = (1, 1)$, and that it is the Appell hypergeometric series F_1

$$\sum_{m_1, m_2=0}^{\infty} \frac{(\alpha; m_1+m_2)(\beta_1; m_1)(\beta_2; m_2)}{(\gamma; m_1+m_2)m_1!m_2!} x^{m_1} y^{m_2},$$

when $(k, l) = (1, 2)$. It is shown in [Kit1] that under the condition that any of

$$a_i, c - \sum_{i=1}^{k} a_i, b_i, c - \sum_{j=1}^{l} b_j$$

is not integral, $F(a, b, c; z)$ admits two integral representations of Euler type

$$F(a, b, c; z) = \frac{\Gamma(c)}{\Gamma(c - \sum_{i=1}^{k} a_i) \prod_{i=1}^{k} \Gamma(a_i)} \int_{\Delta^k} u^{\alpha}(s, z) \varphi^k(s)$$
$$= \frac{\Gamma(c)}{\Gamma(c - \sum_{j=1}^{l} b_j) \prod_{j=1}^{l} \Gamma(b_j)} \int_{\Delta^l} u^{-\alpha}(s', z) \varphi^l(s'), \qquad (5.1.1)$$

where $s = (s_0, \ldots, s_{k-1}), s' = (s'_1, \ldots, s'_l)$,

$$u^{\alpha}(s, z) = \left(\prod_{i=1}^{k} {s_{i-1}}^{a_i}\right) \left(1 - \sum_{i=1}^{k} s_{i-1}\right)^{c - \Sigma a_i}$$

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$$\times \left(\prod_{j=1}^{l} \left(1 - \sum_{i=1}^{k} z_{ij} s_{i-1} \right)^{-b_j} \right),$$

$$u^{-\alpha}(s', z) = \left(\prod_{i=1}^{k} \left(1 - \sum_{j=1}^{l} z_{ij} s'_j \right)^{-a_i} \right)$$

$$\times \left(\prod_{j=1}^{l} {s'_j}^{b_j} \right) \left(1 - \sum_{j=1}^{l} s'_j \right)^{c-\Sigma b_j},$$

$$\alpha = (\alpha_0, \ldots, \alpha_{n+1})$$

$$= \left(a_1, \ldots, a_k, c - \sum_{i=1}^{k} a_i, -b_1, \ldots, -b_l, -c + \sum_{j=1}^{l} b_j \right),$$

$$\varphi^k(s) = \frac{\mathrm{d}s_0 \wedge \cdots \wedge \mathrm{d}s_{k-1}}{(\prod_{i=0}^{k-1} s_i)(1 - \sum_{i=0}^{k-1} s_i)},$$

$$\varphi^l(s') = \frac{\mathrm{d}s'_1 \wedge \cdots \wedge \mathrm{d}s'_l}{(\prod_{j=1}^{l} s'_j)(1 - \sum_{j=1}^{l} s'_j)},$$

$$\Delta^k = \left\{ s \in \mathbb{R}^k \mid s_0, \ldots, s_{k-1}, 1 - \sum_{i=0}^{k-1} s_i > 0 \right\},$$

$$\Delta^l = \left\{ s' \in \mathbb{R}^l \mid s'_1, \ldots, s'_l, 1 - \sum_{j=1}^{l} s'_j > 0 \right\}$$

and the branch $u^{\alpha}(s, z)$ on Δ^k and $u^{-\alpha}(s'z)$ on Δ^l are defined by assigning arguments near to zero for all linear forms of s in $u^{\alpha}(s, z)$ on Δ^k and for those of s' in $u^{-\alpha}(s'z)$ on Δ^l. The identity (5.1.1) implies that

$$\frac{\Gamma(c-b)\Gamma(b)}{\Gamma(c-a)\Gamma(a)} \int_0^1 {s_0}^a (1 - s_0)^{c-a}$$

$$\times (1 - z s_0)^{-b} \frac{\mathrm{d}s_0}{s_0(1 - s_0)}$$

$$= \int_0^1 (1 - z s'_1)^{-a} {s'_1}^b (1 - s'_1)^{c-b} \frac{\mathrm{d}s'_1}{s'_1(1 - s'_1)},$$

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when $(k,l)=(1,1)$, and that

$$\frac{\Gamma(c-b_1-b_2)\Gamma(b_1)\Gamma(b_2)}{\Gamma(c-a)\Gamma(a)}\int_0^1 {s_0}^a(1-s_0)^{c-a}$$

$$\times(1-z_1s_0)^{-b_1}(1-z_2s_0)^{-b_2}\frac{\mathrm{d}s_0}{s_0(1-s_0)}$$

$$=\int_{\Delta^2}(1-z_1s_1'-z_2s_2')^{-a}{s_1'}^{b_1}{s_2'}^{b_2}$$

$$\times(1-s_1'-s_2')^{c-b_1-b_2}\frac{\mathrm{d}s_1'\wedge\mathrm{d}s_2'}{s_1's_2'(1-s_1'-s_2')},$$

when $(k,l)=(1,2)$.

5.2. We show the identity (5.1.1) between the k-fold integral and the l-fold integral in the previous section by picking up the top-left component of (3.1.1) in our main theorem. Take a $(k\times l)$-matrix z near to $z[\xi_k]$. Our main theorem and Lemma 2.1.2 says

$$F^+_{I_0I_0}(\alpha,[x_z])=c(I_0,I_0)F^-_{I_0^\perp I_0^\perp}(-\alpha,[y_z]),\tag{5.2.1}$$

where

$$I_0=\{0,1,\ldots,k\},\qquad I_0^\perp=\{k+1,\ldots,n,n+1\},$$

x_z and y_z are in (2.1.1) and

$$c(I_0,I_0)=V(\alpha)\frac{(-\alpha_{k+1})\ldots(-\alpha_n)}{(2\pi\sqrt{-1})^l}\frac{1}{\mathrm{e}^{2\pi\sqrt{-1}l(\alpha_0+\cdots+\alpha_k)}}$$

$$\times\prod_{j=1}^{l}\frac{\mathrm{e}^{2\pi\sqrt{-1}\alpha_{k+j}}-1}{\mathrm{e}^{2\pi\sqrt{-1}(l+1-j)\alpha_{k+j}}}.$$

By using the formula

$$\Gamma(c)\Gamma(-c)=\frac{2\pi\sqrt{-1}}{-c}\frac{\mathrm{e}^{\pi\sqrt{-1}c}}{\mathrm{e}^{2\pi\sqrt{-1}c}-1},$$

the constant $c(I_0,I_0)$ can be written as

$$c(I_0,I_0)=\frac{\Gamma(\alpha_0)\ldots\Gamma(\alpha_k)}{\Gamma(-\alpha_{k+1})\ldots\Gamma(-\alpha_{n+1})}$$

$$\times \prod_{i=0}^{k} \mathrm{e}^{-\pi\sqrt{-1}(i-k+l)\alpha_i} \prod_{j=1}^{l} \mathrm{e}^{-\pi\sqrt{-1}(l+1-j)\alpha_{k+j}}. \tag{5.2.2}$$

Put $(s_0, \ldots, s_{k-1}) := (t_0/t_k, \ldots, t_{k-1}/t_k)$. Since

$$\varphi^k(s) = \tau_{x_z}^*(\varphi_{I_0})(t, x_z), \qquad \Delta^k = \Delta_{I_0}(x_z)$$

and the argument of $(-1)^{k-i}s_i = L_i(t, x_z)/L_{n+1}(t, x_z)(0 \leqslant i \leqslant k-1)$ is assigned near to $(k-i)\pi$ on Δ^k, we have

$$\int_{\Delta^k} u^\alpha(s, z)\varphi^k(s)$$

$$= \left(\prod_{i=0}^{k} \mathrm{e}^{\pi\sqrt{-1}(k-i)\alpha_i}\right) \cdot D(x_z) \cdot F^+_{I_0 I_0}(\alpha, [x_z]). \tag{5.2.3}$$

Put $(s'_1, \ldots, s'_l) := (t_1/t_0, \ldots, t_l/t_0)$ and note that

$$\varphi^l(s') = \tau_{y_z}^*(\varphi_{I_0^\perp})(t, y_z), \qquad \Delta^l = \Delta_{I_0^\perp}(y_z).$$

Since the argument of

$$(-1)^{j-1}s'_j = L_{k+j}(t, y_z)/L_k(t, y_z)$$

$$= \frac{L_{k+j}(t, y_z)/L_{n+1}(t, y_z)}{L_k(t, y_z)/L_{n+1}(t, y_z)} \quad (1 \leqslant j \leqslant l)$$

is assigned near to $(j-1)\pi$ and that of $1 - s'_1 - \cdots - s'_l = L_{n+1}(t, y_z)/L_k(t, y_z)$ is assigned near to $l\pi$ on Δ^l, we have

$$\int_{\Delta^l} u^{-\alpha}(s', z)\varphi^l(s')$$

$$= \left(\prod_{j=1}^{l+1} \mathrm{e}^{-\pi\sqrt{-1}(j-1)\alpha_{k+j}}\right) \cdot D(y_z) \cdot F^-_{I_0^\perp I_0^\perp}(-\alpha, [y_z]). \tag{5.2.4}$$

Since $D(x_z) = D(y_z)$ and $\sum_{j=0}^{n+1} \alpha_j = 0$, we have

$$\int_{\Delta^k} u^\alpha(s, z)\varphi^k(s) \Big/ \int_{\Delta^l} u^{-\alpha}(s', z)\varphi^l(s')$$

$$= \left(\left(\prod_{i=0}^{k} \mathrm{e}^{\pi\sqrt{-1}(k-i)\alpha_i}\right) \cdot F^+_{I_0 I_0}(\alpha, [x_z])\right) \Big/$$

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$$\left(\left(\prod_{j=1}^{l+1} \mathrm{e}^{-\pi\sqrt{-1}(j-1)\alpha_{k+j}}\right) \cdot F^{-}_{I_0^{\perp} I_0^{\perp}}(-\alpha, [y_z])\right)$$

$$= c(I_0, I_0) \prod_{i=0}^{k} \mathrm{e}^{\pi\sqrt{-1}(k-i)\alpha_i} \prod_{j=1}^{l+1} \mathrm{e}^{\pi\sqrt{-1}(j-1)\alpha_{k+j}}$$

$$= \frac{\Gamma(\alpha_0)\dots\Gamma(\alpha_k)}{\Gamma(-\alpha_{k+1})\dots\Gamma(-\alpha_{n+1})}.$$

Hence, we conclude the argument by proving the identity (0.1) in a rigorous way.

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