TWISTED ORBIFOLD GROMOV–WITTEN INVARIANTS

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Abstract. Let $\mathcal{X}$ be a smooth proper Deligne–Mumford stack over $\mathbb{C}$. One can define twisted orbifold Gromov–Witten invariants of $\mathcal{X}$ by considering multiplicative invertible characteristic classes of various bundles on the moduli spaces of stable maps $\mathcal{X}_{g,n,d}$, cupping them with evaluation and cotangent line classes, and then integrating against the virtual fundamental class. These are more general than the twisted invariants introduced by Tseng. We express the generating series of the twisted invariants in terms of the generating series of the untwisted ones. We derive the corollaries which are used in a paper with Givental about the quantum K-theory of a complex compact manifold X.

§1. Introduction and statement of results

Twisted Gromov–Witten invariants were introduced in [9] for manifold target spaces and extended in [19] to the case of orbifolds. The original motivation was to express Gromov–Witten invariants of complete intersections (*twisted* ones) in terms of the Gromov–Witten invariants of the ambient space (the *untwisted* ones). In addition, they were used in [8] to express Gromov–Witten invariants with values in cobordism in terms of cohomological Gromov–Witten invariants.

Our results incorporate and generalize all of the above: we consider three types of twisting classes. These are multiplicative cohomological classes of bundles of the form $\pi_* E$, where π is the universal family of the moduli space of stable maps to an orbifold $\mathcal{X}$. The main tool in the computations is the Grothendieck–Riemann–Roch theorem for stacks of [18], applied to the morphism π; this gives differential equations satisfied by the generating functions of the twisted Gromov–Witten invariants. To the genus 0 Gromov–Witten potential of an orbifold $\mathcal{X}$ one can associate an overruled Lagrangian cone in a symplectic space $\mathcal{H}$, as explained in Section 2. Solving

the differential equations for each type of twisting has an interpretation in terms of the geometry of the cone: change its position by a symplectic transformation, a translation of the origin, and a change of polarization of $\mathcal{H}$. Our motivation comes from studying the quantum K-theory of a manifold X (see [12]), more precisely, from trying to express Euler characteristics on the (virtual) orbifolds $X_{0,n,d}$ in terms of cohomological Gromov–Witten invariants. However, they have other applications, for instance, recovering the work of [8] on quantum extraordinary cohomology.

In [17], Teleman studies a group action on 2-dimensional quantum field theories. Our results match his, if the field theories come from Gromov–Witten theory.

Let $\mathcal{X}$ be a compact orbifold. Moduli spaces of orbimaps to orbifolds have been constructed in [7] in the setup of symplectic orbifolds and in [4] in the context of Deligne–Mumford stacks. Informally, the domain curve is allowed to have nontrivial orbifold structure at the marked points and nodes. We denote the moduli spaces of degree d maps of genus g with n marked points by $\mathcal{X}_{g,n,d}$.

Just like in the case of manifold target spaces, there are evaluation maps $\overline{\mathrm{ev}}_i$ at the marked points. Although it is clear how these maps are defined on geometric points, it turns out that to have well-defined morphisms of Deligne–Mumford stacks, the target of the evaluation maps is the rigidified inertia stack of $\mathcal{X}$.

We first define a related object, the inertia stack $I\mathcal{X}$, as follows. Around any point $x \in \mathcal{X}$ there is a local chart $(\widetilde{U}_x, G_x)$ such that locally $\mathcal{X}$ is represented as the quotient of $\widetilde{U}_x$ by G_x. Consider the set of conjugacy classes $(1) = (h_x^1), (h_x^2), \ldots, (h_x^{n_x})$ in G_x. Define

$$I\mathcal{X} := \left\{ \left(x, (h_x^i)\right) \mid i = 1, 2, \ldots, n_x \right\}.$$

Pick an element h_x^i in each conjugacy class. Then a local chart on $I\mathcal{X}$ is given by

$$\coprod_{i=1}^{n_x} \widetilde{U}_x^{(h_x^i)} / Z_{G_x}(h_x^i),$$

where $Z_{G_x}(h_x^i)$ is the centralizer of h_x^i in G_x.

The rigidified inertia stack, which we denote $\overline{I\mathcal{X}}$, is defined by taking the quotient at $(x,(g))$ of the automorphism group by the cyclic subgroup generated by g. So, whereas a local chart at $(x,(g))$ on $I\mathcal{X}$ is given by

$\widetilde{U_g}/Z_{G_x}(g)$, on $\overline{I\mathcal{X}}$ a local chart is $\widetilde{U_g}/[Z_{G_x}(g)/\langle g \rangle]$. It is in general disconnected, even if $\mathcal{X}$ is connected. We write $\overline{I\mathcal{X}} := \coprod_{\mu} \bar{\mathcal{X}}_\mu$. The distinguished component corresponding to the identity is a copy of $\mathcal{X}$, and throughout we will label it $\mathcal{X}_0$ to distinguish it from other components of $I\mathcal{X}$. We denote by $\iota : I\mathcal{X} \to I\mathcal{X}$ the involution which maps $(x, (g))$ to $(x, (g^{-1}))$. It descends to an involution on $\overline{I\mathcal{X}}$, which we also denote ι. We write $\mathcal{X}_{\mu^I} := \iota(\mathcal{X}_\mu)$. There is a natural map $q : I\mathcal{X} \to \mathcal{X}$.

The orbifold Poincaré pairing on $I\mathcal{X}$ is defined for $a \in H^*(\mathcal{X}_\mu, \mathbb{C})$, $b \in H^*(\mathcal{X}_{\mu^I}, \mathbb{C})$ as

$$(a, b)_{\mathrm{orb}} := \int_{\mathcal{X}_\mu} a \cup \iota^* b.$$

Here $I\mathcal{X}$ and $\overline{I\mathcal{X}}$ have the same geometric points (coarse spaces); hence, we can identify the rings $H^*(I\mathcal{X}, \mathbb{C})$ and $H^*(\overline{I\mathcal{X}}, \mathbb{C})$. This allows us to pretend that the cohomological pullbacks by the maps $\overline{\mathrm{ev}}_i$ have domain $H^*(I\mathcal{X}, \mathbb{C})$. We can use the maps $\overline{\mathrm{ev}}_i$ to decompose $\mathcal{X}_{g,n,d}$ as a union of closed and open substacks:

$$\mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)} := \mathcal{X}_{g,n,d} \cap (\overline{\mathrm{ev}}_1)^{-1}(\bar{\mathcal{X}}_{\mu_1}) \cap \cdots \cap (\overline{\mathrm{ev}}_n)^{-1}(\bar{\mathcal{X}}_{\mu_n}).$$

For each i we denote by $\bar{\psi}_i = c_1(\bar{L}_i)$, where the line bundle $\bar{L}_i$ has fiber over each point $(\mathcal{C}, x_1, \ldots, x_n, f)$, the cotangent line to the coarse curve C at x_i.

We denote the universal family by $\pi : \mathcal{U}_{g,n,d} \to \mathcal{X}_{g,n,d}$. Here $\mathcal{U}_{g,n,d}$ can be identified with $\bigcup_{(\mu_1,\ldots,\mu_n)} \mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0)}$. Since the extra marked point on the universal family has trivial orbifold structure, the map $\overline{\mathrm{ev}}_{n+1}$ lands in $\mathcal{X}_0$. We will write ev_{n+1} throughout. The moduli spaces $\mathcal{X}_{g,n,d}$ have perfect obstruction theory and are equipped with virtual fundamental classes $[\mathcal{X}_{g,n,d}] \in H_*(\mathcal{X}_{g,n,d}, \mathbb{Q})$. Orbifold Gromov–Witten invariants are obtained by integrating $\bar{\psi}_i$ and evaluation classes on these cycles. We use correlator notation:

$$\langle a_1 \bar{\psi}^{k_1}, \ldots, a_n \bar{\psi}^{k_n} \rangle_{g,n,d} := \int_{[\mathcal{X}_{g,n,d}]} \prod_{i=1}^{n} \mathrm{ev}_i^* a_i \bar{\psi}_i^{k_i}.$$

Their generating series are functions on a suitable infinite-dimensional vector space $\mathcal{H}_+$, which we describe below. Let $\Lambda := \mathbb{C}[[Q]]$ be the Novikov

ring which is a completion of the semigroup ring of degrees of holomorphic curves in $\mathcal{X}$, and let

$$\mathcal{H} := H^*(I\mathcal{X}, \Lambda)((z)).$$

We equip $\mathcal{H}$ with the symplectic form

$$\Omega(\mathbf{f}, \mathbf{g}) := \oint_{z=0} \big(\mathbf{f}(z), \mathbf{g}(-z)\big)_{\text{orb}} dz.$$

Consider the following polarization of $\mathcal{H}$:

$$\mathcal{H}_+ := H^*(I\mathcal{X}, \mathbb{C})[[z]] \qquad \text{and} \qquad \mathcal{H}_- := z^{-1} H^*(I\mathcal{X}, \mathbb{C})[z^{-1}].$$

Let $\mathbf{t}(z) \in \mathcal{H}_+$. The genus g descendant potential and the total descendant potential are defined as

$$\mathcal{F}_{\mathcal{X}}^g\big(\mathbf{t}(z)\big) = \sum_{d,n} \frac{Q^d}{n!} \big\langle \mathbf{t}(\bar{\psi}), \ldots, \mathbf{t}(\bar{\psi}) \big\rangle_{g,n,d},$$

$$\mathcal{D}_{\mathcal{X}}(\mathbf{t}) = \exp\Big(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g(\mathbf{t})\Big),$$

respectively. Then $\mathcal{D}_{\mathcal{X}}$ is a well-defined formal function on $\mathcal{H}_+$ taking values in $\Lambda \otimes \mathbb{C}[[\hbar, \hbar^{-1}]]$. Also it is well known that the differential of the genus 0 potential gives rise to a cone $\mathcal{L}^H \subset \mathcal{H}$ with nice geometric properties (see Theorem 2.6).

Twisted Gromov–Witten invariants are obtained from the usual ones by systematically inserting in the correlators multiplicative classes of certain bundles. For a vector bundle E, a general multiplicative class is of the form

$$\mathcal{A}(E) = \exp\Big(\sum_{k \geq 0} s_k \operatorname{ch}_k E\Big).$$

We want to consider three types of twistings, each by several possibly different multiplicative characteristic classes:

- twistings by a finite number of multiplicative classes $\mathcal{A}_\alpha(\pi_*(\operatorname{ev}_{n+1}^* E))$, where $E \in K^0(\mathcal{X})$;
- twistings by classes $\mathcal{B}_\beta$ (kappa classes) of the form

$$\mathcal{B}_{g,n,d} = \prod_{\beta=1}^{i_B} \mathcal{B}_\beta\big(\pi_*\big(f_\beta(L_{n+1}^{-1}) - f_\beta(1)\big)\big),$$

where L_{n+1} is the cotangent line bundle at the extra marked point on the universal curve, f_β are polynomials with coefficients in $\mathrm{ev}_{n+1}^* K^0(\mathcal{X})$, and 1 is the trivial line bundle; and

- twistings by nodal classes $\mathcal{C}_\delta$ of the form

$$\mathcal{C}_{g,n,d} = \prod_\mu \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu \big(\pi_* (\mathrm{ev}_{n+1}^* F_{\delta\mu} \otimes i_{\mu*} \mathcal{O}_{\mathcal{Z}_\mu}) \big),$$

where $F_{\delta\mu} \in K^0(\mathcal{X})$; see Section 2 for the precise definition of $\mathcal{Z}_\mu$—roughly speaking, it parameterizes nodes with fixed orbifold type; we denote by i_μ the corresponding inclusion $\mathcal{Z}_\mu \to \mathcal{U}_{g,n,d}$, so we allow different types of twistings localized near the loci $\mathcal{Z}_\mu$.

We will refer to these as type $\mathcal{A}, \mathcal{B}, \mathcal{C}$ twistings, respectively. So a twisted Gromov–Witten invariant will be an integral of the form

$$\int_{[\mathcal{X}_{g,n,d}]} \prod_{i=1}^n \mathrm{ev}_i^* a_i \bar{\psi}_i^{k_i} \mathcal{A}(\cdot) \mathcal{B}(\cdot) \mathcal{C}(\cdot).$$

These can be packed in generating series—the twisted potentials $\mathcal{F}_{\mathcal{A},\mathcal{B},\mathcal{C}}^g$, $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$, which we can regard as functions on the same space $\mathcal{H}_+$. We postpone the precise definitions to Section 2. We will write $\mathcal{D}_{\mathcal{A},\mathcal{B}}, \mathcal{L}_{\mathcal{A}}$, and so forth, for objects associated to twisted Gromov–Witten invariants of the types specified in notation.

The main theorems of the paper describe how the twistings change the potentials and the corresponding Lagrangian cones $\mathcal{L}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ (which we define in Section 2).

THEOREM 1.1. *The cone $\mathcal{L}_{\mathcal{A}}$ is obtained from $\mathcal{L}^H$ after rotation by a symplectic transformation*

$$\mathcal{L}_{\mathcal{A}} = \Big(\prod_\alpha \Delta_\alpha \Big) \mathcal{L}^H.$$

We write explicit formulas for each Δ_α in Remark 1.5.

Let now $\mathbf{L}_z$ be a line bundle with first Chern class z.

THEOREM 1.2. *The twisting by the classes $\mathcal{B}_{g,n,d}$ has the same effect as a translation on the Fock space:*

$$(1.1) \qquad \mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}(\mathbf{t}) = \mathcal{D}_{\mathcal{A},\mathcal{C}} \Big(\mathbf{t} + z - z \prod_{i=1}^{i_B} \mathcal{B}_\beta \Big(-\frac{f_\beta(\mathbf{L}_z^{-1}) - f_\beta(1)}{\mathbf{L}_z - 1} \Big) \Big) \cdot K_{\mathcal{B}},$$

where $K_{\mathcal{B}}$ is a constant discussed in the proof.

A related result for manifold target spaces is in [14].

THEOREM 1.3. *The potential $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ satisfies the differential equation*

$$\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}} = \exp\Big(\frac{\hbar}{2} \sum_{a,b,\alpha,\beta,\mu} A^{\mu}_{a,\alpha;b;\beta} \partial^{\alpha,\mu}_{a} \partial^{\beta,\mu^{I}}_{b}\Big) \mathcal{D}_{\mathcal{A},\mathcal{B}},$$

where the coefficients $A^{\mu}_{a,\alpha;b;\beta}$ are defined by (4.12) in Section 4. This is equivalent to considering the potential $\mathcal{D}_{\mathcal{A},\mathcal{B}}$ as a generating function with respect to a new polarization $\mathcal{H} = \mathcal{H}_{+} \oplus \mathcal{H}_{-,\mathcal{C}}$. We give a precise linear transformation of Darboux coordinates on $\mathcal{H}$ in (4.24).

A few remarks are in order at this point.

REMARK 1.4. The study of the K-theoretic Gromov–Witten invariants of a manifold X in [12] leads naturally to considering these twisted Gromov–Witten invariants. Briefly put, to compute K-theoretic Gromov–Witten invariants of X in terms of cohomological ones, one needs to consider cohomological integrals twisted by certain Todd-like classes (see Section 6) of the (virtual) tangent bundle of $X_{0,n,d}$. Proposition 5.3 expresses this tangent bundle as a sum of three contributions—one of each type.

REMARK 1.5. Theorem 1.1 is a rather straightforward generalization of the main theorem in [19], the only difference being that we consider more than one class $\mathcal{A}_{\alpha}$. If the twisting data $\mathcal{A}$ are given by the multiplicative class $\mathcal{A}(\cdot) = \exp(\sum s_k \operatorname{ch}_k(\cdot))$ and by $E \in K^0(\mathcal{X})$, then the symplectic transformation Δ is defined as

$$\Delta := \exp\Big(\sum_{k \geq 0} s_k \Big(\sum_{m \geq 0} \frac{(A_m)_{k+1-m} z^{m-1}}{m!} + \frac{\operatorname{ch}_k(E^{(0)})}{2}\Big)\Big),$$

where by $(A_m)_j$ we mean the degree j part of operators of ordinary multiplication by certain elements $A_m \in H^*(I\mathcal{X})$. To define A_m we introduce more notation. Let r_{μ} be the order of each element in the conjugacy class which is labeled by $\mathcal{X}_{\mu}$. The restriction of the bundle E to $\mathcal{X}_{\mu}$ decomposes into characters; let $E^{(l)}_{\mu}$ be the subbundle on which every element of the conjugacy class acts with eigenvalue $e^{2\pi i l / r_{\mu}}$. The Bernoulli polynomials $B_m(x)$ are defined by

$$\frac{t e^{tx}}{e^t - 1} = \sum_{m \geq 0} \frac{B_m(x) t^m}{m!}.$$

Then

$$(A_m)_{|\mathcal{X}_\mu} := \sum_{l=0}^{l=r_\mu-1} B_m\Big(\frac{l}{r_\mu}\Big) \operatorname{ch}(E_\mu^{(l)}).$$

The symplectic operator in Theorem 1.1 is just the product of Tseng's operators Δ_α associated to each $\mathcal{A}_\alpha$.

REMARK 1.6. The decomposition

$$H^*(I\mathcal{X}, \mathbb{C})((z^{-1})) = \oplus H^*(\mathcal{X}_\mu, \mathbb{C})((z^{-1}))$$

is preserved by the action of this loop group element. The element A_m acts by cup product multiplication on each $H^*(\mathcal{X}_\mu)$.

REMARK 1.7. Theorem 1.1 can be extended to a statement about the total descendant potential using the quantization formalism of [10]. It reads:

$$\mathcal{D}_\mathcal{A}(\mathbf{q}) \approx \prod_\alpha \widehat{\Delta}_\alpha \mathcal{D}_\mathcal{X}(\mathbf{q}),$$

where $\widehat{\Delta}$ denotes the quantization of the operator Δ and where the symbol $\approx$ means that the two sides are equal up to a (precisely determined) scalar factor.

REMARK 1.8. Another way to obtain a basis for the new space $\mathcal{H}_{-,\mathcal{C}}$ of the new polarization from Theorem 1.3 is the following. For each μ, let the series $u_\mu(z)$ be defined by

$$\frac{z}{u_\mu(z)} = \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu\big((q^*F_{\delta\mu})_\mu \otimes (-\mathbf{L}_{-z})\big).$$

Moreover, define the Laurent series $v_{k,\mu}$, $k = 0, 1, 2, \dots$ by

$$\frac{1}{u_\mu(-x-y)} = \sum_{k\geq 0} \big(u_\mu(x)\big)^k v_{k,\mu}\big(u(y)\big),$$

where we expand the left-hand side in the region where $|x| < |y|$. Then $\mathcal{H}_{-,\mathcal{C}} = \bigoplus_\mu \mathcal{H}_{-,\mathcal{C}}^\mu$, and each $\mathcal{H}_{-,\mathcal{C}}^\mu$ is spanned by $\{\varphi_{\alpha,\mu} v_{k,\mu}(u(z))\}$, where $\{\varphi_{\alpha,\mu}\}$ runs over a basis of $H^*(\mathcal{X}_\mu, \mathbb{C})$ and k runs from 0 to ∞.

The rest of the paper is structured as follows. Section 2 is used to introduce the main objects of study: the moduli spaces $\mathcal{X}_{g,n,d}$ and the Gromov–Witten theory of $\mathcal{X}$, the symplectic space $\mathcal{H}$, and the (twisted and untwisted) Gromov–Witten potentials. Section 3 contains the technical results which are the core of the computations—mainly how the twisting cohomological classes pull back on the universal family and the locus of nodes. We are then ready to prove Theorems 1.1, 1.2, and 1.3, which we do in Section 4. In Section 5 we use the results to give a concise proof of the fake quantum Hirzebruch–Riemann–Roch theorem: this was done in [8] by a very long calculation. In Section 6 we extract the corollaries which are used in [12] on quantum K-theory. Finally, in the appendix we state Toën's Grothendieck–Riemann–Roch theorem for stacks, which applied to the universal family is the starting point in the computation.

§2. Orbifold Gromov–Witten theory

Throughout this article, $\mathcal{X}$ will be a proper smooth Deligne–Mumford stack over $\mathbb{C}$ with projective coarse moduli space.

We now recall the definitions of orbicurve and of orbifold stable maps of [7, Section 2] and [4, Section 4]. The idea to extend the definition of a stable map to an orbifold target space is quite natural. One then notices that in order to obtain compact moduli spaces parameterizing these objects, one has to allow orbifold structure on the domain curve at the nodes and marked points (see, e.g., [1]).

DEFINITION 2.1. A nodal n-pointed orbicurve $(\mathcal{C}, x_1, x_2, \ldots, x_n)$ is a nodal marked complex curve such that

- $\mathcal{C}$ has trivial orbifold structure on the complement of the marked points and nodes;
- locally near a marked point, $\mathcal{C}$ is isomorphic to $[\mathrm{Spec}\ \mathbb{C}[z]/\mathbb{Z}_r]$, for some r, and the generator of $\mathbb{Z}_r$ acts by $z \mapsto \zeta z$, $\zeta^r = 1$;
- locally near a node, $\mathcal{C}$ is isomorphic to $[\mathrm{Spec}\ (\mathbb{C}[z,w]/(zw))/\mathbb{Z}_r]$, and the generator of $\mathbb{Z}_r$ acts by $z \mapsto \zeta z$, $w \mapsto \zeta^{-1}w$; we call this action *balanced* at the node.

We now define twisted stable maps.

DEFINITION 2.2. An n-pointed, genus g, degree d orbifold stable map is a representable morphism $f : \mathcal{C} \to \mathcal{X}$, whose domain is an n-pointed genus

g orbicurve $\mathcal{C}$ such that the induced morphism of the coarse moduli spaces $C \to X$ is a stable map of degree d.

We denote the moduli space parameterizing n-pointed, genus g, degree d orbifold stable maps by $\mathcal{X}_{g,n,d}$. It is proved in [5, Theorem 1.4.1] that $\mathcal{X}_{g,n,d}$ is a proper Deligne–Mumford stack. Just like the case of stable maps to manifolds, there are evaluation maps at the marked points, but these land naturally in the *rigidified* inertia orbifold of $\mathcal{X}$, which we denote $\overline{I\mathcal{X}}$.

EXAMPLE 2.3. If $\mathcal{X}$ is a global quotient Y/G, then the strata of $I\mathcal{X}$ are $Y^g/C_G(g)$ and the strata of $\overline{I\mathcal{X}}$ are $\bar{\mathcal{X}}_{(g)} := Y^g/\overline{C_G(g)}$, where $\overline{C_G(g)} = C_G(g)/\langle g \rangle$ for each conjugacy class $(g) \subset G$.

See [3, Section 4.4] and [4, Section 3.4] for the definition of $\overline{I\mathcal{X}}$ in the category of stacks.

We decompose $\mathcal{X}_{g,n,d}$ according to the target of the evaluation maps:

$$\mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)} := \mathcal{X}_{g,n,d} \cap (\overline{\mathrm{ev}}_1)^{-1}(\bar{\mathcal{X}}_{\mu_1}) \cap \cdots \cap (\overline{\mathrm{ev}}_n)^{-1}(\bar{\mathcal{X}}_{\mu_n}).$$

Since we work with cohomology with complex coefficients, we consider the cohomological pullbacks by the maps $\overline{\mathrm{ev}}_i$ having domain $H^*(I\mathcal{X}, \mathbb{C})$. Here $I\mathcal{X}$ and $\overline{I\mathcal{X}}$ have the same coarse spaces, which implies that both spaces have the same cohomology rings with rational coefficients. In fact, there is a map $\Pi: I\mathcal{X} \to \overline{I\mathcal{X}}$, which maps a point $(x, (g))$ to $(x, (\bar{g}))$. If r_i is the order of the automorphism group of x_i, then define

$$\mathrm{ev}_i^* : H^*(I\mathcal{X}, \mathbb{C}) \to H^*(\mathcal{X}_{g,n,d}, \mathbb{C}),$$
$$a \mapsto r_i^{-1}(\overline{\mathrm{ev}}_i)^*(\Pi_* a).$$

Notice that if a marked point x_i has trivial orbifold structure, $\overline{\mathrm{ev}}_i$ lands in the distinguished component $\mathcal{X}_0$ of $\overline{I\mathcal{X}}$. The universal family can be therefore identified with the diagram

$$\mathcal{U}_{g,n,d} := \bigcup_{(\mu_1,\ldots,\mu_n)} \mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0)} \xrightarrow{\mathrm{ev}_{n+1}} \mathcal{X}$$
$$\pi \downarrow$$
$$\mathcal{X}_{g,n,d}$$

In the universal family $\mathcal{U}_{g,n,d}$ lies the divisor of the ith marked point $\mathcal{D}_i$: its points parameterize maps whose domain has a distinguished node

separating two orbicurves $\mathcal{C}_0$ and $\mathcal{C}_1$; $\mathcal{C}_1$ has genus 0 and carries only three special points—the node, the ith marked point, and the $(n+1)$st marked point—and is mapped with degree 0 to $\mathcal{X}$. We write

$$\mathcal{D}_{i,(\mu_1,\ldots,\mu_n)} := \mathcal{D}_i \cap \mathcal{X}_{g,n+1,d,(\mu_1,\ldots,\mu_n,0)}.$$

We denote by σ_i the corresponding inclusions.

Let $\mathcal{Z}$ be the locus of nodes in the universal family. It has codimension 2 in $\mathcal{U}_{g,n,d}$. Denote by $p: \widetilde{\mathcal{Z}} \to \mathcal{Z}$ the double cover over $\mathcal{Z}$ given by a choice of $+,-$ at the node. For the inclusion of a stratum

$$\mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathcal{Z} \hookrightarrow \mathcal{X}_{g,n+1,d},$$

we will denote by p_i ($i=1,2$) the projections

$$p_i: \mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathcal{X}_{g_i,n_i+1,d_i}.$$

We denote by $\mathcal{Z}^{\mathrm{irr}}, \mathcal{Z}^{\mathrm{red}}$ the loci of nonseparating nodes and separating nodes, respectively, and by $i^{\mathrm{irr}}, i^{\mathrm{red}}$ the respective inclusion maps. Moreover, we will need to keep track of the orbifold structure of the node. We denote by $\mathcal{Z}_\mu$ the locus of nodes where the evaluation map at one branch lands in $\bar{\mathcal{X}}_\mu$ and by i_μ the corresponding inclusions.

The moduli spaces $\mathcal{X}_{g,n,d}$ have perfect obstruction theory (see [4]). According to [6] this yields virtual fundamental classes:

$$[\mathcal{X}_{g,n,d}] \in H_*(\mathcal{X}_{g,n,d}, \mathbb{Q}).$$

We define $\bar{\psi}_i$ to be the first Chern classes of line bundles whose fibers over each point $(\mathcal{C}, x_1, \ldots, x_n, f)$ are the cotangent spaces at x_i to the *coarse curve* C. Gromov–Witten invariants are obtained by intersecting $\bar{\psi}$ and evaluation classes against the virtual fundamental class. We write

$$\langle a_1\bar{\psi}^{k_1}, \ldots, a_n\bar{\psi}^{k_n}\rangle_{g,n,d} := \int_{[\mathcal{X}_{g,n,d}]} \prod_{i=1}^{n} \mathrm{ev}_i^*(a_i)\bar{\psi}_i^{k_i}.$$

REMARK 2.4. The moduli spaces $\mathcal{X}_{g,n,d}$ and the evaluation maps differ from those considered in [19]. However, the Gromov–Witten invariants agree, since integration in [19] is done over a weighted virtual fundamental class.

Let $\mathbb{C}[[Q]]$ be the Novikov ring which is the formal power series completion of the semigroup ring of degrees of holomorphic curves in X. (For more on Novikov rings, see [16].) We define the ground ring $\Lambda := \mathbb{C}[[Q]]$ and

$$\mathcal{H} := H^*(I\mathcal{X}, \Lambda)((z)).$$

We endow $\mathcal{H}$ with the symplectic form:

$$\Omega(\mathbf{f}, \mathbf{g}) := \oint_{z=0} \big(\mathbf{f}(z), \mathbf{g}(-z)\big)_{\mathrm{orb}} \, dz.$$

The polarization of $\mathcal{H}$,

$$\mathcal{H}_+ := H^*(I\mathcal{X}, \Lambda)[[z]], \qquad \mathcal{H}_- := z^{-1} H^*(I\mathcal{X}, \Lambda)[z^{-1}],$$

identifies $\mathcal{H}$ with $T^*\mathcal{H}_+$.

REMARK 2.5. This choice of polarization is different from the one in most places in literature. The reason is that in applying these results to quantum K-theory, we need that $e^z \in \mathcal{H}_+$ (see [12, Section 6] for details).

Let $\{\varphi_\alpha\}$ and $\{\varphi^\beta\}$ be dual bases in $H^*(I\mathcal{X}, \Lambda)$. We introduce Darboux coordinates $\{p_a^\alpha, q_b^\beta\}$ on $\mathcal{H}$, and we write

$$\mathbf{p}(z) = \sum_{a,\alpha} p_a^\alpha \varphi_\alpha (-z)^{-a-1} \in \mathcal{H}_-,$$

$$\mathbf{q}(z) = \sum_{b,\beta} q_b^\beta \varphi^\beta z^b \in \mathcal{H}_+.$$

We equip $\mathcal{H}$ with the Q-adic topology. Let

$$\mathbf{t}(z) := t_0 + t_1 z + \cdots \in H^*(I\mathcal{X}, \Lambda)[[z]].$$

Then the genus g and the total potential are defined to be

$$\mathcal{F}^g\big(\mathbf{t}(z)\big) = \sum_{d,n} \frac{Q^d}{n!} \big\langle \mathbf{t}(\bar\psi), \ldots, \mathbf{t}(\bar\psi) \big\rangle_{g,n,d},$$

$$\mathcal{D}\big(\mathbf{t}(z)\big) = \exp\Big(\sum_{g \geq 0} \hbar^{g-1} \mathcal{F}^g\big(\mathbf{t}(z)\big)\Big),$$

respectively. For $\mathbf{t}(z) \in \mathcal{H}_+$, we call the translation $\mathbf{q}(z) := \mathbf{t}(z) - \mathbf{1}z$ the *dilaton shift*. We regard the total descendant potential as a formal function on $\mathcal{H}_+$ in a neighborhood of $-\mathbf{1}z$ taking values in $\mathbb{C}[[Q, \hbar, \hbar^{-1}]]$.

The graph of the differential of $\mathcal{F}^0$ defines a formal germ of a Lagrangian submanifold of $\mathcal{H}$:

$$\mathcal{L}^H := \left\{ (\mathbf{p}, \mathbf{q}), \mathbf{p} = d_{\mathbf{q}} \mathcal{F}^0 \right\} \in \mathcal{H}.$$

THEOREM 2.6 (see [11]). *The submanifold $\mathcal{L}^H$ is (the formal germ of) a Lagrangian cone with vertex at the origin such that each tangent space T is tangent to $\mathcal{L}^H$ exactly along zT.*

The class of cones satisfying properties of Theorem 2.6 is preserved under the action of symplectic transformations on $\mathcal{H}$ which commute with multiplication by z. We call these symplectomorphisms *loop group elements*. They are matrix-valued Laurent series in z:

$$S(z) = \sum_{i \in \mathbb{Z}} S_i z^i,$$

where $S_i \in \operatorname{End}(H^*(I\mathcal{X}) \otimes \Lambda)$. Being a symplectomorphism amounts to

$$S(z) S^*(-z) = I,$$

where I is the identity matrix and S^* is the adjoint of S. Differentiating the relation above at the identity, we see that infinitesimal loop group elements R satisfy

$$R(z) + R^*(-z) = 0.$$

We now introduce twisted Gromov–Witten invariants. For a bundle E we will denote by $\mathcal{A}(E), \mathcal{B}(E), \mathcal{C}(E)$ general multiplicative classes of E. These are of the form

$$\exp\left(\sum_{k \geq 0} s_k \operatorname{ch}_k(E) \right).$$

We then define the classes $\mathcal{A}_{g,n,d}, \mathcal{B}_{g,n,d}, \mathcal{C}_{g,n,d} \in H^*(\mathcal{X}_{g,n,d})$ as products of possibly *different* multiplicative classes of bundles:

$$\mathcal{A}_{g,n,d} = \prod_{\alpha=1}^{i_A} \mathcal{A}_\alpha \big(\pi_*(\operatorname{ev}^* E_\alpha) \big),$$

$$\mathcal{B}_{g,n,d} = \prod_{\beta=1}^{i_B} \mathcal{B}_\beta \big(\pi_* \big(f_\beta(L_{n+1}^{-1}) - f_\beta(1) \big) \big),$$

$$\mathcal{C}_{g,n,d} = \prod_{\mu} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu \big(\pi_*(\operatorname{ev}_{n+1}^* F_{\delta\mu} \otimes i_{\mu*} \mathcal{O}_{\mathcal{Z}_\mu}) \big).$$

Here f_i are polynomials with coefficients in $\mathrm{ev}_{n+1}^* K^0(\mathcal{X})$ and the bundles $E_\alpha, F_{\delta\mu}$ are elements of $K^0(\mathcal{X})$. To keep notation simple we write

$$\Theta_{g,n,d} := \mathcal{A}_{g,n,d} \cdot \mathcal{B}_{g,n,d} \cdot \mathcal{C}_{g,n,d}.$$

Twisted Gromov–Witten invariants are

$$\langle a_1 \bar{\psi}^{k_1}, \ldots, a_n \bar{\psi}^{k_n}; \Theta \rangle_{g,n,d} := \int_{[\mathcal{X}_{g,n,d}]} \prod_{i=1}^{n} \mathrm{ev}_i^*(a_i) \bar{\psi}_i^{k_i} \cdot \Theta_{g,n,d}.$$

Their generating series is the twisted potential $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$:

$$\mathcal{F}_{\mathcal{A},\mathcal{B},\mathcal{C}}^g(\mathbf{t}) := \sum_{d,n} \frac{Q^d}{n!} \langle \mathbf{t}(\bar{\psi}), \ldots, \mathbf{t}(\bar{\psi}); \Theta \rangle_{g,n,d},$$

$$\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}} := \exp\Big(\sum_g \hbar^{g-1} \mathcal{F}_{\mathcal{A},\mathcal{B},\mathcal{C}}^g\Big).$$

We view $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ as a formal function on $\mathcal{H}_+^{\mathcal{A},\mathcal{B},\mathcal{C}}$.

The symplectic vector space $(\mathcal{H}^{\mathcal{A},\mathcal{B},\mathcal{C}}, \Omega_{\mathcal{A},\mathcal{B},\mathcal{C}})$ is defined as $\mathcal{H}^{\mathcal{A},\mathcal{B},\mathcal{C}} = \mathcal{H}$, but with a different symplectic form:

$$\Omega_{\mathcal{A},\mathcal{B},\mathcal{C}}(\mathbf{f},\mathbf{g}) := \oint_{z=0} \big(\mathbf{f}(z), \mathbf{g}(-z)\big)_{\mathcal{A}} \, dz,$$

where $(\,,\,)_{\mathcal{A}}$ is the *twisted* pairing given for $a, b \in H^*(I\mathcal{X})$ by

$$(a,b)_{\mathcal{A}} := \langle a, b, 1; \Theta \rangle_{0,3,0}.$$

REMARK 2.7. We briefly discuss the case $(g,n,d) = (0,3,0)$. According to [3] in this case the evaluation maps lift to $\mathrm{ev}_i : \mathcal{X}_{0,3,0} \to I\mathcal{X}$. The spaces $\mathcal{X}_{0,3,0,(\mu_1,\mu_2,0)}$ are empty unless $\mu_2 = \mu_1^I$, in which case they can be identified with $\mathcal{X}_{\mu_1}$, with the evaluation maps being $\mathrm{ev}_1 = \mathrm{id} : \mathcal{X}_{\mu_1} \to \mathcal{X}_{\mu_1}$, $\mathrm{ev}_2 = \iota : \mathcal{X}_{\mu_1} \to \mathcal{X}_{\mu_1^I}$, and ev_3 is the inclusion of $\mathcal{X}_{\mu_1}$ in $\mathcal{X}$.

REMARK 2.8. On $\mathcal{X}_{0,3,0}$ there are no twistings of type $\mathcal{B}$ (the corresponding pushforwards are trivial for dimensional reasons) or of type $\mathcal{C}$ (there are no nodal curves). Hence, the twisted pairing depends only on the $\mathcal{A}$ classes.

For a bundle E on $\mathcal{X}_\mu$ we denote by E_{inv} the subbundle invariant under the action of the group element associated to $\mathcal{X}_\mu$. According to the previous two remarks, we can rewrite the pairing as

$$(a,b)_\mathcal{A} := \int_{I\mathcal{X}} a \cdot \iota^* b \cdot \prod_\alpha \mathcal{A}_\alpha\big((q^* E_\alpha)_{\mathrm{inv}}\big).$$

There is a rescaling map

$$(\mathcal{H}^{\mathcal{A},\mathcal{B},\mathcal{C}}, \Omega_{\mathcal{A},\mathcal{B},\mathcal{C}}) \to (\mathcal{H}, \Omega),$$

$$a \mapsto a\sqrt{\prod_\alpha \mathcal{A}_\alpha\big((q^* E_\alpha)_{\mathrm{inv}}\big)}$$

which identifies the symplectic spaces. We denote by $\mathcal{D}_{\mathcal{A},\mathcal{B}}, \mathcal{D}_\mathcal{A}$, and so on, the potentials twisted only by classes of type occurring in the notation and by

$$[\mathcal{X}_{g,n,d}]^{\mathrm{tw}} := [\mathcal{X}_{g,n,d}] \cap \Theta_{g,n,d}.$$

§3. Technical prerequisites

The computations in the proof of the theorems rely on pulling back the correlators on the universal orbicurve and on the locus of nodes. Hence, we need to know how the classes $\Theta_{g,n,d}$ behave under such pullbacks. The reader can skip this (unavoidably technical) section. To not make the statements and their proofs even more ugly, we assume throughout this section that i_μ^{red} denotes the inclusion of a *single* nodal stratum in the moduli space $\mathcal{X}_{g,n+1,d}$. Otherwise, (3.2), (3.6), and (3.9) (and their proofs) need on the right-hand side summation after all tuples $g_1 + g_2 = g$, $d_1 + d_2 = d$, $n_1 + n_2 = n$. The result which we will use in the proofs of the theorems is as follows.

PROPOSITION 3.1. *The following equalities hold:*

1. $\pi^*[\mathcal{X}_{g,n,d}]^{\mathrm{tw}}$

$$= [\mathcal{X}_{g,n+1,d}]^{\mathrm{tw}} \cdot \prod_{\beta=1}^{i_B} \mathcal{B}_\beta\Big(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1}\Big)$$

(3.1)
$$+ \sum_{j=1}^{n} [\mathcal{X}_{g,n+1,d}]^{\mathrm{tw}} \cdot \Big(\prod_{\delta=1}^{i_{\mu_j}} \mathcal{C}_\delta^{\mu_j}\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j}\big) - 1\Big)$$

$$+ \sum_\mu [\mathcal{X}_{g,n+1,d}]^{\mathrm{tw}} \cdot \Big(\prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu}) \otimes i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu}\big) - 1\Big),$$

2. $(\pi \circ i_\mu^{\mathrm{red}} \circ p)^*[\mathcal{X}_{g,n,d}]^{\mathrm{tw}}$

(3.2)
$$= \frac{p_1^*([\mathcal{X}_{g_1,n_1+1,d_1}]^{\mathrm{tw}}) \cdot p_2^*([\mathcal{X}_{g_2,n_2+1,d_2}]^{\mathrm{tw}})}{(\mathrm{ev}_+^* \times \mathrm{ev}_-^*)\Delta_{\mu*} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu((q^*F_{\delta\mu})_\mu) \otimes (L_+L_- - 1))},$$

3. $(\pi \circ i_\mu^{\mathrm{irr}} \circ p)^*[\mathcal{X}_{g,n,d}]^{\mathrm{tw}}$

(3.3)
$$= \frac{[\mathcal{X}_{g-1,n+2,d}]^{\mathrm{tw}}}{(\mathrm{ev}_+^* \times \mathrm{ev}_-^*)\Delta_{\mu*} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu((q^*F_{\delta\mu})_\mu) \otimes (L_+L_- - 1))}.$$

Proof. All the equalities follow from the corresponding statements about the classes $\mathcal{A}, \mathcal{B}, \mathcal{C}$ separately, which we state and prove below. Formula (3.1) follows from (3.5), (3.8), and (3.31) combined with some more cancellation; namely, the terms in (3.31) supported on $\mathcal{D}_j$ and $\mathcal{Z}$ are killed by the correction factor in (3.8) which is of the form $1 + \psi_{n+1} \cdot \ldots$. The untwisted virtual fundamental classes satisfy $\pi^*[\mathcal{X}_{g,n,d}] = [\mathcal{X}_{g,n+1,d}]$.

Formulas (3.2) and (3.3) follow from the corresponding Propositions 3.3, 3.4, and 3.9 for each of the classes $\mathcal{A}_{g,n,d}$, $\mathcal{B}_{g,n,d}$, and $\mathcal{C}_{g,n,d}$ combined with the splitting axiom in orbifold Gromov–Witten theory for the untwisted fundamental classes $[\mathcal{X}_{g,n,d}]$, which we briefly review below. Let $\mathfrak{M}_{g,n}^{\mathrm{tw}}$ be the stack of genus g twisted curves with n marked points. There is a natural map

$$\mathrm{gl} : \mathfrak{D}^{\mathrm{tw}}(g_1; n_1 \mid g_2; n_2) \to \mathfrak{M}_{g,n}^{\mathrm{tw}}$$

induced by gluing two families of twisted curves into a reducible curve with a distinguished node. Here $\mathfrak{D}^{\mathrm{tw}}(g_1; n_1 \mid g_2, n_2)$ is defined as in [4, Section 5.1]. This induces a Cartesian diagram:

$$\begin{array}{ccc}
\mathfrak{D}_{g,n}^{\mathrm{tw}}(\mathcal{X}) & \longrightarrow & \mathcal{X}_{g,n,d} \\
\downarrow & & \downarrow \\
\mathfrak{D}^{\mathrm{tw}}(g_1; n_1 \mid g_2; n_2) & \xrightarrow{\mathrm{gl}} & \mathfrak{M}_{g,n}^{\mathrm{tw}}
\end{array}$$

There is a natural map

$$\mathfrak{g} : \bigcup_{d_1+d_2=d} \mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathfrak{D}_{g,n}^{\mathrm{tw}}(\mathcal{X}).$$

Then the diagram

$$
\begin{array}{ccc}
\mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2} \subset \mathcal{Z} & \longrightarrow & \overline{I\mathcal{X}} \\
\downarrow & & \Delta \downarrow \\
\mathcal{X}_{g_1,n_1+1,d_1} \times \mathcal{X}_{g_2,n_2+1,d_2} & \xrightarrow{\mathrm{ev}_+ \times \breve{\mathrm{ev}}_-} & \overline{I\mathcal{X}} \times \overline{I\mathcal{X}}
\end{array}
$$

gives

$$
(3.4) \qquad \sum_{d_1+d_2=d} \Delta^! \big([\mathcal{X}_{g_1,n_1+1,d_1}] \times [\mathcal{X}_{g_2,n_2+1,d_2}] \big) = \mathfrak{g}^* \big(\mathrm{gl}^! ([\mathcal{X}_{g,n,d}]) \big).
$$

For details and proofs of the statements. we refer the reader to [4, Proposition 5.3.1]. The only modification we have made is that we consider the class of the diagonal with respect to the *twisted* pairing on $\mathcal{X}_{0,3,0,(\mu_1,\mu_2,0)}$. This cancels the factor $\mathrm{ev}_\Delta^*(\mathcal{A}_{0,3,0})$ in (3.2) and (3.3).

Informally, relation (3.4) says that the restriction of the virtual fundamental class of $\mathcal{X}_{g,n,d}$ to $\mathcal{Z}$ coincides with the pushforward of the product of virtual fundamental classes under the gluing morphisms. Hence, integration on $\mathcal{Z}$ factors nicely as products of integrals on the two separate moduli spaces. □

The rest of the section is devoted to proving pullback results about each type of twisting class separately.

LEMMA 3.2. *Consider the following diagram:*

$$
\begin{array}{ccc}
\mathcal{X}_{g,n+\circ+\bullet,d,(\mu_1,\ldots,\mu_n,0,0)} & \xrightarrow{\pi_1} & \mathcal{X}_{g,n+\bullet,d,(\mu_1,\ldots,\mu_n,0)} \\
\pi_2 \downarrow & & \pi_2 \downarrow \\
\mathcal{X}_{g,n+\circ,d,(\mu_1,\ldots,\mu_n,0)} & \xrightarrow{\pi_1} & \mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)}
\end{array}
$$

where π_1 forgets the $(n+1)$st marked point (which we denoted $\circ$) and π_2 forgets the $(n+2)$nd marked point (denoted $\bullet$), and let $\alpha \in K^0(\mathcal{X}_{g,n+\circ,d,(\mu_1,\ldots,\mu_n,0)})$. Then $\pi_2^* \pi_{1*} \alpha = \pi_{1*} \pi_2^* \alpha$.

Proof. For simplicity of notation, we suppress the labeling $(\mu_1,\ldots,\mu_n)$ in the proof. Consider the fiber product

$$
\mathcal{F} := \mathcal{X}_{g,n+\circ,d} \times_{\mathcal{X}_{g,n,d}} \mathcal{X}_{g,n+\bullet,d};
$$

denote by p_1, p_2 the projections from $\mathcal{F}$ to the factors, and denote by $\varphi : \mathcal{X}_{g,n+\circ+\bullet,d} \to \mathcal{F}$ the morphism induced by π_1, π_2. The φ is a birational

map: it has positive-dimensional fibers along the locus where the two extra marked points hit another marked point or a node. This locus has codimension 2—this in particular shows that $\mathcal{F}$ is normal. We will prove that

$$\varphi_*(\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}}) = \mathcal{O}_{\mathcal{F}}.$$

By definition of K-theoretic pushforward,

$$\varphi_*\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}} = R^0\varphi_*\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}} - R^1\varphi_*\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}}.$$

It is easy to see that $R^0\varphi_*(\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}}) = \mathcal{O}_{\mathcal{F}}$ as quasi-coherent sheaves. (This is true for every proper birational map with normal target.) We only have to prove that $R^1 = 0$, which we do by looking at the stalks

$$(R^1\varphi_*\mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}})_x = H^1\big(\varphi^{-1}(x), \mathcal{O}_{\mathcal{X}_{g,n+\circ+\bullet,d}|\varphi^{-1}(x)}\big).$$

If the fiber over x is a point, there is nothing to prove. If x is in the blow-up locus, the fiber is a (possibly weighted) $\mathbb{P}^1$. A calculation in [4, Theorem 7.2.1] shows that

$$\chi(\mathcal{C}, \mathcal{O}_{\mathcal{C}}) = 1 - g,$$

where g is the arithmetic genus of the coarse curve C. This shows that $H^1(\varphi^{-1}(x), \mathcal{O}) = 0$. We have $p_{1*}p_2^*\alpha = \pi_2^*\pi_{1*}\alpha$ because the diagram

$$
\begin{array}{ccc}
\mathcal{F} & \xrightarrow{\ p_1\ } & \mathcal{X}_{g,n+\bullet,d,(\mu_1,\ldots,\mu_n,0)} \\
{\scriptstyle p_2}\downarrow & & \downarrow{\scriptstyle \pi_2} \\
\mathcal{X}_{g,n+\circ,d,(\mu_1,\ldots,\mu_n,0)} & \xrightarrow{\ \pi_1\ } & \mathcal{X}_{g,n,d,(\mu_1,\ldots,\mu_n)}
\end{array}
$$

is a fiber square. Therefore,

$$\pi_{1*}\pi_2^*\alpha = p_{1*}\varphi_*(\varphi^*p_2^*\alpha) = p_{1*}p_2^*\alpha\varphi_*(\mathcal{O}) = p_{1*}p_2^*\alpha = \pi_2^*\pi_{1*}\alpha.$$

We need to know how the classes $\mathcal{A}_{g,n,d}, \mathcal{B}_{g,n,d}, \mathcal{C}_{g,n,d}$ behave under pullback by the morphisms π and $\pi \circ i \circ p$. $\qquad\qquad\qquad\qquad\square$

PROPOSITION 3.3. *The following identities hold:*

(3.5)   a. $\pi^*\mathcal{A}_{g,n,d} = \mathcal{A}_{g,n+1,d}$,

(3.6)   b. $(\pi \circ i^{\mathrm{red}} \circ p)^*\mathcal{A}_{g,n,d} = \dfrac{p_1^*\mathcal{A}_{g_1,n_1+1,d_1} \cdot p_2^*\mathcal{A}_{g_2,n_2+1,d_2}}{\mathrm{ev}_\Delta^*\mathcal{A}_{0,3,0}}$,

(3.7)   c. $(\pi \circ i^{\mathrm{irr}} \circ p)^*\mathcal{A}_{g,n,d} = \dfrac{\mathcal{A}_{g-1,n+2,d}}{\mathrm{ev}_\Delta^*\mathcal{A}_{0,3,0}}$.

Denote by $E_{g,n,d} := \pi_*(\mathrm{ev}^*_{n+1}E)$. Then it is shown in [19, Lemma B.0.9] that

a. $\pi^* E_{g,n,d} = E_{g,n+1,d}$,

b. $(\pi \circ i^{\mathrm{red}} \circ p)^* E_{g,n,d} = p_1^*(E_{g_1,n_1+1,d_1}) + p_2^*(E_{g_2,n_2+1,d_2}) - \mathrm{ev}^*_\Delta(q^* E_{\mathrm{inv}})$,

c. $(\pi \circ i^{\mathrm{irr}} \circ p)^* E_{g,n,d} = E_{g-1,n+2,d} - \mathrm{ev}^*_\Delta(q^* E_{\mathrm{inv}})$.

The identities then follow by multiplicativity of the classes $\mathcal{A}_\alpha$. We regard the class $\mathcal{A}_{0,3,0}$ as an element of $H^*(I\mathcal{X}, \mathbb{Q})$. We can then pull it back by the diagonal evaluation morphism ev_Δ at the node.

PROPOSITION 3.4. *The following hold:*

$$(3.8) \qquad \text{a.} \quad \pi^* \mathcal{B}_{g,n,d} = \mathcal{B}_{g,n+1,d} \cdot \prod_{\beta=1}^{i_B} \mathcal{B}_\beta\left(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1} \right),$$

$$(3.9) \qquad \text{b.} \quad (\pi \circ i^{\mathrm{red}} \circ p)^* \mathcal{B}_{g,n,d} = p_1^* \mathcal{B}_{g_1,n_1+1,d_1} \cdot p_2^* \mathcal{B}_{g_2,n_2+1,d_2},$$

$$(3.10) \qquad \text{c.} \quad (\pi \circ i^{\mathrm{irr}} \circ p)^* \mathcal{B}_{g,n,d} = \mathcal{B}_{g-1,n+2,d}.$$

Proof. The first identity is a consequence of Lemma 3.2. More precisely, we apply the lemma to the class $\alpha = \mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^{k+1}$. This gives

$$\pi_2^* \pi_{1*}\left[\mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^{k+1}\right]$$
$$= \pi_{1*} \pi_2^*\left[\mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^{k+1}\right]$$
$$= \pi_{1*}\left[\mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^{k+1} - (\sigma_\bullet)_*\left(\mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^k\right)\right]$$
$$= \pi_{1*}\left(\mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^{k+1}\right) - \mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^k.$$

The last equality follows because $\pi_1 \circ \sigma_\bullet = \mathrm{Id}$, and the second equality uses the comparison identity for cotangent line bundles L_i:

$$\pi^*\left((L_i - 1)^{k+1}\right) = (L_i - 1)^{k+1} - \sigma_{i*}\left[(L_i - 1)^k\right].$$

But both morphisms π_1, π_2 can be identified with the universal orbicurve π. Hence, we deduce that

$$(3.11) \qquad \begin{aligned} &\pi^* \pi_*\left(\mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^{k+1}\right) \\ &\quad = \pi_*\left(\mathrm{ev}^*_{n+2}(E)(L_{n+2} - 1)^{k+1}\right) - \mathrm{ev}^*_{n+1}(E)(L_{n+1} - 1)^k, \end{aligned}$$

or, more generally, if we expand

$$f_\beta(L_{n+1}^{-1}) - f_\beta(1) = \sum_{k \geq 0} a_k (L_{n+1} - 1)^{k+1},$$

then

$$\text{(3.12)} \quad \begin{aligned} \pi^* \pi_* \big(f_\beta(L_{n+1}^{-1}) - f_i(1)\big) \\ = \pi_* \big(f_\beta(L_{n+2}^{-1}) - f_i(1)\big) - \frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1}. \end{aligned}$$

Then (3.8) follows because $\mathcal{B}_\beta$ are multiplicative classes:

$$\begin{aligned} \pi^* \mathcal{B}_\beta \big(\pi_* \big(f_\beta(L_{n+1}^{-1}) - f_\beta(1)\big)\big) \\ = \mathcal{B}_\beta \big(\pi^* \pi_* \big(f_\beta(L_{n+1}^{-1}) - f_\beta(1)\big)\big) \\ = \mathcal{B}_\beta \Big(\pi_* \big(f_\beta(L_{n+2}^{-1}) - f_\beta(1)\big) - \frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1}\Big) \\ = \mathcal{B}_\beta \big(\pi_* \big(f_\beta(L_{n+2}^{-1}) - f_\beta(1)\big)\big) \cdot \mathcal{B}_\beta \Big(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1}\Big). \qquad \square \end{aligned}$$

EXAMPLE 3.5. In the case $f_\beta = \mathrm{ev}_{n+1}^*(E_\beta) \otimes L_{n+1}^{-1}$ (which is the only one we will need), we have

$$\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1} = -E_\beta L_{n+1}^{-1},$$

and relation (3.8) reads

$$\text{(3.13)} \qquad \pi^* \mathcal{B}_{g,n,d} = \mathcal{B}_{g,n+1,d} \cdot \prod_{\beta=1}^{i_B} \mathcal{B}_\beta(E_\beta \otimes L_{n+1}^{-1}).$$

Relation (3.9) follows from the identity

$$\begin{aligned} (\pi \circ i_{\mathrm{red}})^* \big[\pi_* \big(f(L_{n+1}^{-1}) - f(1)\big)\big] \\ = p_1^* \big[\pi_* \big(f(L_{n_1+2}^{-1}) - f(1)\big)\big] + p_2^* \big[\pi_* \big(f(L_{n_2+2}^{-1}) - f(1)\big)\big], \end{aligned}$$

which we prove below. By linearity, it is enough to prove the result for $f = (L_{n+1} - 1)^{k+1}$ for $k \geq 0$. Assume for now that $k \geq 1$. Relation (3.11) gives

$$\text{(3.14)} \qquad \pi^* \pi_* (L_{n+1} - 1)^{k+1} = \pi_* (L_{n+2} - 1)^{k+1} - (L_{n+1} - 1)^k.$$

When we apply $p^* i_{\text{red}}^*$ to this relation, the second summand in the right-hand side of (3.14) vanishes because L_{n+1} is trivial on $\widetilde{\mathcal{Z}}$. Therefore,

$$p^* i_{\text{red}}^* \pi^* \pi_* (L_{n+1} - 1)^{k+1} = (i_{\text{red}} \circ p)^* \pi_* (L_{n+2} - 1)^{k+1}.$$

Let $\mathcal{X}_{g_1, n_1+1, d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2, n_2+1, d_2}$ be a stratum of $\mathcal{Z}$. If we denote by $\pi : \mathcal{U}'_{g,n,d} \to \mathcal{U}_{g,n,d}$ the universal curve, then we have a fiber diagram

$$
\begin{array}{ccc}
\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 & \xrightarrow{\;\;i\;\;} & \mathcal{U}'_{g,n,d} \\
\pi \downarrow & & \pi \downarrow \\
\mathcal{X}_{g_1, n_1+1, d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2, n_2+1, d_2} & \xrightarrow{\;\;i\;\;} & \mathcal{U}_{g,n,d}
\end{array}
$$

Here $\mathcal{Z}_1$ and $\mathcal{Z}_3$ are the universal curves over the factors $\mathcal{X}_{g_1, n_1+1, d_1}$ and $\mathcal{X}_{g_2, n_2+1, d_2}$. So, using

$$(3.15) \qquad i_{\text{red}}^* \pi_* (L_{n+2} - 1)^{k+1} = \pi_* i_{\text{red}}^* (L_{n+2} - 1)^{k+1},$$

we see that the contribution of the strata $\mathcal{Z}_1$ and $\mathcal{Z}_3$ above is

$$(3.16) \qquad p_1^* \big[\pi_* \big(f(L_{n_1+2}^{-1}) - f(1) \big) \big] + p_2^* \big[\pi_* \big(f(L_{n_2+2}^{-1}) - f(1) \big) \big].$$

So if we show that the contribution from $\mathcal{Z}_2$ is 0, we are done. The curve $\mathcal{Z}_2$ is the universal curve over the factor $\mathcal{X}_{0,3,0}$; hence, it is a fiber product $\mathcal{X}_{g_1, n_1+1, d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,4,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2, n_2+1, d_2}$. The fibers of the map $\mathcal{Z}_2 \to \mathcal{Z}$ are (weighted) $\mathbb{P}^1$. However, the class L_{n+2} (consider it as the cotangent line $L_1 \in K^0(\bar{M}_{0,4})$) is a cotangent line at a point with trivial orbifold structure, so we can use Lee's formula in [15], which in this particular case reads

$$(3.17) \qquad \chi(\bar{M}_{0,4}, L_1^k) = k + 1.$$

Hence, the Euler characteristics of $(L_{n+2} - 1)^{k+1}$ are

$$
\begin{aligned}
\chi\big(&\bar{M}_{0,4}, (L_1 - 1)^{k+1} \big) \\
&= \sum_{i=0}^{k+1} (i+1)(-1)^{k+1-i} \binom{k+1}{i} \\
&= \sum_{i=0}^{k+1} (-1)^{k+1-i} \binom{k+1}{i} + (k+1) \sum_{i=1}^{k+1} (-1)^{k+1-i} \binom{k}{i-1} = 0 + 0 = 0.
\end{aligned}
$$

This almost proves the statement. We are left with the case $k = 0$, which is slightly different. The sum above equals 1, but this is canceled by the -1 in the second term of (3.14). Relation (3.9) follows then from the multiplicativity of the classes $\mathcal{B}_\beta$. A similar computation shows relation (3.10).

LEMMA 3.6. *Let* $F \in K^0(\mathcal{X})$. *Then*

a. $\quad \pi^* \pi_* i_{\mu*} \big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu} \big)$

(3.18)
$$= \pi_* i_{\mu*} \big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu} \big)$$
$$- \sum_{j, \mu_j = \mu} \mathrm{ev}_{n+1}^*(F) \otimes \sigma_{j*} \mathcal{O}_{\mathcal{D}_j} - i_{\mu*} \big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu} \big),$$

b. $\quad (\pi \circ i \circ p)^* \big(\pi_* i_{\mu*} \big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu} \big) \big)$

(3.19)
$$= p_1^* \big(\pi_* i_{\mu*} \big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu} \big) \big)$$
$$+ p_2^* \big(\pi_* i_{\mu*} \big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu} \big) \big) + \big(\mathrm{ev}_{n+1}^* F \otimes (1 - L_+ L_-) \big).$$

REMARK 3.7. Before delving into the technicalities of the proof, we try a heuristic explanation of why the rather ugly formulas should be true.

• Assume for now that F is the trivial bundle $\mathbb{C}$. The nodal locus $\mathcal{Z}$ separates nodes in the following sense. Above a point of $\mathcal{X}_{g,n,d}$ representing a nodal curve with k nodes lie exactly k points of $\mathcal{Z}$. This is very similar to the way the normalization of a nodal curve $\widetilde{C} \to C$ separates the nodes. But the structure sheaves of $\widetilde{C}$ and C differ (in K-theory) by skyscraper sheaves at the preimages of nodes. That is pretty much what the first formula expresses: the pullback of the structure sheaf of the codimension 1 stratum of nodal curves in $\mathcal{X}_{g,n,d}$ equals the structure sheaf of the nodal locus in the universal family, minus a copy of the structure sheaf of $\mathcal{Z}$ (which has codimension 2 in the universal family) itself. The terms supported on the divisors $\mathcal{D}_j$ are subtracted because they are nodes in the universal family, but they lie over the whole space $\mathcal{X}_{g,n,d}$. We will see that the presence of the class $\mathrm{ev}_{n+1}^*(F)$ does not complicate things too much.

• For the second formula, think of $\pi_* i_{\mu*} \alpha$ as a class supported on a codimension 1 subvariety. We pull it back along the map (πi), which is like restricting to another codimension 1 subvariety. If parameter varieties intersect along a codimension 2 cycle (represented by curves with two nodes), then they contribute $p_i^*(\pi_* i_{\mu*} \alpha)$ to (3.19). If they are the same subvariety, then α gets multiplied with the Euler class of the normal bundle of it in the ambient space, which is $1 - L_+ L_-$.

Proof of Lemma 3.6. Denote by $\mathcal{Z}_\bullet$, $\mathcal{Z}_\circ$, (resp., $\mathcal{Z}_{\bullet\circ}$) the nodal loci living inside the corresponding moduli spaces (and by $\mathcal{Z}_{\circ,\mu}$ and so forth the ones with nodes of specific orbifold type) in the following diagram (we write $\bar{\mu}$ for the sequence $(\mu_1, \ldots, \mu_n)$):

$$
\begin{array}{ccccc}
\pi_2^{-1}(\mathcal{Z}_{\circ,\mu}) & \xrightarrow{i_\mu} & \bigcup_{\bar{\mu}} \mathcal{X}_{g,n+\circ+\bullet,d,(\bar{\mu},0,0)} & \xrightarrow{\pi_1} & \bigcup_{\bar{\mu}} \mathcal{X}_{g,n+\bullet,d,(\bar{\mu},0)} \\
\pi_2 \downarrow & & \pi_2 \downarrow & & \pi_2 \downarrow \\
\mathcal{Z}_{\circ,\mu} & \xrightarrow{i_\mu} & \bigcup_{\bar{\mu}} \mathcal{X}_{g,n+\circ,d,(\bar{\mu},0)} & \xrightarrow{\pi_1} & \mathcal{X}_{g,n,d}
\end{array}
$$

Remember that $\mathcal{Z}_{\circ,\mu}$ is defined as the total range of the gluing map, as follows. (For simplicity we omit in the notation the stratum parameterizing self-intersecting curves; the proof carries through word by word.)

$$
\mathcal{X}_{g_1,n_1+1,d_1} \times_{\bar{\mathcal{X}}_\mu \times \bar{\mathcal{X}}_{\mu I}} \mathcal{X}_{0,3,0} \times_{\bar{\mathcal{X}}_\mu \times \bar{\mathcal{X}}_{\mu I}} \mathcal{X}_{g_2,n_2+1,d_2} \to \mathcal{Z}_\circ \hookrightarrow \mathcal{X}_{g,n+\circ,d}.
$$

We will compute $\pi_2^*(\pi_{1*} i_{\mu*}(\mathrm{ev}_\circ^*(F) \otimes \mathcal{O}_{\mathcal{Z}_{\circ,\mu}}))$.

The square on the left is a fiber diagram; hence, $i_* \pi_2^* = \pi_2^* i_*$. For the one on the right, we have proved that $\pi_2^* \pi_{1*} = \pi_{1*} \pi_2^*$. Therefore,

$$
(3.20) \qquad \pi_2^*\big(\pi_{1*} i_{\mu*}(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_{\circ,\mu}})\big) = \pi_{1*} i_{\mu*} \pi_2^*\big(\mathrm{ev}_\circ^*(F) \otimes \mathcal{O}_{\mathcal{Z}_{\circ,\mu}}\big).
$$

However,

$$
\pi_2^*(\mathrm{ev}_\circ^* F \otimes \mathcal{O}_{\mathcal{Z}_{\circ,\mu}}) = \mathrm{ev}_\circ^* F \otimes \mathcal{O}_{\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})}.
$$

The space $\pi_2^{-1}(\mathcal{Z}_{\circ,\mu}) := \mathcal{Z}_{\circ,1} \cup \mathcal{Z}_{\circ,2} \cup \mathcal{Z}_{\circ,3}$ is a singular space, where each codimension 2 stratum is the universal curve over one factor of $\mathcal{Z}_{\circ,\mu}$, and they intersect along two codimension 3 strata—call them $\mathcal{Z}_{12}$ and $\mathcal{Z}_{23}$:

$$
\mathcal{Z}_{12} = \mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2},
$$

where the two rational components carry the points $\bullet$, $\circ$ and two nodes. Figure 1 schematically represents each of these five strata. We can write the structure sheaf of $\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})$ as

$$
\mathcal{O}_{\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})} = \mathcal{O}_{\mathcal{Z}_{\circ,1}} + \mathcal{O}_{\mathcal{Z}_{\circ,3}} + \mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}}.
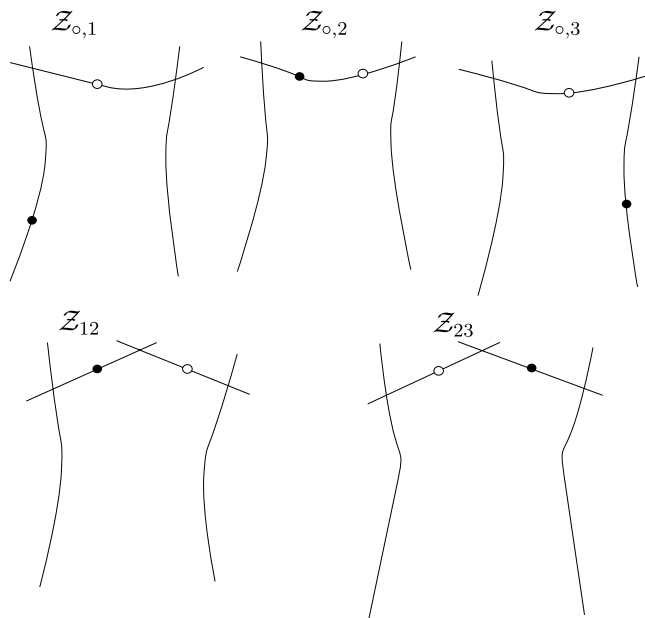$$

Figure 1: Strata of $\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})$.

We tensor this with the class $\mathrm{ev}_\circ^* F$, keeping in mind that on the strata $\mathcal{Z}_{\circ,2}, \mathcal{Z}_{12}, \mathcal{Z}_{23}$ $\mathrm{ev}_\circ = \mathrm{ev}_\bullet$,

$$
(3.21) \quad
\begin{aligned}
&\mathrm{ev}_\circ^* F \otimes \mathcal{O}_{\pi_2^{-1}(\mathcal{Z}_{\circ,\mu})} \\
&= \mathrm{ev}_\circ^* F \otimes [\mathcal{O}_{\mathcal{Z}_{\circ,1}} + \mathcal{O}_{\mathcal{Z}_{\circ,3}}] + \mathrm{ev}_\bullet^* F \otimes [\mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}}].
\end{aligned}
$$

We plug (3.21) in (3.20) and get

$$
(3.22) \quad
\begin{aligned}
&\pi_2^* \big(\pi_{1*} i_{\mu*} \big(\mathrm{ev}_\circ^*(F) \otimes \mathcal{O}_{\mathcal{Z}_{\circ,\mu}}\big)\big) \\
&= \pi_{1*} i_{\mu*} \big[\mathrm{ev}_\circ^* F(\mathcal{O}_{\mathcal{Z}_{\circ,1}} + \mathcal{O}_{\mathcal{Z}_{\circ,3}}) + \mathrm{ev}_\bullet^* F(\mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}})\big].
\end{aligned}
$$

We now notice that the union of $\mathcal{Z}_{\circ,1}$ and $\mathcal{Z}_{\circ,3}$ is almost $\mathcal{Z}_{\bullet\circ,\mu}$, but not quite. There are strata

$$
\mathcal{X}_{g,n,d} \times_{\bar{\mathcal{X}}_\mu} \mathcal{X}_{0,3,0} \times_{\bar{\mathcal{X}}_\mu} \mathcal{X}_{0,3,0},
$$

which are in $\mathcal{Z}_{\bullet\circ,\mu}$, but they are missing from $\mathcal{Z}_{\circ,1} \cup \mathcal{Z}_{\circ,3}$ because the map $\pi_2 \circ i_\mu$ contracts one rational tail. These are mapped by $\pi_1 \circ i_\mu$ isomorphically

to divisors $\mathcal{D}_j \in \mathcal{X}_{g,n+\bullet,d}$. There is one such stratum for each j such that $\mu_j = \mu$. Hence, we can write

$$(3.23) \quad \begin{aligned} &\pi_{1*}i_{\mu*}[\mathrm{ev}_\circ^* F\mathcal{O}_{\mathcal{Z}_{\circ,1}} + \mathrm{ev}_\circ^* F\mathcal{O}_{\mathcal{Z}_{\circ,3}}] \\ &\qquad = \pi_{1*}i_{\mu*}\big(\mathrm{ev}_\circ^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu}\big) - \sum_{j,\mu_j=\mu} \mathrm{ev}_\circ^*(F) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j}. \end{aligned}$$

The codimension 3 strata $\mathcal{Z}_{12}$ and $\mathcal{Z}_{23}$ are mapped by $\pi_1 i_\mu$ isomorphically to $\mathcal{Z}_{\bullet,\mu}$. As for $\mathcal{Z}_{\circ,2}$, this is a $\mathbb{P}^1$ fibration over $\mathcal{Z}_{\bullet,\mu}$. When we push forward, we integrate the structure sheaf of (weighted) $\mathbb{P}^1$. This equals 1, as already explained. At the end of the day, we see that the last three terms in (3.22) contribute

$$(3.24) \quad \pi_{1*}i_{\mu*}\big[\mathrm{ev}_\bullet^* F(\mathcal{O}_{\mathcal{Z}_{\circ,2}} - \mathcal{O}_{\mathcal{Z}_{12}} - \mathcal{O}_{\mathcal{Z}_{23}})\big] = -\mathrm{ev}_\bullet^* F \otimes i_{\mu*}\mathcal{O}_{\mathcal{Z}_{\bullet,\mu}}.$$

Adding up (3.23) with (3.24) and identifying $\pi_1 = \pi_2 = \pi$ and $\mathrm{ev}_\circ = \mathrm{ev}_{n+1}$ proves the first equality in the lemma.

For the second equality, we first prove the following.

LEMMA 3.8. *Let $j : \mathcal{Z} \hookrightarrow \mathcal{U}_{g,n,d}$ be the codimension 2 nodal locus. Then*

$$(3.25) \quad \begin{aligned} &j^* \pi_* i_{\mu*}(\mathrm{ev}_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}) \\ &\qquad = p_1^* \pi_* i_{\mu*}(\mathrm{ev}_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}) \\ &\qquad\quad + p_2^* \pi_* i_{\mu*}(\mathrm{ev}_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}) + (2 - L_+ - L_-)\mathrm{ev}_{n+1}^*(F). \end{aligned}$$

Proof. Let $\mathcal{U}'_{g,n,d}$ be the universal curve over $\mathcal{U}_{g,n,d}$. The universal curve over $\mathcal{Z}$ is a union of three types of strata, depending on which component the extra marked point on $\mathcal{U}'_{g,n,d}$—which we denote $\bullet$—lies on

$$\mathcal{Z}_1 = \mathcal{X}_{g_1,n_1+1+\bullet,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2},$$
$$\mathcal{Z}_2 = \mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3+\bullet,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2},$$
$$\mathcal{Z}_3 = \mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1+\bullet,d_2}.$$

The diagram below is a fiber square:

$$\begin{array}{ccc} \mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 & \xrightarrow{\;\;j\;\;} & \mathcal{U}'_{g,n,d} \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ \mathcal{Z} & \xrightarrow{\;\;j\;\;} & \mathcal{U}_{g,n,d} \end{array}$$

Hence, $j^* \pi_* i_{\mu*} \alpha = \pi_* j^* i_{\mu*} \alpha$. To compute $j^* i_{\mu*} \alpha$, we form the following fiber diagram:

$$
\begin{array}{ccc}
\bar{\mathcal{Z}} & \xrightarrow{\ \ j\ \ } & \mathcal{Z}_{\bullet,\mu} \\
\pi \downarrow & & \pi \downarrow \\
\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3 & \xrightarrow{\ \ j\ \ } & \mathcal{U}'_{g,n,d}
\end{array}
$$

The space $\bar{\mathcal{Z}}$ is simply the intersection of $\mathcal{Z}_1 \cup \mathcal{Z}_2 \cup \mathcal{Z}_3$ with $\mathcal{Z}_{\bullet,\mu}$. Where the intersection is transversal, one can simply write $j^* i_{\mu*} \alpha = i_{\mu*} j^* \alpha$. On components where the intersection is not transversal, there is some excess bundle N and $j^* i_{\mu*} \alpha = i_{\mu*} e(N) j^* \alpha$. The strata $\mathcal{Z}_1$ and $\mathcal{Z}_3$ intersect the nodal locus $\mathcal{Z}_{\bullet,\mu}$ in $\mathcal{U}'_{g,n,d}$ transversely along codimension 4 strata, which can be seen as the nodal locus in $\mathcal{X}_{g_1,n_1+1+\bullet,d_1}$ and $\mathcal{X}_{g_2,n_2+1+\bullet,d_2}$, respectively. Hence, the contribution to (3.25) is

$$
p_1^* \pi_* i_{\mu*}(\mathrm{ev}_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}) + p_2^* \pi_* i_{\mu*}(\mathrm{ev}_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu}).
$$

On the other hand, $\mathcal{Z}_2$ intersects $\mathcal{Z}_{\bullet,\mu}$ along two codimension 3 strata of the form

$$
\mathcal{Z}_1 = \mathcal{X}_{g_1,n_1+1,d_1} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{0,3,0} \times_{\overline{I\mathcal{X}}} \mathcal{X}_{g_2,n_2+1,d_2}.
$$

Each gives a 1-dimensional excess normal bundle with Euler classes $1 - L_+$ and $1 - L_-$, respectively. They project isomorphically to $\mathcal{Z}$ downstairs. Hence, they contribute

$$
(2 - L_+ - L_-)\mathrm{ev}_{n+1}^*(F).
$$

Adding up, we get (3.25). $\qquad\square$

We now prove formula (3.19) in Lemma 3.6. It falls out easily by combining (3.18) with Lemma 3.8. More precisely, we take i^* of (3.18): the first term is computed in Lemma 3.8, the part supported on $\mathcal{D}_j$ vanishes, and

$$
(3.26) \qquad i_\mu^* i_{\mu*} \mathcal{O}_{\mathcal{Z}_\mu} = e(N) = (1 - L_-)(1 - L_+),
$$

where N is the normal bundle of $\mathcal{Z}_\mu$ in the ambient space. When we add this with (3.25) we get

$$
(\pi \circ i)^* \pi_* i_{\mu*}\big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu}\big)
$$
$$
(3.27) \qquad = p_1^*\big(\pi_* i_{\mu*}\big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu}\big)\big)
$$
$$
\qquad\quad + p_2^*\big(\pi_* i_{\mu*}\big(\mathrm{ev}_{n+1}^*(F) \otimes \mathcal{O}_{\mathcal{Z}_\mu}\big)\big) + \mathrm{ev}_{n+1}^* F \otimes (1 - L_+ L_-),
$$

as stated. $\qquad\square$

PROPOSITION 3.9. *The following hold:*

a. $\pi^* \mathcal{C}_{g,n,d} = \mathcal{C}_{g,n+1,d} \cdot \prod_{j=1}^{n} \prod_{\delta=1}^{i_{\mu_j}} \mathcal{C}_\delta^{\mu_j}\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j}\big)$

(3.28)

$$\cdot \prod_{\mu} \prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu}) \otimes (i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu})\big),$$

b. $p^*(i_\mu^{\mathrm{red}})^* \pi^* \mathcal{C}_{g,n,d}$

$$= (p_1^* \mathcal{C}_{g_1,n_1+1,d_1}^\mu \cdot p_2^* \mathcal{C}_{g_2,n_2+1,d_2}^\mu)$$

(3.29)

$$\cdot (\mathrm{ev}_+^* \times \mathrm{ev}_-^*)\Delta_{\mu*}\Big(\prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu\big((q^*F_{\delta\mu})_\mu\big) \otimes (1 - L_+L_-)\Big),$$

c. $p^*(i_\mu^{\mathrm{irr}})^* \pi^* \mathcal{C}_{g,n,d}$

$$= \mathcal{C}_{g-1,n+2,d}^\mu$$

(3.30)

$$\cdot (\mathrm{ev}_+^* \times \mathrm{ev}_-^*)\Delta_{\mu*}\Big(\prod_{\delta=1}^{i_\mu} \mathcal{C}_\delta^\mu\big((q^*F_{\delta\mu})_\mu\big) \otimes (1 - L_+L_-)\Big).$$

Proof. The equalities (3.28) and (3.29) are immediate consequences of (3.18) and (3.19) and of the multiplicativity of the classes $\mathcal{C}_{g,n,d}$. We will use (3.28) in a different form, transforming the product into a sum:

$$\pi^* \mathcal{C}_{g,n,d} = \mathcal{C}_{g,n+1,d} \cdot \prod_{j=1}^{n} \prod_{\delta=1}^{i_C} \big(1 + \mathcal{C}_\delta^{\mu_j}\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j}\big) - 1\big)$$

$$\cdot \prod_{\mu} \Big(1 + \prod_{\delta=1}^{i_{C_\mu}} \mathcal{C}_\delta^\mu\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu}) \otimes i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu}\big) - 1\Big)$$

(3.31)

$$= \mathcal{C}_{g,n+1,d} + \sum_j \mathcal{C}_{g,n+1,d}$$

$$\cdot \prod_{\delta=1}^{i_{C_{\mu_j}}} \big(\mathcal{C}_\delta^{\mu_j}\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu_j}) \otimes \sigma_{j*}\mathcal{O}_{\mathcal{D}_j}\big) - 1\big)$$

$$+ \sum_\mu \mathcal{C}_{g,n+1,d} \cdot \Big(\prod_{\delta=1}^{i_{C_\mu}} \mathcal{C}_\delta^\mu\big(-\mathrm{ev}_{n+1}^*(F_{\delta\mu}) \otimes i_{\mu*}\mathcal{O}_{\mathcal{Z}_\mu}\big) - 1\Big).$$

This happens because the classes $\mathcal{C}_\delta^\mu(.) - 1$ are supported on $\mathcal{D}_i$ and $\mathcal{Z}$, and $\mathcal{D}_i \cdot \mathcal{D}_j = \mathcal{D}_i \cdot \mathcal{Z}_\mu = 0$ if $i \neq j$. (We will use the same trick in (4.6) below.) $\quad\square$

We conclude the section by doing a short Grothendieck–Riemann–Roch computation which will turn out to be useful in the next section.

LEMMA 3.10. *Let* $F \in K^0(\mathcal{X})$. *Then*

$$
\begin{aligned}
(3.32) \qquad & \mathrm{ch}\big(\pi_* i_{\mu*}(\mathrm{ev}_{n+1}^* F \otimes \mathcal{O}_{\mathcal{Z}_\mu})\big) \\
& = \pi_* i_{\mu*}\big(\mathrm{ch}(\mathrm{ev}_{n+1}^* F) \cdot \mathrm{Td}^\vee(-L_+ \otimes L_-)\big).
\end{aligned}
$$

Proof. Recall that $r(\mu)$ is the order of the distinguished node on $\mathcal{Z}_\mu$. We will simply write r throughout the proof.

We apply Toën's theorem (Theorem A.4) to the map $f = \pi \circ i$. The map π is given in local coordinates near $\mathcal{Z}_\mu$ by

$$
(z, x, y)/\mathbb{Z}_r \times \mathbb{Z}_r \mapsto (z, xy)/\mathbb{Z}_r,
$$

where z is a vector coordinate along $\mathcal{Z}_\mu$ and $\mathcal{Z}_\mu$ is given by $x = y = 0$. The generator of $\mathbb{Z}_r \times \mathbb{Z}_r$ acts on the (x, y) plane as follows. We have $(x, y) \mapsto (\zeta^a x, \zeta^b y)$ and necessarily by multiplication by ζ^{a+b} on the base, so in this local description If maps r copies of the point $(z, 0, 0)$ to $(z, 0)$ on the base. Each copy has weight $1/r$ due to their orbifold structures. The relative tangent bundle is $-L_+^{-1} \otimes L_-^{-1}$ because the coordinate on the base is $\varepsilon = xy$ and is invariant with respect to the $\mathbb{Z}_r$-action. This proves the statement. $\quad\square$

§4. Proofs of theorems

Proof of Theorem 1.1. This is an easy consequence of Tseng's result and of the commutativity of the operators Δ_α. $\quad\blacksquare$

Proof of Theorem 1.2. Remember that $\mathcal{B}_{g,n,d}$ is a product of i_B multiplicative characteristic classes. We will prove the statement using induction on i_B. The case $i_B = 0$ is trivial. Assuming that the statement holds for $i_B - 1$, we will prove the infinitesimal version of the proposition for i_B. Namely, assume that the twisting class $\mathcal{B}_{i_B}$ is

$$
\mathcal{B}_{i_B} = \exp\Big(\sum_{l \geq 1} v_l \, \mathrm{ch}_l \, \pi_* \big(f(L_{n+1}^{-1}) - f(1)\big)\Big).
$$

We compute

$$
\frac{\partial \mathcal{D}_{A,B,C}}{\partial v_l} \mathcal{D}_{A,B,C}^{-1}
$$
$$
= \sum_{d,n} \frac{Q^d \hbar^{g-1}}{n!} \left\langle \prod_{i=1}^{n} \mathbf{t}(\bar{\psi}_i) \cdot \mathrm{ch}_l \, \pi_* \big(f(L_{n+1}^{-1}) - f(1) \big) \cdot \Theta_{g,n,d} \right\rangle_{g,n,d}.
$$

(4.1)

To compute $\mathrm{ch}_l \, \pi_*(f(L_{n+1}^{-1}) - f(1))$ above, we apply Toën's Grothendieck–Riemann–Roch theorem to the morphism π to get

$$
(4.2) \qquad \mathrm{ch}\big(\pi_* \big(f(L_{n+1}^{-1}) - f(1) \big) \big) = I \pi_* \big(\widetilde{\mathrm{ch}} \big(f(L_{n+1}^{-1}) - f(1) \big) \mathrm{Td}^\vee(\Omega_\pi) \big).
$$

Notice that $\widetilde{\mathrm{ch}} = \mathrm{ch}$ because the last marked point is not an orbifold point. We have

$$
(4.3) \qquad\qquad \widetilde{\mathrm{ch}} \big(f(L_{n+1}^{-1}) - f(1) \big) = f(e^{-\psi_{n+1}}) - f(1).
$$

In our situation, there are three strata on the universal curve which get mapped to $\mathcal{X}_{g,n,d,(\mu_1,\dots,\mu_n)}$:

- the total space $\mathcal{X}_{g,n+1,d,(\mu_1,\dots,\mu_n,0)}$;
- the locus of marked points $\mathcal{D}_{j,(\mu_1,\dots,\mu_n)}$;
- the nodal loci $\mathcal{Z}_\mu$ where $\mu \neq 0$; that is, the node is an orbifold point.

But the expression on the right-hand side in (4.3) is a multiple of ψ_{n+1}, and ψ_{n+1} vanishes on the locus of marked points $\mathcal{D}_j$ and on the locus of nodes $\mathcal{Z}$. Hence, only the total space contributes to the Grothendieck–Riemann–Roch theorem. Exact sequences very similar to (5.7) and (5.5) in Section 5 allow us to write the sheaf of relative differentials (see also [19]):

$$
(4.4) \qquad\qquad \Omega_\pi = L_{n+1} - \bigoplus_{j=1}^{n} (\sigma_j)_* \mathcal{O}_{\mathcal{D}_{j,(\mu_1,\dots,\mu_n)}} - i_* L.
$$

Keeping in mind that the bundle L defined in Section 5 has trivial Chern character, we get

$$
(4.5) \qquad \mathrm{Td}^\vee(\Omega_\pi) = \mathrm{Td}^\vee(L_{n+1}) \prod_{j=1}^{n} \mathrm{Td}^\vee(-\sigma_{j*} \mathcal{O}_{\mathcal{D}_{j,(\mu_1,\dots,\mu_n)}}) \mathrm{Td}^\vee(-i_* \mathcal{O}_{\mathcal{Z}}).
$$

We now use the fact that $\psi_{n+1} \cdot \mathcal{D}_j = \psi_{n+1} \cdot \mathcal{Z} = 0$ (recall that $\mathrm{Td}^\vee(L_{n+1}) - 1$ is a multiple of ψ_{n+1}) to rewrite the product above as a sum:

$$
(4.6) \quad
\begin{aligned}
\mathrm{Td}^\vee(\Omega_\pi) = {}& \mathrm{Td}^\vee(L_{n+1}) + \sum_{j=1}^n \big(\mathrm{Td}^\vee(-\sigma_{j*}\mathcal{O}_{\mathcal{D}_{j,(\mu_1,\ldots,\mu_n)}}) - 1\big) \\
& + \mathrm{Td}^\vee(-i_*\mathcal{O}_{\mathcal{Z}}) - 1.
\end{aligned}
$$

The last $n+1$ summands are classes supported on $\mathcal{D}_j$ and $\mathcal{Z}$, so they are killed by the presence of ψ in $f(e^{-\psi_{n+1}}) - f(1)$. After all these cancellations, we see that

$$
(4.7) \quad \mathrm{ch}\big(\pi_*\big(f(L_{n+1}^{-1}) - f(1)\big)\big) = \pi_*\big(\big(f(e^{-\psi_{n+1}}) - f(1)\big) \cdot \mathrm{Td}^\vee(L_{n+1})\big).
$$

Here we see that formula (4.7) is a linear combination of kappa classes $K_{aj} = \pi_*(\mathrm{ev}_{n+1}^*\varphi_a \psi_{n+1}^{j+1})$. Now we pull the correlators back on the universal orbicurve. It is essential here that the corrections in the $\mathcal{C}_{g,n,d}$ classes are also supported on $\mathcal{D}_j$ and $\mathcal{Z}$ (as we can see from (3.31)), and that the presence of ψ_{n+1} kills them. Therefore (we denote by $[f]_l$ the homogeneous part of degree l of f),

$$
(4.8) \quad
\begin{aligned}
\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}^{-1} & \frac{\partial \mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}}{\partial v_l} \\
&= \sum_{d,n,g} \frac{Q^d \hbar^{g-1}}{n!} \int_{[\mathcal{X}_{g,n+1,d}]} \prod_{i=1}^n \Big(\sum_{k_i \ge 0} \big(\mathrm{ev}_i^*(t_{k_i}) \cdot \bar{\psi}_i^{k_i}\big)\Big) \cdot \Theta_{g,n+1,d} \\
&\quad \cdot \big[\big(f(e^{-\psi_{n+1}}) - f(1)\big) \cdot \mathrm{Td}^\vee(L_{n+1})\big]_{l+1} \\
&\quad \cdot \prod_{\beta=1}^{i_B} \mathcal{B}_\beta\Big(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1}\Big) \\
&\quad - \int_{\mathcal{X}_{0,3,0}} \varphi_a \psi_3^{m+1}(\cdots) - \int_{\mathcal{X}_{1,1,0}} \varphi_a \psi_1^{m+1}(\cdots).
\end{aligned}
$$

The correction terms occur because the spaces $\mathcal{X}_{0,3,0}$ and $\mathcal{X}_{1,1,0}$ are not universal families. Notice that the first correction is always 0 for dimensional reasons and that the second is not equal to 0 only for $m = \deg(\varphi_a) = 0$ (again for dimensional reasons). If we denote this contribution by K_{l,i_B}, then the constant $K_{\mathcal{B}}$ in Theorem 1.2 equals $\prod_{i,l} e^{K_{l,i}}$. This will not play any further role.

So the new twisting by the class $\mathcal{B}_{i_B}$ has the same effect as the translation

$$\mathbf{t}_{\mathcal{B}}(z) = \mathbf{t}(z) + z - z\prod_{\gamma=1}^{i_B}\mathcal{B}_\beta\Big(-\frac{f_\beta(\mathbf{L}_z^{-1}) - f_\beta(1)}{\mathbf{L}_z - 1}\Big)$$

because both potentials satisfy the same differential equation. To see this, differentiate the potential $\mathcal{D}_{\mathcal{A},\mathcal{B}}(\mathbf{t}_{\mathcal{B}}(z))$ in v_l:

(4.9)
$$\frac{\partial\mathcal{D}_{\mathcal{A},\mathcal{B}}(\mathbf{t}_B(z))}{\partial v_l}\mathcal{D}_{\mathcal{A},\mathcal{B}}^{-1}$$
$$= \sum_{d,n,g}\frac{Q^d\hbar^{g-1}}{n!}\int_{[\mathcal{X}_{g,n+1,d}]}\prod_{i=1}^n\Big(\sum_{k_i\geq 0}\big(\mathrm{ev}_i^*(t_{k_i})\cdot\bar\psi_i^{k_i}\big)\Big)$$
$$\cdot\psi_{n+1}\,\mathrm{ch}_l\Big(\frac{f(L_{n+1}^{-1}) - f(1)}{L_{n+1} - 1}\Big)$$
$$\cdot\Theta_{g,n+1,d}\cdot\prod_{\beta=1}^{i_B}\mathcal{B}_\beta\Big(-\frac{f_\beta(L_{n+1}^{-1}) - f_\beta(1)}{L_{n+1} - 1}\Big).$$

However,

(4.10)
$$\psi_{n+1}\,\mathrm{ch}_l\Big(\frac{f(L_{n+1}^{-1}) - f(1)}{L_{n+1} - 1}\Big) = \psi_{n+1}\Big[\frac{f(e^{-\psi_{n+1}}) - f(1)}{e^\psi - 1}\Big]_l$$
$$= \Big[\psi_{n+1}\frac{f(e^{-\psi_{n+1}}) - f(1)}{e^\psi - 1}\Big]_{l+1}$$
$$= \big[\big(f(e^{-\psi_{n+1}}) - f(1)\big)\cdot\mathrm{Td}^\vee(L_{n+1})\big]_{l+1}$$

because

$$\mathrm{Td}^\vee(L_{n+1}) = \frac{\psi_{n+1}}{e^{\psi_{n+1}} - 1}.$$

Plugging (4.10) in (4.9), we see that (4.9) and (4.8) are of exactly the same form. The potentials also satisfy the same initial condition at $\mathbf{v} = 0$ by the induction hypothesis. $\square$

Proof of Theorem 1.3. We will prove that

(4.11)
$$\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}} = \exp\Big(\frac{\hbar}{2}\sum_{a,b,\alpha,\beta,\mu}A_{a,\alpha;b;\beta}^\mu\partial_a^{\alpha,\mu}\partial_b^{\beta,\mu^I}\Big)\mathcal{D}_{\mathcal{A},\mathcal{B}},$$

where $A^\mu_{a,\alpha;b,\beta}$ are the coefficients of the expansion

$$\sum_{a,b} A^\mu_{a,\alpha;b,\beta}\varphi_{\alpha,\mu}\bar\psi_+^a \otimes \varphi_{\beta,\mu^I}\bar\psi_-^b$$

(4.12)
$$= \frac{\Delta_{\mu*}(\prod_{\delta=1}^{i_\mu}\mathcal{C}_\delta^\mu((q^*F_{\delta\mu})_\mu \otimes (1-\mathbb{L}_z))-1)}{-\psi_+ - \psi_-}$$

$$\in H^*(\mathcal{X}_\mu,\mathbb{Q})[\bar\psi_+] \otimes H^*(\mathcal{X}_{\mu^I},\mathbb{Q})[\bar\psi_-].$$

Here $\psi_+ = c_1(L_+)$, $\psi_- = c_1(L_-)$, and $\Delta_\mu : \mathcal{X}_\mu \to \mathcal{X}_\mu \otimes \mathcal{X}_{\mu^I}$ is the composition $(\mathrm{Id}\times\iota)\circ\Delta$. The map

$$\Delta_{\mu_*} : H^*(\mathcal{X}_\mu,\mathbb{Q}) \to H^*(\mathcal{X}_\mu,\mathbb{Q}) \otimes H^*(\mathcal{X}_{\mu^I},\mathbb{Q})$$

extends naturally to a map, which we abusively also call $\Delta_{\mu*}$:

$$\Delta_{\mu_*} : H^*(\mathcal{X}_\mu,\mathbb{Q})[z] \to H^*(\mathcal{X}_\mu,\mathbb{Q})[\bar\psi_+] \otimes H^*(\mathcal{X}_{\mu^I},\mathbb{Q})[\bar\psi_-],$$

by mapping $z \mapsto \psi_+ \otimes 1 + 1 \otimes \psi_-$, and the right-hand side of (4.12) should be understood in this way.

We will prove (4.11) using induction on the total number $\sum_\mu i_\mu$ of twisting classes $\mathcal{C}_\delta^\mu$. If $\sum i_\mu = 0$, then the equality is trivial. Let now $\sum i_\mu \geq 1$. Assuming that (4.11) is true for $\sum i_\mu - 1$, we will prove the infinitesimal version of the theorem for $\sum i_\mu$. More precisely, fix a μ_0, and let the multiplicative class $\mathcal{C}^{\mu_0}$ (we omit the lower index) be of the form

(4.13)
$$\mathcal{C}^{\mu_0}(E) = \exp\Big(\sum_l w_l \,\mathrm{ch}_l(E)\Big).$$

As we vary the coefficients w_l, we obtain a family of elements in the Fock space. We prove (4.11) by showing that both sides satisfy the same differential equations with the same initial condition. Notice that the induction hypothesis ensures that both sides of (4.11) satisfy the same initial condition at $\mathbf{w} = 0$. Moreover, $\partial\mathcal{D}_{\mathcal{A},\mathcal{B}}/\partial w_l = 0$, so on the right-hand side only the coefficients $A^{\mu_0}_{a,\alpha;b,\beta}$ depend on w_l. So if we denote the right-hand side by $\mathcal{G}$ and differentiate it, we get

(4.14)
$$\frac{\hbar}{2}\sum_{a,b}\frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l}\partial_a^{\alpha,\mu_0}\partial_b^{\beta,\mu_0^I}\mathcal{G} = \frac{\partial}{\partial w_l}\mathcal{G}.$$

To compute $\partial A^{\mu_0}_{a,\alpha;b,\beta}/\partial w_l$, we differentiate in w_l relation (4.12) to get

$$\sum_{a,\alpha;b,\beta} \frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l} \varphi_{\alpha,\mu_0} \bar{\psi}^a_+ \otimes \varphi_{\beta,\mu_0^I} \bar{\psi}^b_-$$

$$(4.15) \qquad = \frac{-1}{\psi_+ + \psi_-} \cdot \Delta_{\mu_0*}$$

$$\cdot \left(\mathrm{ch}_l\big((q^*F)_{\mu_0}(1 - L_+L_-)\big) \prod_{\delta=1}^{i_{\mu_0}} \mathcal{C}^{\mu_0}_\delta\big((q^*F)_{\mu_0}(1 - L_+L_-)\big) \right).$$

However,

$$(4.16) \qquad \mathrm{ch}_l\big((q^*F)_{\mu_0}(1 - L_+L_-)\big) = \big[\mathrm{ch}(q^*F)_{\mu_0}(1 - e^{\psi_+ + \psi_-})\big]_l;$$

hence,

$$\sum_{a,\alpha;b,\beta} \frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l} \varphi_{\alpha,\mu_0} \bar{\psi}^a_+ \otimes \varphi_{\beta,\mu_0^I} \bar{\psi}^b_-$$

$$(4.17) \qquad = \frac{-1}{\psi_+ + \psi_-} \cdot \Delta_{\mu_0*}$$

$$\cdot \left(\big[\mathrm{ch}(q^*F)_{\mu_0}(1 - e^{\psi_+ + \psi_-})\big]_l \prod_{\gamma=1}^{i_{\mu_0}} \mathcal{C}^{\mu_0}_\delta\big((q^*F)_{\mu_0}(1 - L_+L_-)\big) \right).$$

Below we prove that $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ satisfies the same second-order differential equation. The partial derivative of $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ with respect to w_l equals

$$\mathcal{D}^{-1}_{\mathcal{A},\mathcal{B},\mathcal{C}} \frac{\partial \mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}}{\partial w_l}$$

$$(4.18) \qquad = \sum_{d,n} \frac{Q^d \hbar^{g-1}}{n!}$$

$$\cdot \big\langle \mathbf{t}(\bar{\psi}_1), \ldots, \mathbf{t}(\bar{\psi}_n); \mathrm{ch}_l \, \pi_* \big(\mathrm{ev}^*_{n+1}(F) \otimes i_{\mu_0*} \mathcal{O}_{\mathcal{Z}_\mu}\big) \cdot \Theta_{g,n,d} \big\rangle_{g,n,d}.$$

Lemma 3.10 shows that

$$(4.19) \quad \mathrm{ch}_l \, \pi_* \big(\mathrm{ev}^*_{n+1}(F) \otimes i_{\mu_0*} \mathcal{O}_{\mathcal{Z}}\big) = \pi_* i_{\mu_0*} \left[\mathrm{ev}^*_{n+1} \mathrm{ch}(F) \cdot \frac{e^{\psi_+ + \psi_-} - 1}{\psi_+ + \psi_-} \right]_{l-1}.$$

Using (4.19) and the formula

$$\int_{[\mathcal{X}_{g,n,d}]} (\pi_* i_* a) \cdot b = \int_{[\mathcal{Z}]} a \cdot (\pi \circ i)^* b,$$

we pull back the right-hand side of (4.18) on $\mathcal{Z}$. Moreover, we use Proposition 3.1 to pull back the correlators on the factors $\mathcal{X}_{g_1,n_1+1,d_1} \times \mathcal{X}_{g_2,n_2+1,d_2}$.

The classes $[\mathcal{X}_{g,n,d}]^{\text{tw}}$ pull back as in formulas (3.2) and (3.3). As a consequence, we see that if we define the coefficients $A^{\mu_0,l}_{a,\alpha;b,\beta}$ by

(4.20)
$$\sum_{a,b,\alpha,\beta} A^{\mu_0,l}_{a,\alpha;b,\beta} \varphi_{\alpha,\mu_0} \bar{\psi}^a_+ \otimes \varphi_{\beta,\mu^I_0} \bar{\psi}^b_-$$
$$= \Delta_{\mu_0 *}\left(\left[\text{ch}(q^* F)_{\mu_0} \cdot \frac{e^{\psi_+ + \psi_-} - 1}{\psi_+ + \psi_-}\right]_{l-1}\right.$$
$$\left.\cdot \left(\prod_{\delta=1}^{i_{\mu_0}} \mathcal{C}_\delta\big((q^* F)_{\mu_0} \otimes (1 - L_+ L_-)\big)\right)\right),$$

we can express (4.18) as

(4.21)
$$\mathcal{D}^{-1}_{\mathcal{A},\mathcal{B},\mathcal{C}} \frac{\partial \mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}}{\partial w_l}$$
$$= \sum_{g_i,n_i,d_i} \frac{Q^{d_1+d_2} \hbar^{g_1+g_2-1}}{n_1! n_2!}$$
$$\cdot \sum_{a,b,\alpha,\beta} \frac{1}{2} \langle \mathbf{t}, \ldots, \mathbf{t}, A^{\mu_0,l}_{a,\alpha;b,\beta} \varphi_{\alpha,\mu_0} \bar{\psi}^a_+ ; \Theta_{g_1,n_1+1,d_1}\rangle_{g_1,n_1+1,d_1}$$
$$\cdot \langle \mathbf{t}, \ldots, \mathbf{t}, \varphi_{\beta,\mu^I_0} \bar{\psi}^b_- ; \Theta_{g_2,n_2+1,d_2}\rangle_{g_2,n_2+1,d_2}$$
$$+ \sum_{\substack{g,n,d \\ a,b,\alpha,\beta}} \frac{1}{2} \frac{Q^d \hbar^{g-1}}{n!}$$
$$\cdot \langle \mathbf{t}, \ldots, \mathbf{t}, A^{\mu_0,l}_{a,\alpha;b,\beta} \varphi_{\alpha,\mu_0} \bar{\psi}^a_+, \varphi_{\beta,\mu^I_0} \bar{\psi}^b_- ; \Theta_{g-1,n+2,d}\rangle_{g-1,n+2,d}.$$

Hence, the generating function $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ satisfies the equation

(4.22)
$$\frac{\partial \mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}}{\partial w_l} = \frac{\hbar}{2} \sum_{a,b} A^{\mu_0,l}_{a,\alpha;b,\beta} \partial^{\alpha,\mu_0}_a \partial^{\beta,\mu^I_0}_b \mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}.$$

Comparing (4.17) with (4.20), we see that

$$(4.23) \qquad \frac{\partial A^{\mu_0}_{a,\alpha;b,\beta}}{\partial w_l} = A^{\mu_0,l}_{a,\alpha;b,\beta}.$$

Therefore, both sides of (4.11) satisfy the same partial differential equation. The theorem follows. $\qquad\square$

REMARK 4.1. According to [8, pp. 91–95], this change of generating function corresponds to a change of polarization; namely, we regard the potential $\mathcal{D}_{\mathcal{A},\mathcal{B},\mathcal{C}}$ as an element of the Fock space $\mathcal{H}_{\mathcal{C}} = \mathcal{H}_+ \oplus \mathcal{H}_{-,\mathcal{C}}$. The corresponding element in $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ with the usual polarization is $\mathcal{G}$. If $\{q_a^{\alpha,\mu}, p_b^{\beta,\mu}\}$ and $\{\bar{q}_a^{\alpha,\mu}, \bar{p}_b^{\beta,\mu}\}$ are Darboux coordinate systems on $\mathcal{H}$ and $\mathcal{H}_{\mathcal{C}}$, respectively, then this change of polarization is given in coordinates by

$$(4.24) \qquad \begin{aligned} p_b^{\beta,\mu} &= \bar{p}_b^{\beta,\mu}, \\ \bar{q}_a^{\alpha,\mu} &= q_a^{\alpha,\mu} - \sum_{a,b} A^{\mu}_{a,\alpha;b,\beta} p_b^{\beta,\mu}. \end{aligned}$$

EXAMPLE 4.2. Let $\mathcal{X}$ be a manifold, and let $\mathcal{C}(\pi_* i_* \mathcal{O}_{\mathcal{Z}}) = \mathrm{Td}(-\pi_* i_* \mathcal{O}_{\mathcal{Z}})^\vee$. Then $A_{a,\alpha;b,\beta}$ do not depend on α or β, and we have

$$\mathcal{C}(1 - L_+ L_-) = \mathrm{Td}^\vee(L_+ L_-) = \frac{-\psi_+ - \psi_-}{1 - e^{\psi_+ + \psi_-}}.$$

This gives

$$\sum_{a,b} A_{a,\alpha,b,\beta} \psi^a \psi^b = \frac{1}{\psi_+ + \psi_-} - \frac{1}{e^{\psi_+ + \psi_-} - 1}.$$

According to [8, Section 2.3.2], the expansion of

$$\frac{1}{1 - e^{\psi_+ + \psi_-}} = \sum_{k \geq 0} \frac{e^{k\psi_+}}{(1 - e^{\psi_+})^{k+1}} (e^{\psi_-} - 1)^k$$

gives a Darboux basis on $\mathcal{H}_{\mathcal{C}}$ in the sense of Theorem 1.3; that is, $\varphi_a(e^{k\psi_+}) / ((1 - e^{\psi_+})^{k+1})$ span $\mathcal{H}_-$.

§5. Quantum fake Hirzebruch–Riemann–Roch

As a first application, we recover the quantum Hirzebruch–Riemann–Roch theorem of [8], which expresses the potential of the fake cobordism theory in terms of the cohomological one. Throughout this section, X will be a compact complex manifold.

We first briefly review some basic background facts on complex-oriented cohomology theories. A more detailed review is given in [8].

DEFINITION 5.1. A complex-oriented cohomology theory is a multiplicative cohomology theory E^* together with a choice of element $u_E \in E^2(\mathbb{CP}^\infty)$ such that if $j : \mathbb{CP}^1 \to \mathbb{CP}^\infty$ is the inclusion, then $j^*(u_E)$ is the standard generator of $E^2(\mathbb{CP}^1)$.

We denote the ground ring by $R_E := E^*(pt)$. One can define Chern classes satisfying the usual axioms such that $j^*(u_E)$ is the first Chern class of the Hopf bundle. The Chern–Dold character is the unique multiplicative natural transformation

$$\mathrm{ch}_E : E^*(X) \to H^*(X, R_E),$$

which is the identity if $X = \{pt\}$.

In particular, $\mathrm{ch}_E(u_E)$ is a power series in z, where z is the standard orientation of $H^*(X, R_E)$. We denote it $u_E(z)$. The Todd class is the unique multiplicative class which for a line bundle L is

$$\mathrm{Td}_E(L) := \frac{c_1(L)}{u_E(c_1(L))}.$$

We now fix the cohomology theory to be complex cobordism MU^*. For a given i, $MU^i(X)$ is defined as

$$MU^i(X) := \lim_{j \to \infty} \left[\Sigma^j X, MU(i+j) \right],$$

where $[\ ,\ ]$ denotes homotopy classes of maps, $\Sigma^j X$ is the iterated reduced suspension of X, and $MU(k)$ are the Thom spaces.

Cobordism is universal among complex-oriented cohomology theories in the following sense. For any other cohomology (E, u_E) there is a unique natural transformation $MU \to E$ which maps u to u_E. (We will write u, R, etc., instead of u_{MU}, R_{MU}.) If X has complex dimension n, $MU^i(X)$ can be identified with the complex bordism group $MU_{2n-i}(X)$. This is Poincaré

duality for complex cobordism and bordism. The image of u under the Chern–Dold map is a formal power series $u(z)$, where z is the first Chern class of the universal line bundle.

The ground ring of the cobordism is $R := MU^*(pt) = \mathbb{C}[p_1, p_2, \ldots]$ (we tensored with $\mathbb{C}$), where p_i is the class of the map $\mathbb{CP}^i \to pt$. For a local complete intersection map $f : X \to Y$, there is a pushforward f_* and a Hirzebruch–Riemann–Roch theorem which says that the diagram

$$
\begin{array}{ccc}
MU^*(X) & \xrightarrow{\mathrm{ch}_{MU} \cdot \mathrm{Td}(T_f)} & H^*(X, R) \\
{\scriptstyle f_*}\Big\downarrow & & {\scriptstyle f_*}\Big\downarrow \\
MU^*(Y) & \xrightarrow{\quad\mathrm{ch}_{MU}\quad} & H^*(Y, R)
\end{array}
$$

is commutative. We define fake cobordism-valued Gromov–Witten invariants to be given by the above theorem applied to the morphisms $X_{g,n,d} \to \{pt\}$.

Denote by $\mathcal{T}_{g,n,d}$ the virtual tangent bundle to $X_{g,n,d}$. The genus g descendant cobordism-valued potential (called *extraordinary potential* in [8]) is defined as

$$
\mathcal{F}^g_{MU} := \sum_{d,n} \frac{Q^d}{n!} \int_{[X_{g,n,d}]} \prod_{i=1}^{n} \Big(\sum_{k \geq 0} \mathrm{ch}_{MU}\big(\mathrm{ev}_i^* t_k u(\psi_i)^k \big) \Big) \cdot \mathrm{Td}_{MU}(\mathcal{T}_{g,n,d}).
$$

It is a formal function of

$$
\mathbf{t}(u) := \sum_{k \geq 0} t_k u^k \in MU^*(X)[[u]],
$$

which takes values in the ring $R[[Q]]$. The total extraordinary potential is

$$
\mathcal{D}_{MU} := \exp\Big(\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}^g_{MU} \Big).
$$

We define $\mathcal{U}$ to be the space

$$
\mathcal{U} := MU^*\big(X, \mathbb{C}[[Q]] \big)[[u]].
$$

The symplectic form on $\mathcal{U}$ is

$$
\Omega_{MU}(\mathbf{f}, \mathbf{g}) := \oint_{z=0} \big(\mathbf{f}\big(u(z)\big), \mathbf{g}\big(u(-z)\big) \big)_{MU} \, dz,
$$

with the pairing

$$(\alpha, \beta)_{MU} = \int_X \mathrm{ch}_{MU}(\alpha) \cdot \mathrm{ch}_{MU}(\beta) \cdot \mathrm{Td}_{MU}(T_X).$$

The space $\mathcal{U}_+$ of the polarization on $\mathcal{U}$ is defined to include all power series in u. If we expand

$$\frac{1}{u(-x-y)} = \sum_{k \geq 0} u^k(x) v_k\big(u(y)\big),$$

then $\mathcal{U}_-$ is defined as the span of all $\phi_\alpha v_k(u)$ for all $k \geq 0$, $\phi_\alpha \in MU^*(X)$. It is shown in [8] that these two subspaces realize a polarization of $\mathcal{U}$. To show how the extraordinary potential is related to the cohomological one, we define a modification of $\mathcal{H}$:

$$\mathcal{H}_{MU} := H^*\big(X, R[[Q]]\big)((z)).$$

The pairing and symplectic form on $\mathcal{H}_{MU}$ (henceforth denoted $\mathcal{H}$) are defined in the obvious way. The map

$$\widetilde{\mathrm{ch}}_{MU} : \mathcal{U} \to \mathcal{H},$$

$$\sum_k t_k u^k \mapsto \sqrt{\mathrm{Td}_{MU}(T_X)}\Big(\sum_k \mathrm{ch}_{MU}(t_k) u^k(z)\Big)$$

is a symplectomorphism which maps $\mathcal{U}_+$ to $\mathcal{H}_+$, but it does not map $\mathcal{U}_-$ to $\mathcal{H}_-$. Let

$$(5.1) \qquad \mathbf{q}(z) = \sqrt{\mathrm{Td}_{MU}(T_X)}\big(\mathbf{t}(z) + u(-z)\big).$$

We regard $\mathcal{F}^0_{MU}, \mathcal{D}_{MU}$ as functions of $\mathbf{q}(z)$ (hence a function on $\mathcal{H}_+$) via the identifications above. Let $\widehat{\nabla}$ be the quantized linear symplectic transformation

$$\widehat{\nabla} := \exp(A_{a,\alpha;b,\beta} g^{\alpha\beta} \partial_a^\alpha \partial_b^\beta),$$

with $A_{a,\alpha;b,\beta}$ given in Example 4.2. Let

$$(5.2) \qquad \Delta := \exp\Big(\sum_{m \geq 0} \sum_{l=0}^{\dim(X)} s_{2m-1+l} \frac{B_{2m}}{(2m)!} \mathrm{ch}_l(T_X) z^{2m-1}\Big),$$

where the s_k are defined by

$$\exp\Big(\sum_{k \geq 1} s_k \frac{x^k}{k!}\Big) = \frac{x}{u(x)} \in H^*(X, R).$$

THEOREM 5.2. *We then have*

$$\mathcal{D}_{MU} \approx \widehat{\nabla}\widehat{\Delta}\mathcal{D}.$$

The proof will be a consequence of the description of the virtual tangent bundles to $X_{g,n,d}$ as linear combinations of classes of type $\mathcal{A}, \mathcal{B}, \mathcal{C}$, as follows.

PROPOSITION 5.3. *We have*

$$(5.3) \qquad \mathcal{T}_{g,n,d} := \pi_* \mathrm{ev}_{n+1}^*(T_X - 1) - \pi_*(L_{n+1}^{-1} - 1) - (\pi_* i_* \mathcal{O}_{\mathcal{Z}})^{\vee}.$$

Proof. We follow closely the computation in the dissertation [8]. However, the proof there, while leading to the same formula, is a bit imprecise in assuming that L_{n+1} restricted to $\mathcal{Z}$ is the trivial line bundle. Recall that $\mathcal{Z}$ is the nodal locus in the universal family and that it is parameterized by $\widetilde{\mathcal{Z}}$, which is a fiber product of moduli spaces of maps of lower genus. The gluing map $\widetilde{\mathcal{Z}} \to \mathcal{Z}$ is generically 2 to 1. The symmetry on $\widetilde{\mathcal{Z}}$ permuting the two marked points which become the node after gluing acts nontrivially on the fibers of $i^* L_{n+1}$ above the fixed-point locus. Hence, $i^* L_{n+1}$ is a nontrivial (orbi)bundle on $\mathcal{Z}$. We denote it by L. More precisely, let L' be the $\mathbf{Z}_2$ equivariant line bundle on $\widetilde{\mathcal{Z}}$ which is $\widetilde{\mathcal{Z}} \times \mathbb{C}$ as a set and on which $-1 \in \mathbb{Z}_2$ acts by

$$(x, v) \mapsto (-1 \cdot x, -v) \quad \text{for } x \in \widetilde{\mathcal{Z}}; v \in \mathbb{C}.$$

Then $L = L'/\mathbf{Z}_2$.

Let $(C, x_1, \ldots, x_n)$ be a point in $X_{g,n,d}$, and let D be the divisor of marked points $D = \mathcal{D}_1 + \cdots + \mathcal{D}_n$. Then (see [8] and the references therein)

$$(5.4) \qquad \begin{aligned} \mathcal{T}_{g,n,d} &= \pi_*(\mathrm{ev}^* T_X) - H^0\big(C, \Omega_{\pi}^{\vee}(-D)\big) + H^1\big(C, \Omega_{\pi}^{\vee}(-D)\big) \\ &= \pi_*(\mathrm{ev}^* T_X) - \pi_*\big(\Omega_{\pi}^{\vee}(-D)\big). \end{aligned}$$

Roughly the first summand accounts for deformations of the map, the second for infinitesimal automorphisms of the curve $(C, x_1, \ldots, x_n)$, and the third for deformations of the complex structure of C and smoothing of the nodes. Denote by ω_{π} the dualizing sheaf of the universal family. According to [8], we have the exact sequence

$$(5.5) \qquad 0 \to \omega_{\pi} \to L_{n+1} \to \oplus_j \sigma_{j*}(\mathcal{O}_{\mathcal{D}_j}) \to 0.$$

Using Serre duality and the relation given by the above exact sequence, the second summand in (5.4) becomes

$$(5.6) \qquad -\pi_*\big(\Omega_\pi^\vee(-D)\big) = \big[\pi_*\big(\Omega_\pi(D) \otimes \omega_\pi\big)\big]^\vee = \big[\pi_*(\Omega_\pi \otimes L_{n+1})\big]^\vee.$$

There is an exact sequence

$$(5.7) \qquad\qquad\qquad 0 \to \Omega_\pi \to \omega_\pi \to i_*L \to 0.$$

First, notice that Ω_π and ω_π coincide away from $\mathcal{Z}$. Near a point of $\mathcal{Z}$, the map π can be described locally by

$$\pi : (z, x, y) \to (z, xy),$$

where z is a (vector) coordinate on $\widetilde{\mathcal{Z}}$ viewed as an orbifold chart for $\mathcal{Z}$, and the symmetry $-1 \in \mathbb{Z}_2$ interchanges x and y. Locally, sections of ω_π have the form

$$f(z, x, y)\frac{dx \wedge dy}{d(xy)},$$

and sections of Ω_π are of the form $g(z, x, y)\, dx + h(z, x, y)\, dy$, where we impose the relation $x\, dy + y\, dx = 0$. There is a natural inclusion:

$$\Omega_\pi \to \omega_\pi$$

$$g(z, x, y)\, dx + h(z, x, y)\, dy \mapsto \big(xg(z, x, y) - yh(z, x, y)\big)\frac{dx \wedge dy}{d(xy)}.$$

Sections in the cokernel are represented by elements of the form

$$\alpha(z)\frac{dx \wedge dy}{d(xy)}.$$

This is identified with i_*L because the symmetry acts nontrivially on $dx \wedge dy$. This establishes (5.7).

We now use (5.7) to rewrite $\Omega_\pi = \omega_\pi - i_*L$ and then plug in (5.6):

$$(5.8) \qquad \big[\pi_*(\Omega_\pi \otimes L_{n+1})\big]^\vee = \big[\pi_*(\omega_\pi \otimes L_{n+1})\big]^\vee - \big[\pi_*\big(i_*(L) \otimes L_{n+1}\big)\big]^\vee.$$

The first term in (5.8) equals $-\pi_*[L_{n+1}^{-1}]$ by Serre duality again. Replacing in (5.8), we get

$$(5.9) \qquad \big[\pi_*(\Omega_\pi \otimes L_{n+1})\big]^\vee = -\pi_*[L_{n+1}^{-1}] - \big[\pi_*i_*(L \otimes i^*L_{n+1})\big]^\vee.$$

But $i^*L_{n+1} = L$, and $L^2 = 1$. Hence, the last term in (5.9) is $-(\pi_*i_*\mathcal{O}_\mathcal{Z})^\vee$. Formula (5.3) then follows by plugging (5.9) in (5.4). $\qquad\qquad\square$

Proof of Theorem 5.2. We regard the Todd class Td_{MU} as a family of multiplicative classes depending on the parameters s_i. Then the twisting theorems apply:

- twisting by $\mathrm{Td}_{MU}(\pi_* \mathrm{ev}_{n+1}^*(T_X - 1))$ corresponds to acting by the operator $\widehat{\Delta}$ on the potential $\mathcal{D}$ according to Remark 1.7;
- twisting by $\mathrm{Td}_{MU}(-\pi_*(L_{n+1}^{-1} - 1))$ accounts for the dilaton shift (5.2) according to Theorem 1.2;
- twisting by the class $\mathrm{Td}_{MU}(-(\pi_* i_* \mathcal{O}_{\mathcal{Z}})^\vee)$, which according to the proof of Theorem 1.3 and Example 4.2 is tantamount to acting on the potential by the operator $\widehat{\nabla}$. □

By looking only at genus 0, we easily deduce the following.

COROLLARY 5.4. *The graph of the generating series $\mathcal{F}_{MU}^0$, viewed as a formal function of $\mathbf{q}(z)$ with respect to the polarization*

$$(5.10) \qquad \mathcal{H}_{MU} = \mathcal{H}_+ \oplus \left\{ \phi_\alpha v_k\big(u(z)\big) \mid k \geq 0, \phi_\alpha \in H^*(X, R) \right\},$$

is a Lagrangian cone $\mathcal{L}_{MU}$. It is obtained from the cohomological cone $\mathcal{L}^H$ after rotating by the symplectic transformation Δ.

§6. Applications to the Gromov–Witten theory of $X \times B\mathbb{Z}_m$

In this section we apply the results to the Gromov–Witten theory of the orbifold $X \times B\mathbb{Z}_m$, where X is a smooth complex manifold. The motivation lies in the study of the quantum K-theory of X. The results in this section are used in [12, Section 8].

Let G be a finite group which acts trivially on X, and let $\mathcal{X} = X \times BG$, the stack-theoretic quotient. We denote by $[\gamma_i]$ the conjugacy class of $\gamma_i \in G$ and by $C(\gamma)$ the centralizer of γ. The inertia stack of X/G is the disjoint union $\coprod_i ([\gamma_i], X/C(\gamma_i))$. Therefore,

$$H^*\big(I(X/G), \mathbb{C}\big) = \bigoplus_{[\gamma_i]} H^*(X, \mathbb{C}).$$

Denote by $e_{[\gamma_i]} := 1 \in H^*(([\gamma_i], pt/C([\gamma_i])))$. A basis of $H^*(([\gamma_i], X/C(\gamma_i)))$ is given by $\varphi_a \times e_{[\gamma_i]}$, where $\{\varphi_a\}$ is a basis of $H^*(X, \mathbb{C})$. The Poincaré pairing is given by

$$(\varphi_a \times e_{[\gamma_i]}, \varphi_b \times e_{[\gamma_j]}) = \frac{\delta_{[\gamma_i][\gamma_j^{-1}]}}{|C(\gamma_i)|} \int_X \varphi_a \smile \varphi_b.$$

The J function is defined as

$$J_{\mathcal{X}}(t,-z) = -z + t(z)$$

(6.1)
$$+ \sum_{n,d} \frac{Q^d}{n!} \phi_a \Big\langle \frac{\widetilde{\phi^a}}{-z-\bar{\psi}_1}, t(\bar{\psi}_2), \ldots, t(\bar{\psi}_n) \Big\rangle_{n,d}^{X/G},$$

where $\{\phi_a\}, \{\widetilde{\phi^a}\}$ are dual bases. We use [13, Proposition 3.4] to express the correlators in terms of correlators on $X_{0,n,d}$. In fact, there is a finite degree map $(X \times BG)_{0,n,d,([\gamma_1],\ldots,[\gamma_n])} \to X_{0,n,d}$. In [13] it is shown that the degree equals

$$\frac{|\chi_0^G(\boldsymbol{\gamma})|}{|G|},$$

where

$$\chi_0^G(\boldsymbol{\gamma}) := \Big\{ (\sigma_1,\ldots,\sigma_n) \,\Big|\, 1 = \prod_{j=1}^{n} \sigma_j, \sigma_j \in [\gamma_j] \text{ for all } j \Big\}.$$

Since the $\bar{\psi}$ classes in the correlators are pullbacks of ψ classes from the coarse curve, it follows that

(6.2) $$\Big\langle \prod_i \bar{\psi}_i^{k_i}\big(\mathrm{ev}_i^*(t_i \times e_{[\gamma_i]})\big) \Big\rangle_{0,n,d}^{X/G} = \frac{|\chi_0^G(\gamma)|}{|G|} \Big\langle \prod_i \psi_i^{k_i} \mathrm{ev}_i^*(t_i) \Big\rangle_{0,n,d}^{X},$$

where $t_i \in H^*(X)$.

From now on, let $G = \mathbb{Z}_m$, and let ζ be a primitive mth root of unity. Denote by td_ζ the multiplicative class defined for line bundles L by

$$td_\zeta(L) := \frac{1}{1 - \zeta e^{-c_1(L)}}.$$

We twist the cohomological potential of $\mathcal{X}$ with three types of twisting classes as follows.

- The type $\mathcal{A}$ classes we take to be

$$td\big(\pi_*\mathrm{ev}^*(T_X)\big) \prod_{k=1}^{m-1} td_{\zeta^k}\big(\pi_*\mathrm{ev}^*(T_X \otimes \mathbb{C}_{\zeta^k})\big).$$

For a function $s(x)$, the Euler–Maclaurin asymptotics of $\prod_{r=1}^{\infty} e^{s(x-rz)}$ are given by

$$\sum_{r=1}^{\infty} s(x - rz) = \left(\sum_{r=1}^{\infty} e^{-rz\partial_x}\right) s(x)$$

$$= \frac{z\partial_x}{e^{z\partial_x} - 1}(z\partial_x)^{-1} s(x)$$

$$= \frac{s^{(-1)}(x)}{z} - \frac{s(x)}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} s^{(2k-1)}(x) z^{2k-1},$$

where $s^k = d^k s/dx^k$, s^{-1} is the antiderivative $\int_0^x s(t)\, dt$, and B_{2k} are Bernoulli numbers. The effect of the type $\mathcal{A}$ twisting is as follows.

COROLLARY 6.1. *The cone rotates by the loop group element*

$$\mathcal{L}^{\mathrm{tw}} = \prod_{j=0}^{m-1} (\square_j) \mathcal{L}_{\mathcal{X}},$$

where we think of $\mathcal{L}_{\mathcal{X}}$ as a product of m copies of $\mathcal{L}_X$, and each operator $\square_j$ acts on the copy corresponding to the sector labeled by g^j. Let $[kj/m]$ denote the greatest integer less than kj/m. The operators in the statement are Euler–Maclaurin expansions of the products

$$\square_0 = \prod_i \prod_{r=1}^{\infty} \frac{x_i - rz}{1 - e^{-mx_i + mrz}},$$

$$\square_j = \prod_{k=0}^{m-1} \prod_i \prod_{r=1}^{\infty} \frac{x_i - rz}{1 - \zeta^k e^{-x_i + rz - (kj/m - [kj/m])z}}.$$

Proof. The corollary follows by application of [19, Corollary 4.2.3] to the twisting data described above. □

• The type $\mathcal{B}$ classes we take to be

$$td\big(\pi_*(1 - L_{n+1}^{-1})\big) \prod_{k=1}^{m-1} td_{\zeta^k}\big(\pi_*(1 - L_{n+1}^{-1} \otimes \mathrm{ev}^* \mathbb{C}_{\zeta^k})\big).$$

COROLLARY 6.2. *The dilaton shift changes from $q(z) = \mathbf{t}(z) - z$ to $q(z) = \mathbf{t}(z) - (1 - e^{mz})$.*

Proof. We apply Theorem 1.2 to the potential $\mathcal{F}$.

In our case $f_\beta = -\mathrm{ev}^*_{n+1}(\mathbb{C}_\zeta) \otimes L^{-1}_{n+1}$, we have

$$\frac{f_\beta(L^{-1}_{n+3}) - f_\beta(1)}{L_{n+3} - 1} = \mathbb{C}_\zeta L^{-1}_{n+3},$$

so according to Theorem 1.2 (fix ζ to be primitive mth root of unity), the translation is

$$\mathbf{t}(z) := \mathbf{t}(z) + z - z \prod_{k=0}^{m-1} \mathrm{Td}_{\zeta^k}(-\mathbb{C}_{\zeta^k} \mathbf{L}^{-1}_z)$$

(6.3)

$$:= \mathbf{t}(z) + z - z\frac{1-e^z}{z} \prod_{k=1}^{m-1}(1-\zeta^k e^z) = \mathbf{t}(z) + z - (1 - e^{mz}). \qquad \square$$

• The type $\mathcal{C}$ classes we take to be as follows. We twist by the class $\mathrm{Td}^\vee(-\pi_* i_{g*} \mathcal{O}_{\mathcal{Z}_g})$ the nodal locus $\mathcal{Z}_g$; we twist the locus $\mathcal{Z}_0$ of nonstacky nodes by

$$td^\vee\big(-\pi_*(i_* \mathcal{O}_{\mathcal{Z}_0})\big) \prod_{k=1}^{m-1} td^\vee_{\zeta^k}\big(-\pi_*(i_* \mathcal{O}_{\mathcal{Z}_0} \otimes \mathrm{ev}^* \mathbb{C}_{\zeta^k})\big).$$

We do not twist the other nodal loci.

COROLLARY 6.3. *The nodal twisting changes the polarization in the sectors $(\mathcal{X}, 1)$ and $(\mathcal{X}, g)$ of $I\mathcal{X}$. The new Darboux basis is given by expansions of*

$$\frac{1}{1 - e^{m\psi_+ + m\psi_-}}$$

for $(\mathcal{X}, 1)$ and of

$$\frac{1}{1 - e^{\frac{\psi_+ + \psi_-}{m}}} = \frac{1}{1 - e^{\psi_+ + \psi_-}}$$

for $(\mathcal{X}, g)$.

Proof. According to Theorem 1.3, the coefficients $A^0_{a,\alpha,b,\beta}$ in the untwisted sector are given by

$$-\frac{\prod_{i=0}^{m-1} \mathcal{C}^0_k(1 - L_+ L_-) - 1}{\psi_+ + \psi_-} = -\frac{1}{\psi_+ + \psi_-}\Big(\frac{\psi_+ + \psi_-}{\prod_{k=0}^{m-1}(1 - \zeta^k e^{\psi_+ + \psi_-})} - 1\Big)$$

$$= \frac{1}{\psi_+ + \psi_-} - \frac{1}{e^{m\psi_+ + m\psi_-} - 1}.$$

Then (see Example 4.2 and [8, Section 2.3.2]) the Darboux basis is given by the expansion of $1/(1 - e^{m\psi_+ + m\psi_-})$. In the same way, the coefficients $A^g_{a,\alpha,b,\beta}$ are given by expansion of

$$-\frac{(\mathrm{Td}^\vee(L_+ L_- - 1) - 1)}{\psi_+ + \psi_-} = \frac{1}{\psi_+ + \psi_-} - \frac{1}{e^{(\psi_+ + \psi_-)} - 1},$$

and hence the polarization is given by the expansion of $1/(1 - e^{\bar{\psi}_+/m + \bar{\psi}_-/m})$.

$\square$

Appendix. Grothendieck–Riemann–Roch for stacks

The main tool for proving Theorems 1.1, 1.2, and 1.3 is a generalization of Grothendieck–Riemann–Roch theorem for morphisms of stacks due to Toën [18]. Before stating it, we will introduce more notation.

DEFINITION A.1. Define $Tr : K^0(\mathcal{X}) \to K^0(I\mathcal{X})$ to be the map

$$F \mapsto \oplus \lambda_i(g) F_i$$

on each component $(g, \mathcal{X}_\mu)$ of the inertia stack, where F_i is the decomposition of the g-action and $\lambda_i(g)$ is the eigenvalue of g on F_i.

DEFINITION A.2. Define $\widetilde{\mathrm{ch}} : K^0(\mathcal{X}) \to H^*(I\mathcal{X})$ to be the map $\mathrm{ch} \circ Tr$.

Now each vector bundle E on $\mathcal{X}$ restricts on each connected component $(g, \mathcal{X}_\mu)$ of the inertia stack as the direct sum $E_{\mathrm{inv}} \oplus E_{\mathrm{mov}}$.

DEFINITION A.3. Define $\widetilde{\mathrm{Td}}(E) : K^0(\mathcal{X}) \to H^*(I\mathcal{X})$ to be the class

$$\widetilde{\mathrm{Td}} := \frac{\mathrm{Td}(E_{\mathrm{inv}})}{\mathrm{ch}(Tr \circ \lambda_{-1}(E_{\mathrm{mov}})^\vee)},$$

where λ_{-1} is the operation in K-theory defined as $\lambda_{-1}(V) := \sum_{a \geq 0} (-1)^a \Lambda^a V$. In the following theorem, we assume that the morphism f factors as the composition of a smooth regular immersion followed by a smooth morphism. Then one can define T_f as in the case of local complete intersection morphisms of manifolds.

THEOREM A.4 (see [18]). *Let $f : \mathcal{X} \to \mathcal{Y}$ be a proper morphism of smooth Deligne–Mumford stacks (over $\mathbb{C}$) with quasi-projective coarse moduli spaces. This induces a morphism $If : I\mathcal{X} \to I\mathcal{Y}$. If f factors as stated above, we have*

$$(A.1) \qquad \widetilde{\mathrm{ch}}(f_* E) = If_* \big(\widetilde{\mathrm{ch}}(E) \widetilde{\mathrm{Td}}(T_f) \big).$$

Restricting to the identity component $\mathcal{Y}$ of $I\mathcal{Y}$, we get

$$(A.2) \qquad \mathrm{ch}(f_* E) = If_* \big(\widetilde{\mathrm{ch}}(E) \widetilde{\mathrm{Td}}(T_f)_{|If^{-1}\mathcal{Y}} \big).$$

The universal curve π, to which we apply Theorem A.4, is not necessarily a local complete intersection, so following [19], we proceed as follows. The construction in [2] provides a family of orbicurves

$$(A.3) \qquad \widetilde{\pi} : \mathcal{U} \to \mathcal{M}$$

and an embedding $\mathcal{X}_{g,n,d} \to \mathcal{M}$ satisfying the following properties.
- The family $\mathcal{U} \to \mathcal{M}$ pulls back to the universal family over $\mathcal{X}_{g,n,d}$.
- A vector bundle of the form $\mathrm{ev}_{n+1}^*(E)$ extends to a vector bundle over $\mathcal{U}$.
- The Kodaira–Spencer map $T_m \mathcal{M} \to Ext^1(\mathcal{O}_{\mathcal{U}_m}, \mathcal{O}_{\mathcal{U}_m})$ is surjective for all $m \in \mathcal{M}$.
- The locus $\mathcal{Z} \subset \mathcal{U}$ of the nodes of $\widetilde{\pi}$ is smooth, and $\widetilde{\pi}(\mathcal{Z})$ is a divisor with normal crossings.
- The pullback of the normal bundle $N_{\mathcal{Z}/\mathcal{U}}$ to the double cover $\widetilde{Z}$ given by choice of marked points at the node is isomorphic to the direct sum of the cotangent line bundles at the two marked points.

Thus, technically we apply Grothendieck–Riemann–Roch to $\widetilde{\pi}$ and then cap with the virtual fundamental classes $[\mathcal{X}_{g,n,d}]^{\mathrm{tw}}$. Therefore, in the computations we assume that the universal family π satisfies the above properties.

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