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On the Polyak–Viro Vassiliev Invariant of Degree 4

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Abstract. Using the Polyak–Viro Gauss diagram formula for the degree-4 Vassiliev invariant, we extend some previous results on positive knots and the non-triviality of the Jones polynomial of untwisted Whitehead doubles.

1 Introduction

When some new knot invariants are introduced, one is interested in knowing what properties of knots they measure. For example, the Alexander polynomial can reflect the property of a knot to be fibered or slice, while the Jones polynomial reflects more strongly other properties like being achiral or alternating. Shortly after Vassiliev invariants were introduced [Va], it was shown that whenever any finite number of such invariants admit given values on some knot, they do so also on alternating [S] or slice [Ng] knots (up to the condition of zero Arf invariant), so that Vassiliev invariants are useless in detecting these properties.

Contrarily, via the Gauss diagram formulas of Fiedler [Fi, Fi2, FS] and Polyak–Viro [PV], Vassiliev invariants turned out to have numerous applications to positive and almost positive knots [St, St2]. In particular, the author [St] showed that the (properly normalized) Vassiliev invariants of degree 2 and 3 are bounded below by a multiple of the crossing number of a positive reduced diagram, implying that a positive knot has only finitely many positive (reduced) diagrams.

In this note we prove a similar type of inequality for the degree-4 Vassiliev invariant given in [PV]. The method of proof is similar to the one in [St], but technically more difficult, since the formula lacks the configuration of linked pairs of crossings and contains a negative (coefficient) term, whose contribution must be compensated.

An advantage of the Gauss diagram formulas is that they allow calculation of the invariant in polynomial time with respect to the crossing number of the diagram, and thus allow practical computation for very high crossing number, where the calculation of the link polynomials is considerably slower. Based on such a calculation, we then discuss some applications of the Polyak–Viro formula to untwisted Whitehead doubles and the non-triviality of their Jones polynomial. In particular, we will establish this non-triviality for untwisted Whitehead doubles of knots of up to 15 crossings.

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2 Gauss Sums

2.1 Basic Definitions

We use the Alexander–Briggs notation and the Rolfsen [Ro] tables to distinguish between a knot K and its obverse (mirror image) $!K$. *Projection* is the same as *diagram*, and this means a knot or link diagram. Diagrams are always assumed to be oriented.

We recall briefly the definition of Gauss sum invariants.

Definition 2.1 ([Fi2]) A Gauss diagram of a knot diagram is an oriented circle with arrows connecting points on it mapped to a crossing and oriented from the preimage of the undercrossing (underpass) to the preimage of the overcrossing (overpass).

We will also call the two arrow ends *hook* and *tail*.

Example 2.2 As an example, Figure 1 shows the knot 6_2 in its commonly known projection and the corresponding Gauss diagram.

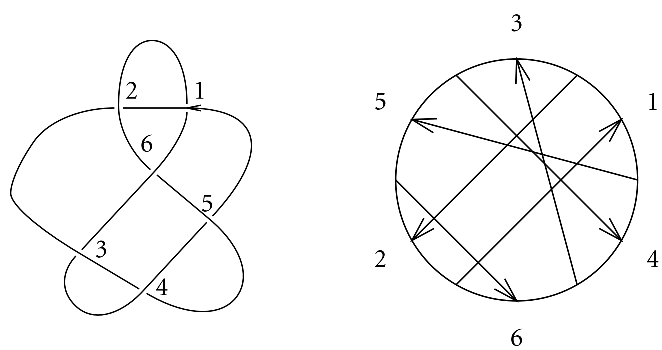

Figure 1: The standard diagram of the knot 6_2 and its Gauss diagram.

Definition 2.3 The *writhe* is a number (± 1), assigned to any crossing in a link diagram. A crossing as in Figure 2(a) has writhe 1 and is called positive. A crossing as in Figure 2(b) has writhe -1 and is called negative.

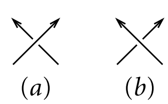

Figure 2

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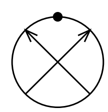
Definition 2.4 A knot is called *positive* if it has a positive diagram, *i.e.,* a diagram with all crossings positive. A knot is called *almost positive* if it is not positive, but has an almost positive diagram, *i.e.,* a diagram with all crossings positive except one.

In addition to their intrinsic knot-theoretical interest, positive knots (or certain sub- or super-classes of them) have been studied in a variety of contexts, including singularity theory [A, BoW], algebraic curves [Ru, Ru2], dynamical systems [Wi] and (in some vague and not yet understood way) in 4-dimensional QFTs [Kr].

A Gauss sum is evaluated by choosing arrows from the Gauss diagram of a knot diagram matching a given configuration, and summing for each such matching choice of arrows a quantity called *weight*. Usually (if not specified, by default) the weight is the product of the writhes of the crossings whose arrows match the configuration. For convenience we will identify a configuration with its evaluation on a fixed specified diagram.

The simplest (non-trivial) Vassiliev invariant is the Casson invariant $v_2 = \nabla_2$, with $\nabla_i = [\nabla]_{z^i}$ being the coefficient of z^i in the Conway polynomial, for which Polyak–Viro gave the simple Gauss sum formula

(1) $v_2 =$  .

Here the point on the circle corresponds to a point on the knot diagram, to be placed arbitrarily except on a crossing.

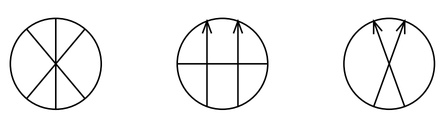
Polyak–Viro and also Fiedler gave other formulas for the degree-3 Vassiliev invariant v_3. To make precise which variation of the degree-3 Vassiliev invariant we mean, we have

$$v_3 = -\frac{1}{12}V_2 - \frac{1}{36}V_3,$$

where V is the Jones polynomial [J], and $V_i := V^{(i)}(1)$. (Note that $v_2 = \nabla_2 = -\frac{1}{6}V_2$.) Fiedler's formula for v_3 [Fi2, FS] reads[1]

$$4v_3 = \sum_{(3,3)} w_p w_q w_r + \sum_{(4,2)0} w_p w_q w_r + \frac{1}{2} \sum_{p,\,q \text{ linked}} (w_p + w_q), \tag{2}$$

where the configurations are



(3, 3) (4, 2)0 p, q linked

Here chords depict arrows which may point in both directions, and w_p denotes the writhe of the crossing p. If two chords a and b intersect, we call the corresponding crossings *linked* and write $a \cap b$ (read "*a intersects b*"). In a linked pair, we call

[1] Note the factor 4 by which (2) differs from the definition in [St]!

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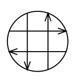
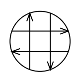
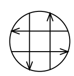
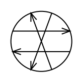
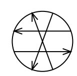
distinguished the crossing whose over-pass is followed by the under-pass of the other crossing when passing the diagram in the orientation direction.

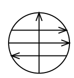
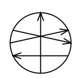
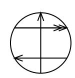
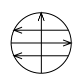
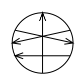
We note that v_3 is asymmetric, *i.e.*, $v_3(!K) = -v_3(K)$, so that achiral knots have zero invariant.

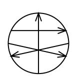
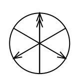
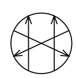
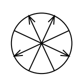
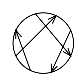
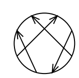
2.2 The Main Result

For the invariants of higher degree the formulas are more complicated, and the analysis becomes more involved. For the degree-4 Vassiliev invariant v_4, the Polyak–Viro formula has the form

(3)

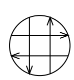
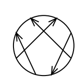
$v_4 =$  $+\,6$  $+\,2$  $+\,3$  $+$ 

$+\,2$  $+\,2$  $+$  $-$  $+$ 

$+$  $+$  $+\,2$  $+\,2$  $+$  $+$  .

Here the weights are taken to be the products of the writhes of the *single* arrows.

For simplicity, denote the diagrams in (3) (without coefficients) by $\langle 1\rangle, \dots, \langle 16\rangle$ in order of occurrence. Thus v_4 is written as $\langle 1\rangle + 6\,\langle 2\rangle + \cdots + \langle 16\rangle$ with

$\langle 1\rangle =$  $, \dots, \langle 16\rangle =$  .

As noted in [PV], v_4 is symmetric (that is, has the same value on mirror images), takes the value 3 on the trefoil(s) and 2 on the figure-eight knot. It is also primitive. However, the values on 3 non-trivial knots are needed to identify the invariant, since the space of symmetric primitive Vassiliev invariants of degree ≤ 4 is 3-dimensional. Using the additional value $v_4 = 25$ on 5_1 (on its usual 5-crossing diagram, all terms in (3) are zero, except $\langle 14\rangle = 5$ and $\langle 12\rangle = 15$) and some calculation, one arrives at the expression

$$v_4 = -\frac{V_4}{144} + 2\nabla_2^2 - \frac{5}{2}\nabla_4 - \frac{V_3}{24} + \frac{\nabla_2}{12}. \tag{4}$$

Our main aim in this note will be to prove the following theorem.

Theorem 2.5 *If D is a positive reduced knot diagram of c crossings, then $v_4(D) \geq 3c/4$.*

From the formulas (1) and (2), it is obvious that on a positive diagram v_2 and v_3 are non-negative, since the formulas in this case basically count the number of matching choices of arrows for some of the configurations. By some more detailed

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arguments we showed that this number is bounded below by a positive multiple of the crossing number of the diagram [St]. To prove Theorem 2.5, a similar analysis is necessary. It is more difficult than for the Vassiliev invariants of lower degree, because linked pairs are not counted and the contribution of a negative (coefficient) term occurs. For this reason and because of the large number of terms, it appears difficult to prove a reasonable estimate of v_4 on almost positive knots. (In a diagram with a negative crossing there are too many configurations of negative contribution to account for.)

2.3 Two Examples

There are examples of knots for which the new inequality is violated, but those for the Vassiliev invariants of degree 2 and 3 shown in [St] are not. Particularly interesting is the knot $16_{1377111}$ in Figure 3. It has $v_2 = 7$ and $v_3 = 7$, so that with $c = 16$ we have $v_2 \geq \frac{c}{4}$ and $v_3 \geq \left\lfloor \frac{c-1}{2} \right\rfloor$. However, $v_4 = 9 < \frac{3}{4}c$. The knot has the positive Conway polynomial $\nabla(\sqrt{z}) = [1]\ 7\ 13\ 9\ 2$ and the (more general) condition of [CM] involving the skein polynomial is also satisfied. The knot also satisfies the equalities

$$\text{(5)} \qquad \min\deg V = \min\deg_v P/2 = \max\deg_z P/2 = \min\deg_a F/2 = g,$$

with P being the skein and F the Kauffman polynomial. (The genus $g = 5$ was determined using [St3].)

The only alternative methods to show non-positivity for this example are the property $\max\deg\nabla = \max\deg_z P$ [Cr], the condition $\min\mathrm{cf}_a F = \min\mathrm{cf}_l P$ [Yo], and the condition on the positivity of the "critical line" coefficients $[F]_{z^l a^m}$ with $m - l$ minimal [Th] (following since positive diagrams are A-adequate). However, these conditions, and also those in (5), involve invariants which are of exponential complexity and are therefore hard to apply for more complicated knots. For example, v_4 takes just a few seconds on diagrams of about 65 crossings, while the calculation of the skein polynomial using the skein method may take up to several days! The genus cannot even always be determined in general, except by the work of Haken, which has proved impractiably complicated.

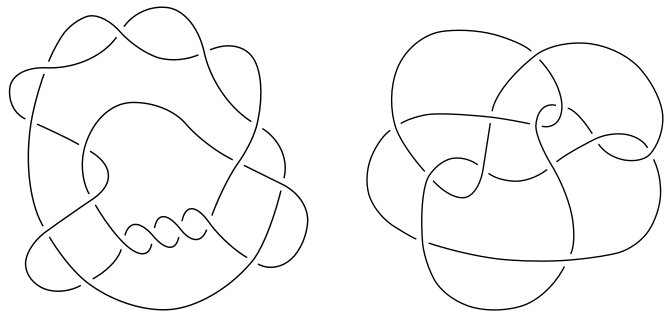
One can find other examples, such as 12_{2089}, for which the equalities in (5) and the above three conditions violated by $16_{1377111}$ are satisfied (in this case $g = 2$ was verified in [St4]), but ours is not.

Thus our criterion is new, and sometimes more effective.

3 Proof of Theorem 2.5

Before we start with the proof, we recall a property and a move of positive diagrams, introduced in [St].

Lemma 3.1 (Extended Even Valence eev(c), see [St]) *In the Gauss diagram of a knot diagram D, any arrow c is intersected by an even number of other arrows. If D is positive, exactly one-half of the arrows intersecting c are distinguished in the linked pair with c (and the other half are not).*

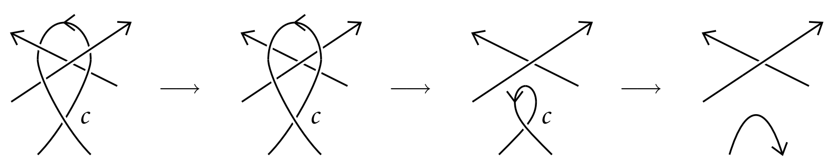


$16_{1377111}$ 12_{2089}

Figure 3: Two knots on which our criterion is rather effective.

Alternatively, we also say that exactly one-half of the arrows intersecting c intersect it in one or the other *direction*.

Definition 3.2 The *loop move* from a knot diagram A to a diagram B, consists in choosing a segment of the line in A between the two passings of a crossing c, such that it has no self-crossings, removing this segment by switching some of the crossings on it (for A positive exactly one-half), and eliminating all reducible crossings thereafter.



On the Gauss diagram, the loop move means choosing an arrow c such that one of the half-arcs into which the ends of c separate the circle contains no endpoints of arrows $b \not\cap c$, then removing c and all $a \cap c$, and finally to deleting all chords b which have become isolated after this removal (these are the chords $b \not\cap c$, such that for all $a \cap b$, also $a \cap c$).

Since the loop move transforms a knot diagram into another knot diagram, its Gauss diagram version preserves the realizability of the Gauss diagram (by a knot diagram). Also, the move preserves the positivity of a knot diagram (and its Gauss diagram).

Proof of Theorem 2.5 We split the proof into three steps, each one estimating the contribution of an appropriate part of the Gauss sum (3), and include some intermediate statements as lemmas.

Step 1: We first show that $v_4 \geq 0$. For this we need to account for the negative (coefficient) term $\langle 9\rangle$. Consider the sum $2\langle 6\rangle + 2\langle 7\rangle + \langle 8\rangle - \langle 9\rangle$ and symmetrize with respect to mirror image, noting that v_4 is invariant under this symmetrization. For the

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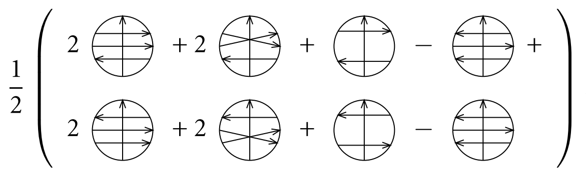
Gauss diagram, mirroring the knot diagram means reversing all arrows and negating the writhe. However, in a positive diagram all weights before and after negating the writhe remain positive, so we can ignore negating the writhe. Then we obtain

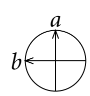
(6)



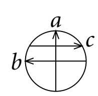
Here we replaced the double arrow in $\langle 8\rangle$ and its arrow-reversed counterpart by a single arrow, since the knot diagram is positive (and the weight does not change), and the Gauss diagrams gain no symmetries after the replacement (contrarily to $\langle 12\rangle$, which would gain a cyclic symmetry of order 3, and whose coefficient would have to be multiplied by 3 accordingly).

We define now the *upper index* $U(a,b)$ of two linked chords a and b as follows.

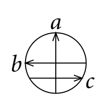
(7)



The arrow b separates the circle into two half-arcs. Count the arrows c (with both ends) on the half-arc of b on which the overpass (arrow hook) of a lies, and which intersect a in the in the opposite direction to that of b.



In the same way define the *lower index* $u(a,b)$, only counting arrows c on the half-arc of b on which the underpass (arrow tail) of a lies.



Now all the configurations in (6) have a distinguished (vertical) arrow, intersecting (as unique arrow) all other arrows in the configuration. Thus the evaluation of (6) on D can be split into the sum for any fixed vertical arrow.

Let a be such an arrow, and let $b_1,\dots,b_r$ be the arrows intersecting a, such that they are distinguished in the linked pair with a (that is, they look like b in (7)). Then set $r' = 2r$, and let $b_{r+1},\dots,b_{r'}$ be the arrows intersecting a, such that a is distinguished in the pair.

For simplicity denote by $\langle\!\langle 1\rangle\!\rangle,\dots,\langle\!\langle 8\rangle\!\rangle$ the diagrams in (6). Now consider $(6)'$, obtained from (6) by reversing the knot orientation. This clearly leaves v_4 invariant too, but for the Gauss diagrams it means to take the mirror image. Accordingly, for $k = 1,\dots,8$ set $\langle\!\langle k'\rangle\!\rangle$ to be the mirror image of $\langle\!\langle k\rangle\!\rangle$. (Arrows that point from left to

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right in $\langle\langle k\rangle\rangle$ point from right to left in $\langle\langle k'\rangle\rangle$ and vice versa.) Then we have

$$\langle\langle 1\rangle\rangle + \langle\langle 2\rangle\rangle = \sum_{i=1}^{r} \binom{U(a,b_i)}{2} \qquad \langle\langle 3\rangle\rangle = \sum_{i=1}^{r} U(a,b_i)$$
$$\langle\langle 5\rangle\rangle + \langle\langle 6\rangle\rangle = \sum_{i=1}^{r} \binom{u(a,b_i)}{2} \qquad \langle\langle 7\rangle\rangle = \sum_{i=1}^{r} u(a,b_i)$$
$$\langle\langle 4'\rangle\rangle = \langle\langle 8'\rangle\rangle = \sum_{i=1}^{r} u(a,b_i)U(a,b_i).$$

Five analogous identities with the sums over $i = r+1,\dots,r'$ hold for the mirrored Gauss diagrams, *i.e.,* reversed knot orientation. Abbreviating $u_i = u(a,b_i)$ and $U_i = U(a,b_i)$, we obtain then

$$\frac{1}{2}\big((6)+(6)'\big) = \frac{1}{4}\Big[\sum_{i=1}^{r'} 2\binom{U_i}{2} + 2\binom{u_i}{2} + U_i + u_i - 2U_iu_i\Big]$$
$$= \frac{1}{4}\sum_{i=1}^{r'}(U_i - u_i)^2 \geq 0.$$

Since the other configurations are non-negative, this shows $v_4 \geq 0$.

As we will need it later, denote the right-hand side in the above estimate

$$\widetilde{w}(a) = \frac{1}{4}\sum_{i=1}^{r'}\big(U(a,b_i) - u(a,b_i)\big)^2,$$

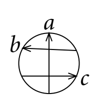
the sum taken over all chords b_i linked with a, distinguished or not.

Now we give a better description of the crossings with $\tilde{w}(a) > 0$.

Lemma 3.3 *$\tilde{w}(a) > 0$ if and only if there exist $b, c \cap a$ in opposite directions with $b \not\cap c$. Moreover, in this case $\tilde{w}(a) \geq \frac{1}{2}$.*

Proof "$\Rightarrow$". If $\tilde{w}(a) > 0$, clearly not all u_i and U_i vanish. Thus such b, c must exist.

"$\Leftarrow$". Let such b and c exist, and fix b and c.

(8) 

We now construct a family $\mathcal{L}$ of chords intersecting a with $b,\ c \in \mathcal{L}$ by induction as follows.

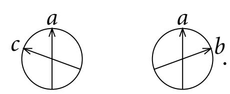
Set $b_0 = b$ and $b_1 = c$ (this indexing is unrelated to the preceding argument). Then assume that for some $k \geq 0$ and $n > 0$, $\mathcal{L}_{k,n} = \{b_i : -k \leq i \leq n\}$ is already defined. If $U(a,b_{-k}) > 0$, set b_{-k-1} to be one of the chords counted by $U(a,b_{-k})$, and if $u(a, b_n) > 0$, set b_{n+1} to be one of the chords counted by $u(a, b_n)$. Otherwise, if $U(a, b_{-k}) = u(a, b_n) = 0$, set $\mathfrak{L} = \mathfrak{L}_{k,n}$.

Then any two arrows in $\mathfrak{L}$ do not intersect, and b_i and b_{i+1} intersect a in opposite directions. Then at least two of the b_i, namely b_{-k} and b_n, have $U(a, b_i) \neq u(a, b_i)$, and thus $\tilde{w}(a) \geq \frac{1}{2}$. ■

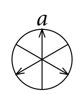
Denote by $(\,\cdot\,)_s$ the symmetrization with respect to *both* orientation and mirroring. Then we just showed that

$$(9) \qquad \big(\, 2\,\langle 6\rangle + 2\,\langle 7\rangle + \langle 8\rangle - \langle 9\rangle \big)_s = \sum_a \tilde{w}(a) \geq \frac{\big|\{a:\ \tilde{w}(a) > 0\}\big|}{2}.$$

Step 2: Now we consider the arrows a with $\tilde{w}(a) = 0$. Clearly all chords $c \cap a$ with c distinguished must intersect all chords $b \cap a$ with b non-distinguished

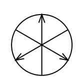
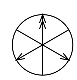


If we have c and b intersecting in such a way that

(10) 

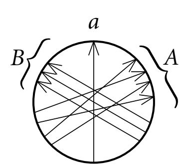
,

then $(a, b, c) \in \langle 12\rangle$ in (3). (Note that in a positive diagram,

 $= \frac{1}{3}$  ,

but it is convenient to retain the distinction of the double arrow as we will soon see.)

Thus consider the case where no b and c with (10) occur. Then we have a picture like this:

(11) 

with

$$(12) \quad A := \{b : b \cap a, a \text{ distinguished}\} \quad \text{and} \quad B := \{b : b \cap a, b \text{ distinguished}\}.$$

Here we drew only a and all $b \cap a$. Because of $eev(a)$ we have $|A| = |B|$.

Consider the arrow c in B whose arrow hook is closest to the arrow hook of a. Since $|A| + 1$ arrows (a including) intersect c in one direction, and from the arrows

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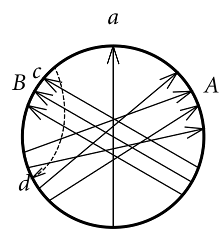
drawn at most $|B| - 1$ intersect c in the other direction, by $eev(c)$ there is at least one other chord d not drawn in (11) (that is, $d \not\pitchfork a$), with $d \cap c$ in the direction opposite to a.

Assume (without loss of generality up to mirroring the diagram and interchanging A and B), that d is on the left of a in (11):



We have drawn d also to intersect arrows in A, as we will now justify.

Lemma 3.4 *d intersects arrows in A* and *B.*

Proof Assume d intersects arrows only in B, but not in A. Then all arrows $d' \not\pitchfork d$ on the opposite side of d to a also intersect (possibly arrows in B, but) no arrows in A. By performing loop moves at a suitable choice of such d' we can achieve that they all disappear, that is, there is no $d' \not\pitchfork d$ on the opposite side of d to a. Call this new positive diagram D''. Applying in D'' a loop move on d gives a positive diagram D'.

We had $|A| = |B|$ in D, and the moves from D to D' did not affect A, but at least the loop move on d deleted (at least) one arrow from B. (We take A and B in D' to be defined as in (12), in the same way as for D.) But then $|B| < |A|$ in D', and thus $eev(a)$ is violated, a contradiction to D' being positive. ■

It follows from the lemma that there exist $c \in A$, $e \in B$ or $e \in A$, $c \in B$, such that $d \cap c, e$ and $d \cap c$ in the opposite direction to a. Then $(a, c, d, e) \in \langle 4 \rangle \cup \langle 5 \rangle$.

In summary, we showed in Step 2 that if $\tilde{w}(a) = 0$, then a participates in at least one of the configurations $\langle 12 \rangle$, $\langle 4 \rangle$ or $\langle 5 \rangle$ of (3). In case a participates in $\langle 12 \rangle$, consider the configuration in which a is the double arrow. Thus we have assigned to a a configuration, which we call *matching configuration* for a.

We have that the changes of both orientation (mirroring the Gauss diagram) and mirroring (reversing the arrows) preserve $\langle 12 \rangle$, and interchange $\langle 4 \rangle$ and $\langle 5 \rangle$. Any configuration $\langle 4 \rangle \cup \langle 5 \rangle$ is realized as matching configuration at most twice (a is one of the two arrows that intersect only two of the remaining three arrows in the configuration, and not all three), and after symmetrization appears with coefficient 2. $\langle 12 \rangle$ is realized for at most one a (it must be the double arrow) and appears with coefficient 1.

Denote for simplicity by $\langle 12 \rangle_{\tilde{w}>0}$ and $\langle 12 \rangle_{\tilde{w}=0}$ the number of configurations of type $\langle 12 \rangle$ in which the double arrow a has $\tilde{w}(a) > 0$, resp., $\tilde{w}(a) = 0$. Then we have

$$\left(3\,\langle 4 \rangle + \langle 5 \rangle + \langle 12 \rangle_{\tilde{w}=0}\right)_s \geq \left|\{a : \tilde{w}(a) = 0\}\right|. \tag{13}$$

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Step 3: We come back to the arrows a with $\tilde{w}(a) > 0$ because we can estimate another contribution of theirs to the Gauss sum, which is different from $\tilde{w}$.

Lemma 3.5 *Any a with $\tilde{w}(a) > 0$ participates in $\langle 1\rangle$, $\langle 2\rangle$, $\langle 3\rangle$, or $\langle 12\rangle$.*

Proof Choose chords c and b with (8) from Lemma 3.3. If all d with $d \cap b, c$ intersect b, c in the same direction as a, analogously to the proof of Lemma 3.4, loop moves at b, and possibly previously on some chords $b' \not\cap b$ on the opposite side of b to that of c would create a positive diagram with violated condition $eev(c)$.

Thus there is a d with $d \cap b, c$ in the opposite direction to a. If $d \cap a$, then (b, d, a) or $(c, d, a) \in \langle 12\rangle$. Otherwise, $(a, b, c, d) \in \langle 1\rangle \cup \langle 2\rangle \cup \langle 3\rangle$. ■

Define for each a with $\tilde{w}(a) > 0$ as before a matching configuration. If $a \in \langle 12\rangle$, then set the matching configuration of a to be the one configuration $\langle 12\rangle$ in which a is the double arrow. Otherwise, set the matching configuration of a to be one of the configurations $\langle 1\rangle \cup \langle 2\rangle \cup \langle 3\rangle$ in which a participates.

Since any configuration in $\langle 1\rangle \cup \langle 2\rangle \cup \langle 3\rangle$ is matching for ≤ 4 arrows a, $\langle 12\rangle$ is matching for at most one a (the double arrow), and since the symmetrization preserves $\langle 1\rangle$ and interchanges $\langle 2\rangle$ and $\langle 3\rangle$, we have

$$\Big(\langle 1\rangle + 6\,\langle 2\rangle + 2\,\langle 3\rangle + \langle 12\rangle_{\tilde{w}>0}\Big)_s \geq \frac{\big|\{a : \tilde{w}(a) > 0\}\big|}{4}. \tag{14}$$

Putting (13), (9), and (14) together, and using the symmetry $v_4 = (v_4)_s$, we now obtain

$$\begin{aligned} v_4 &\geq \Big[\Big(\langle 1\rangle + 6\,\langle 2\rangle + 2\,\langle 3\rangle + \langle 12\rangle_{\tilde{w}>0}\Big) + \Big(3\,\langle 4\rangle + \langle 5\rangle + \langle 12\rangle_{\tilde{w}=0}\Big) \\ &\qquad + \Big(2\,\langle 6\rangle + 2\,\langle 7\rangle + \langle 8\rangle - \langle 9\rangle\Big)\Big]_s \\ &\geq \frac{\big|\{a : \tilde{w}(a) > 0\}\big|}{4} + \big|\{a : \tilde{w}(a) = 0\}\big| + \frac{\big|\{a : \tilde{w}(a) > 0\}\big|}{2} \\ &\geq \frac{3}{4}c(D). \end{aligned}$$

■

The positive 4 crossing diagram of the trefoil shows that the constant $\frac{3}{4}$ cannot be improved (at least without additive correction). Also, because of the same type of examples given in [St], one cannot obtain a lower bound for v_4 in general positive diagrams D growing faster than $O(c(D))$.

4 On the Triviality of the Jones Polynomial

We conclude with some applications of the formula (4). From its integrality, evident from (3), it follows that the Jones polynomial determines ∇_4 mod 2. But, in fact, we have the following:

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Proposition 4.1 *A knot K with trivial Jones polynomial has* $4 \mid \nabla_4$.

Proof If $\nabla_2 = 0$ and $4 \nmid \nabla_4$, then we have for the determinant $\det K = V(-1) = \nabla(2i)$, that $\det K = 1 + 16\nabla_4 - 64\big(\nabla_6 - 4\nabla_8 + \cdots\big) \not\equiv 1 \bmod 64$; in particular $\det K \neq 1$, a contradiction. ∎

Corollary 4.2 *If K has $\nabla \neq 1$ and $V = 1$, then $g(K) \geq 3$.*

Proof We have $\nabla_2 = 0$, because $\nabla_2 = -\frac{1}{6}V_2$, and if $g(K) \leq 2$ and $\nabla_4 \neq 0$, we have $\det K = |16\nabla_4 - 1| \neq 1$, a contradiction. Thus either $g \geq 3$, or $g \leq 2$ and $\nabla_4 = 0$, in which case $\nabla = 1$. ∎

Corollary 4.3 *If $V_K = 1$, then* $10 \mid v_4(K)$.

Proof The only term surviving in (4) is $\frac{5}{2}\nabla_4$, and then we apply Proposition 4.1. ∎

We also have an application to twofold (iterated) untwisted Whitehead doubles. Let $w_\pm$ denote the untwisted double operation of knots with positive (resp., negative) clasp, and $w_{\pm_1,\pm_2} = w_{\pm_2} \circ w_{\pm_1}$.

Proposition 4.4 $v_4(w_{\pm,\pm}(K)) = -8v_2(K), \quad v_4(w_{\pm,\mp}(K)) = 8v_2(K).$

Proof We know from [St] that the dualization $w_\pm^*$ of $w_\pm$ is a nilpotent endomorphism of the space of Vassiliev invariants preserving its (degree) filtration. A basis for the space of Vassiliev invariants of degree ≤ 4 is

$$1, \nabla_2, v_3, \nabla_2^2, \nabla_4, v_4. \tag{15}$$

If $w_\pm^*(v_4)$ has a non-zero coefficient of v_4 in the basis (15), then $w_\pm^*$ cannot be nilpotent, since it kills any of the invariants in (15) except v_4 and v_3, but v_3 has smaller degree than v_4. Thus

$$w_\pm^*(v_4) = a_\pm v_3 + (\text{terms depending on } \nabla \text{ only}).$$

Therefore,

$$w_{\pm_1,\pm_2}^* v_4 = a_{\pm_1} w_{\pm_2}^* v_3 = \pm_2 2a_{\pm_1} v_2$$

by [St]. To determine $a_\pm$, it is sufficient to calculate $v_4(w_{\pm,+}(K))$ for some K with $v_2 \neq 0$. Take the figure-eight knot $K = 4_1$ with $v_2 = -1$. Then we find by calculation (in a few seconds, despite that the diagrams of $w_{\pm,+}(K)$ have 78 crossings) that

$$v_4[w_{\pm,+}(4_1)] = \pm 8 = 2a_\pm \cdot -1, \tag{16}$$

whence $a_\pm = \mp 4$. ∎

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Corollary 4.5 *If K is the twofold untwisted Whitehead double of a knot, then $8 \mid v_4(K)$.*

Corollary 4.6 *If K is the twofold untwisted Whitehead double of a knot with $v_2 \neq 0$, then $V_K \neq 1$.*

In particular, by [St, St2] we obtain the following:

Corollary 4.7 *The twofold untwisted Whitehead double of a positive or almost positive knot has non-trivial Jones polynomial.*

Remark 4.8 T. Stanford [S2] has found that

$$\frac{\nabla_2(\nabla_2+1)}{4}+\frac{\nabla_4}{2}+\frac{v_3}{2}$$

is always integral. This statement is equivalent to $\frac{\nabla_2(\nabla_2+1)}{2}+\nabla_4 \equiv v_3 \bmod 2$, which can be established by checking it on a few knots, since the space of (chord, unitrivalent, etc.) diagrams has no 2-torsion in degree ≤ 4, and so a degree-4 Vassiliev invariant mod 2 is the reduction mod 2 of a degree-4 Vassiliev invariant over $\mathbb{Z}$, for which a basis is well known (see (15)).

Using the integrality of v_4, one can obtain from this that

$$\frac{\nabla_2^2}{4}+\frac{7\nabla_2}{12}-\frac{V_4}{144}$$

is always integral. Thus any knot with trivial Conway polynomial satisfies $144 \mid V_4$.

We finish with a simple necessary condition for the untwisted Whitehead double of a knot to have trivial Jones polynomial.

Proposition 4.9 *If $w_+(K)$ or $w_-(K)$ have trivial Jones polynomial, then $v_2(K) = v_3(K) = 0$.*

Proof That $v_2(K)=0$ follows from [St], since $v_3(w_\pm(K)) = \pm 2v_2(K)$. To be shown is that $v_3(K)=0$. We have from the proof of Proposition 4.4 that

$$\begin{aligned} w_+^* v_4 &= a_2^+\nabla_2 + a_{2,2}^+\nabla_2^2 + a_4^+\nabla_4 + a_3^+ v_3, \\ w_-^* v_4 &= a_2^-\nabla_2 + a_{2,2}^-\nabla_2^2 + a_4^-\nabla_4 + a_3^- v_3\,. \end{aligned} \tag{17}$$

Since

$$\begin{aligned} w_+^* v_4(K) &= v_4(w_+(K)) \;=\; v_4(!w_+(K)) \;=\; v_4(w_-(!K)) \;=\; w_-^* v_4(!K) \\ &= a_2^-\nabla_2(!K) + a_{2,2}^-\nabla_2^2(!K) + a_4^-\nabla_4(!K) + a_3^- v_3(!K) \\ &= a_2^-\nabla_2(K) + a_{2,2}^-\nabla_2^2(K) + a_4^-\nabla_4(K) - a_3^- v_3(K)\,, \end{aligned}$$

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by comparing coefficients with (17), we obtain

$$a_2^- = a_2^+, \quad a_{2,2}^- = a_{2,2}^+, \quad a_4^- = a_4^+, \quad a_3^- = -a_3^+.$$

Since we know already that $a_3^{\pm} \neq 0$, it follows that

$$v_4(w_+(K)) \neq v_4(w_-(K)) \iff v_3(K) \neq 0\,.$$

But by a simple skein argument at the clasp of the Whitehead double, if one of $w_+(K)$ or $w_-(K)$ has trivial Jones polynomial, then both do, and since their (trivial) Conway polynomials also coincide, v_4 vanishes on both doubles. This shows the assertion. ∎

Corollary 4.10 *If K is a (prime or composite) knot of ≤ 15 crossings, then $w_+(K)$ and $w_-(K)$ have nontrivial Jones polynomial.*

Proof By the simple skein argument, it suffices to consider only $w_+(K)$, and from a pair of mirror images only one knot K, as $!w_+(K) = w_-(!K)$. Among the 322,033 such knots up to 15 crossings (up to mirroring and orientation), there are only 7116 with $v_2 = v_3 = 0$, and calculating the polynomials of their Whitehead doubles was feasible (even if after some time). ∎

Since v_2 and v_3 are fast to calculate (on each of the aforementioned 78 crossing diagrams of the twofold untwisted Whitehead doubles of the figure-eight-knot, the calculation took less than a second), the condition is applicable in practice also to more complicated examples. It appears that in average about 97% of the knots can be excluded this way.

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