FORMAL MULTIPLICATION OF TRIGONOMETRIC SERIES

Dedicated to the memory of Hanna Neumann

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1. Introduction and Theorems

Let

$$\sum_{n=-\infty}^{\infty} a_n e^{int} \quad \text{and} \quad \sum_{n=+\infty}^{\infty} b_n e^{int} \tag{1}$$

be the given trigonometric series, then the formal product of them is defined by

$$\sum_{n=-\infty}^{\infty} c_n e^{int} \quad \text{with} \quad c_n = \sum_{n=-\infty}^{\infty} a_{n-m} b_m \tag{2}$$

where the last series is supposed to be convergent for every n.

Rajchman [1] proved the following

THEOREM A. *If the two series* (1) *satisfy the conditions*

$$a_n = o(1) \text{ as } |n| \to \infty \text{ and } \sum_{n=-\infty}^{\infty} |n b_n| < \infty, \tag{3}$$

then the formal product (2) *is convergent at the point where the second series of* (1) *converges to zero.*

Recently, Zygmund [2] proved the

THEOREM B. *If the two series in* (1) *satisfy the conditions*

$$a_n = O(1) \quad \text{as} \quad |n| \to \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |b_n| < \infty, \tag{4}$$

then the function

$$f_2(t) = \frac{1}{2} c_0 t^2 - \sum_{n \neq 0} \frac{c_n}{n^2} e^{int} \tag{5}$$

obtained by integrating the series (2) *twice is smooth at each point where the*

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second series of (1) *converges to zero. The function f is called to be smooth at the point t if*

$$\frac{f(t+h)-2f(t)+f(t-h)}{h} \to 0 \quad \text{as} \quad h \to 0.$$

Our first theorem is as follows:

THEOREM 1. *If the two series in* (1) *satisfy the following conditions*

$$a_n = O(|n|^k) \quad \text{as} \quad n \to \infty \quad \text{and} \quad \sum_{n=-\infty}^{\infty} |n^k b_n| < \infty \tag{6}$$

where $0 \leqq k < 1$, *then the function* (5) *is smooth at each point where the second series in* (1) *converges to zero.*

The case $k = 0$ of Theorem 1 reduces to Theorem B.

We can generalize Theorem 1 as follows:

THEOREM 2. *If the two series in* (1) *satisfy the conditions* (6) *where* $k \geqq 0$, *then the function*

$$f_\alpha(t) = P_\alpha(t) + \sum_{n \neq 0} \frac{c_n}{(in)^\alpha} e^{in} \qquad (P_\alpha \text{ being a polynomial})$$

obtained by integrating the series (2) $v = [k]+2$ *times is* α*-smooth at each point where the second series of (1) converges to zero. The* α*-smooth of f at the point t is defined by*

$$\frac{\Delta_{2h}^{\alpha} f(t-\alpha h)}{h^{\alpha-1}} \to 0 \quad \text{as} \quad h \to 0$$

where

$$\Delta_{2h}^{\alpha} f(t-\alpha h) = \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} f(x+2jh-\alpha h).$$

The case $0 \leqq k < 1$ of Theorem 2 reduces to Theorem 1.

2. Proof of Theorem 1

By the assumption, we can suppose that the second series of (1) converges to zero at the origin and shall prove that f_2 is smooth there. Since, by (5),

$$f_2(2h) - 2f_2(0) + f_2(-2h) = o(h) + 4 \sum_{n \neq 0} \frac{c_n}{n^2} \sin^2 nh,$$

we have to prove that

$$P = \sum_{n \neq 0} c_n \frac{\sin^2 nh}{n^2 h} = o(1) \quad \text{as} \quad h \to 0.$$

We shall define $a(u)$ and $b(u)$ on the whole interval $(-\infty, \infty)$ such that they are continuous everywhere, linear in any interval $(n, n+1)$ $(n = 0, \pm 1, \pm 2, \cdots)$ and

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$$a(n) = a_n, \quad b(n) = b_n \quad (n = 0, \pm 1, \pm 2, \cdots).$$

We write

$$(7) \qquad c(u) = \sum_{m=-\infty}^{\infty} b(m)\, a(u-m) \quad (-\infty < u < \infty).$$

Since we suppose $\sum_{m=-\infty}^{\infty} b(m) = 0$, so that

$$(8) \qquad \int_{-\infty}^{\infty} b(v)\, dv = 0.$$

If we write $j(u) = -u + [u] + 1/2$ for all u, then we can write

$$P = \int_{-\infty}^{\infty} c(u) \frac{\sin^2 uh}{u^2 h} du + \int_{\infty}^{\infty} c(u) \frac{\sin^2 uh}{u^2 h} dj(u) + o(1)$$

$$= P_1 + P_2 + o(1) \quad \text{as } h \to 0$$

and, by (7)

$$P_1 = \int_{-\infty}^{\infty} \frac{\sin^2 uh}{u^2 h} du \left(\int_{-\infty}^{\infty} b(v) a(u-v) dv + \int_{-\infty}^{\infty} b(v) a(u-v) dj(v) \right)$$

$$= Q_1 + Q_2.$$

By (6) and (8),

$$Q_1 = \int_{-\infty}^{\infty} b(v)\, dv \int_{-\infty}^{\infty} a(u-v) \frac{\sin^2 uh}{u^2 h} du$$

$$= \int_{-\infty}^{\infty} b(v)\, dv \left(\int_{-\infty}^{\infty} a(u-v) \frac{\sin^2 uh}{u^2 h} du - \int_{-\infty}^{\infty} a(u) \frac{\sin^2 uh}{u^2 h} du \right)$$

$$= \int_{-\infty}^{\infty} b(v)\, dv \left(\int_{-\infty}^{\infty} a(w) \frac{\sin^2 (w+v)h}{(w+v)^2 h} dw - \int_{-\infty}^{\infty} a(w) \frac{\sin^2 wh}{w^2 h} dw \right)$$

$$= h \int_{-\infty}^{\infty} b(v)\, dv \int_{-\infty}^{\infty} a(w)\, dw \int_{wh}^{(w+v)h} d\left(\frac{\sin^2 y}{y^2} \right)$$

$$= h \left(\int_{-1/h}^{1/h} dv + \int_{1/h}^{\infty} dv + \int_{-\infty}^{-1/h} dv \right) = R_1 + R_2 + R_3.$$

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$$R_1 = h \int_{-1/h}^{1/h} b(v)\,dv \left(\int_{-2/h}^{2/h} + \int_{2/h}^{\infty} + \int_{-\infty}^{-2/h} \right) a(w)\,dw \int_{wh}^{(w+v)h} d\left(\frac{\sin^2 y}{y^2}\right)$$

$$= S_1 + S_2 + S_3$$

where

$$|S_1| \leqq Ah^2 \int_{-1/h}^{1/h} |vb(v)|\,dv \int_{-2/h}^{2/h} |w|^k\,dw$$

$$\leqq Ah^{1-k} \int_{-1/h}^{1/h} |vb(v)|\,dv = o(1) \quad \text{as } h \to 0,$$

since

$$h^{1-k} \int_{-\varepsilon/h}^{\varepsilon/h} |vb(v)|\,dv \leqq \varepsilon^{1-k} \int_{-\varepsilon/h}^{\varepsilon/h} |v^k b(v)|\,dv$$

for small ε and

$$\int_{1/h \geqq |v| \geqq \varepsilon/h} |v^k b(v)|\,dv = o(1) \quad \text{as } h \to 0.$$

Further we get

$$|S_2| \leqq A \int_{-1/h}^{1/h} |vb(v)|\,dv \int_{2/h}^{\infty} w^{k-2}\,dw$$

$$\leqq Ah^{1-k} \int_{-1/h}^{1/h} |vb(v)|\,dv = o(1) \quad \text{as } h \to 0,$$

and similarly

$$S_3 = h \int_{-1/h}^{1/h} b(v)\,dv \int_{2/h}^{\infty} a(-w)\,dw \int_{-wh}^{(-w+v)h} d\left(\frac{\sin^2 y}{y^2}\right) = o(1) \quad \text{as } h \to 0.$$

Thus we have proved $R_1 = o(1)$. For the estimation of R_2, we divide the second integral in six parts as follows:

$$R_2 = h \int_{1/h}^{\infty} b(v)\,dv \int_{-\infty}^{\infty} a(w)\,dw \int_{wh}^{(w+v)h} d\left(\frac{\sin^2 y}{y^2}\right)$$

$$= h \int_{1/h}^{\infty} b(v)\,dv \left(\int_{-\infty}^{-v-1/2h} + \int_{-v-1/2h}^{-v+1/2h} + \int_{-v+1/2h}^{-v/2} + \int_{-v/2}^{-1/2h} + \int_{-1/2h}^{1/2h} + \int_{1/2h}^{\infty} \right)$$

$$a(w)\,dw \cdot \int_{wh}^{(w+v)h} d\left(\frac{\sin^2 y}{y^2}\right),$$

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then

$$|R_2| \leqq Ah \int_{1/h}^{\infty} |b(v)|\, dv \left(\int_{-\infty}^{-v-1/2h} \frac{|w|^k}{(w+v)^2h^2} dw + \int_{-v-1/2h}^{-v+1/2h} |w|^k dw \right.$$

$$\left. + \int_{-v+1/2h}^{-v/2} \frac{|w|^k}{(w+v)^2h^2} dw + \int_{-v/2}^{-1/2h} \frac{|w|^k}{w^2h^2} dw + \int_{-1/2h}^{1/2h} |w|^k dw + \int_{1/2h}^{\infty} \frac{w^k}{w^2h^2} dw \right)$$

$$= o(1) \quad \text{as} \quad h \to 0.$$

Similarly we can prove that $R_3 = o(1)$. Collecting above estimations, we get $Q_1 = o(1)$ as $h \to 0$.

We shall now estimate Q_2. Using integration by parts for inner integral,

$$Q_2 = \int_{-\infty}^{\infty} \frac{\sin^2 uh}{u^2 h} du \int_{-\infty}^{\infty} b(v)a(u-v)dj(v)$$

$$= - \int_{-\infty}^{\infty} \frac{\sin^2 uh}{u^2 h} du \int_{-\infty}^{\infty} j(v)\,(b'(v)a(u-v) - b(v)a'(u-v))\, dv$$

$$= - U_1 - U_2.$$

By the definition of $b(u)$,

$$\int_{-\infty}^{\infty} j(v)\, b'(v)\, dv = 0 \quad \text{and} \quad \int_{-\infty}^{\infty} |v|^k |b'(v)|\, dv < \infty$$

and then we can apply the method of estimation of Q_1 to the integral

$$U_1 = \int_{-\infty}^{\infty} j(v)\, b'(v)\, dv \int_{-\infty}^{\infty} \frac{\sin^2 uh}{u^2 h} a(u-v)\, du$$

and we can see $U_1 = o(1)$ as $h \to 0$. On the other hand we write

$$U_2 = \int_{-\infty}^{\infty} b(v) j(v)\, dv \int_{-\infty}^{\infty} a'(u-v) \frac{\sin^2 uh}{u^2 h} du$$

$$= \int_{-\infty}^{\infty} b(v) j(v)\, dv \int_{-\infty}^{\infty} a(u-v)\, d\left(\frac{\sin^2 uh}{u^2 h}\right)$$

$$= \int_{-M}^{M} + \left(\int_{-\infty}^{-M} + \int_{M}^{\infty} \right) = V_1 + V_2$$

for some M, where

$$V_1 = \int_{-M}^{M} b(v) j(v)\, dv \int_{-\infty}^{\infty} a(u-v) \left(\frac{\sin 2uh}{u^2} - \frac{2\sin^2 uh}{u^3 h} \right) du$$

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and then

$$|V_1| = \int_{-M}^{M} |b(v)|\,dv \left(\int_{-\infty}^{-1/h} + \int_{-1/h}^{1/h} + \int_{1/h}^{\infty}\right) |a(u-v)| \left|\frac{\sin 2uh}{u^2} - \frac{2\sin^2 uh}{u^3 h}\right| du$$

$$\leqq \int_{-M}^{M} |b(v)|\,dv \left\{\int_{-\infty}^{-1/h} \left(\frac{1}{|u|^{2-k}} + \frac{1}{h|u|^{3-k}}\right) du\right.$$

$$\left. + h\int_{-1/h}^{1/h} \frac{du}{(|u|+1)^{1-k}} + \int_{1/h}^{\infty} \left(\frac{1}{u^{2-k}} + \frac{1}{hu^{3-k}}\right) du\right\}$$

$$= o(1) \quad \text{as} \quad h \to 0 .$$

Further

$$V_2 = h\left(\int_{-\infty}^{-M} + \int_{M}^{\infty}\right) b(v)j(v)\,dv \int_{-\infty}^{\infty} a\left(\frac{w}{h} - v\right) d\left(\frac{\sin^2 w}{w^2}\right)$$

$$= h\left(\int_{-\infty}^{-M} + \int_{M}^{\infty}\right) b(v)j(v)\,dv \left(\int_{-\infty}^{-2h|v|} + \int_{-2h|v|}^{2h|v|} + \int_{2h|v|}^{\infty}\right)$$

and then

$$|V_2| \leqq Ah\left(\int_{-\infty}^{-M} + \int_{M}^{\infty}\right) |b(v)|\,dv$$

$$\cdot \left(\int_{-\infty}^{-2h|v|} \left|\frac{w}{h}\right|^k \frac{dw}{w^2} + \int_{-2h|v|}^{2h|v|} |v|^k \left|d\left(\frac{\sin^2 w}{w^2}\right)\right| + \int_{2h|v|}^{\infty} \left(\frac{w}{h}\right)^k \frac{dw}{w^2}\right)$$

which may be made smaller when M is taken sufficiently large. Thus we have proved that $Q_2 = o(1)$ as $h \to 0$ and then $P_1 = o(1)$. Finally,

$$P_2 = \int_{-\infty}^{\infty} c(u) \frac{\sin^2 uh}{u^2 h} dj(u)$$

$$= \sum_{m=-\infty}^{\infty} b(m) \int_{-\infty}^{\infty} a(u-m) \frac{\sin^2 uh}{u^2 h} dj(u)$$

$$= \sum_{m=-\infty}^{\infty} b(m) \int_{-\infty}^{\infty} j(u) \left(a'(u-m) \frac{\sin^2 uh}{u^2 h} du + a(u-m)\, d\left(\frac{\sin^2 uh}{u^2 h}\right)\right)$$

$$= -W_1 - W_2 ,$$

where W_2 is estimated similarly to U_2. We shall now estimate W_1. Since the integral of $j(u)a'(u-m)$ over the interval with integral end points vanishes,

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$$\sum_{m=-[1/h]}^{[1/h]} b(m) \int_{-[1/h]}^{[1/h]} j(u)a'(u-m) \frac{\sin^2 uh}{u^2 h} du$$

$$= \sum_{m=-[1/h]}^{[1/h]} b(m) \int_{-[1/h]}^{[1/h]} j(u)\, a'(u-m) \left(\frac{\sin^2 uh}{u^2 h} - h \right) du$$

$$= \sum_{m=-[1/h]}^{[1/h]} b(m) \int_{-[1/h]}^{[1/h]} j(u)\, a'(u-m) \left(\sum_{k=2}^{\infty} \frac{(-1)^{k-1} 2^{2k-1}}{(2k)!} u^{2k-2} h^{2k-1} \right) du$$

$$= \sum_{k=2}^{\infty} \frac{(-1)^{k-1} 2^{2k-1}}{(2k)!} C_k$$

where

$$C_k = h^{2k-1} \sum_{m=-[1/h]}^{[1/h]} b(m) \int_{-[1/h]}^{[1/h]} j(u)\, a'(u-m)\, u^{2k-2}\, du$$

$$= (2k-2)\, h^{2k-1} \sum_{m=-[1/h]}^{[1/h]} b(m) \int_{-[1/h]}^{[1/h]} u^{2k-3}\, du \int_{-[1/h]}^{u} j(v)\, a'(v-m)\, dv$$

$$= (2k-2)\, h^{2k-1} \sum_{m=-[1/h]}^{[1/h]} b(m) \int_{-[1/h]}^{[1/h]} u^{2k-3}\, du \int_{[u]}^{u} j(v)\, a'(v-m)\, dv$$

$$= O\left(h^{1-k} \sum_{m=-[1/h]}^{[1/h]} | b(m) | \right) = O\left(h^{1-k} \sum_{m=-[1/h]}^{[1/h]} | m |^k \, | b(m) | \right)$$

$$= o(1), \quad \text{as} \quad h \to 0,$$

so that we can get

$$\sum_{m=-[1/h]}^{[1/h]} b(m) \int_{-[1/h]}^{[1/h]} j(u)\, a'(u-m) \frac{\sin^2 uh}{u^2 h} du = o(1) \quad \text{as} \quad h \to 0 .$$

On the other hand

$$\sum_{m=-[1/h]}^{[1/h]} b(m) \int_{[1/h]}^{\infty} j(u)\, a'(u-m) \frac{\sin^2 uh}{u^2 h} du$$

$$= \sum_{m=-[1/h]}^{[1/h]} b(m) \int_{[1/h]}^{\infty} \left(\frac{\sin 2uh}{u^2} - \frac{2\sin^2 uh}{u^3 h} \right) \int_{[u]}^{u} j(v)\, a'(v-m)\, dv$$

$$= O\left(\sum_{m=-[1/h]}^{[1/h]} | b(m) | \int_{[1/h]}^{\infty} \frac{du}{u^{2-k}} \right) + O\left(\frac{1}{h} \sum_{m=-[1/h]}^{[1/h]} | b(m) | \int_{[1/h]}^{\infty} \frac{du}{u^{3-k}} \right)$$

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$$= O\left(h^{1-k} \sum_{m=-[1/h]}^{[1/h]} |b(m)|\right) = o(1) \quad \text{as} \quad h \to 0 .$$

Similarly we can estimate rest terms in W_1 and we get $W_1 = o(1)$ as $h \to 0$. Thus we get $P_2 = o(1)$ and then $P = o(1)$, and we have proved the theorem.

3. Proof of Theorem 2

Under the conditions

$$a_n = O(|n|^k), \quad \sum_{n=-\infty}^{\infty} |n^k b_n| < \infty \; (k \geqq 0) \quad \text{and} \quad \sum_{n=-\infty}^{\infty} b_n = 0 ,$$

we have to prove that

$$\Delta_{2h}^{\alpha} f_\alpha(t - \alpha h) = o(h^{\alpha-1}) ,$$

where $\alpha = [k] + 2$ and

$$\Delta_{2h}^{\alpha} f_\alpha(t - \alpha h) = \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} f_\alpha(t + 2jh - \alpha h)$$

$$\sim \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{c_n}{(in)^\alpha} \sum_{j=0}^{\alpha} (-1)^j \binom{\alpha}{j} e^{in(t+2jh-\alpha h)}$$

$$= \sum_{\substack{n=-\infty \\ n \neq 0}}^{\infty} \frac{c_n}{(in)^\alpha} e^{int} \cdot 2^\alpha \sin^\alpha nh .$$

We shall define $a(u)$, $b(u)$ $c(u)$ and $j(u)$ as before. Applying the method of proof of Theorem 1, we can prove that

$$P = \sum_{n \neq 0} c_n \frac{\sin^\alpha nh}{n^\alpha h^{\alpha-1}}$$

$$= \int_{-\infty}^{\infty} c(u) \frac{\sin^\alpha uh}{u^\alpha h^{\alpha-1}} du + \int_{-\infty}^{\infty} c(u) \frac{\sin^\alpha uh}{u^\alpha h^{\alpha-1}} dj(u) + o(1)$$

$$= o(1) \quad \text{as} \quad h \to 0 .$$

References

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