J. Austral. Math. Soc. (*Series A*) 36 (1984), 153–175

SATURATED AND EPIMORPHICALLY CLOSED VARIETIES OF SEMIGROUPS

P. M. HIGGINS

(Received 19 April 1982; revised 15 October 1982)

Communicated by T. E. Hall

Abstract

We establish a necessary condition (E) for a semigroup variety to be closed under the taking of epimorphisms and a necessary condition (S) for a variety to consist entirely of saturated semigroups. Condition (S) is shown to be sufficient for heterotypical varieties and a stronger condition (S') is shown to be sufficient for homotypical varieties.

1980 *Mathematics subject classification* (*Amer. Math. Soc.*): 20 M 07, 20 M 05.
Keywords and phrases: semigroup, variety, epimorphism, dominion, saturated, homotypical, heterotypical.

1. Preliminaries and introduction

Let U, S be semigroups with U a subsemigroup of S. Following Howie and Isbell [17] we say that U *dominates* an element $d \in S$ if for every semigroup T and all pairs of homomorphisms $\alpha, \beta: S \to T$, $\alpha | U = \beta | U$ implies that $d\alpha = d\beta$. The set of all elements of S dominated by U is called the *dominion* of U in S and is denoted by $\mathrm{Dom}(U, S)$. It is easily verified that $\mathrm{Dom}(U, S)$ is a subsemigroup of S containing U.

Let $\alpha: S \to T$ be a semigroup homomorphism. Then α is an *epimorphism* (epi for short) if for every pair of homomorphisms $\beta, \gamma: T \to V$, $\alpha\beta = \alpha\gamma$ implies $\beta = \gamma$. Every onto homomorphism is an epimorphism although the converse is false [5]. One can easily show that a homomorphism $\alpha: S \to T$ is epi if and only if the inclusion $i: S\alpha \to T$ is epi, which is equivalent to the statement that $\mathrm{Dom}(S\alpha, T) = T$.

We say U is *epimorphically embedded* in S if $\mathrm{Dom}(U, S) = S$ while at the other extreme we say U is *closed* in S if $\mathrm{Dom}(U, S) = U$, and that U is *absolutely closed* if U is closed in every containing semigroup S. A semigroup U is *saturated* if $\mathrm{Dom}(U, S) \neq S$ for every properly containing semigroup S. A class of semigroups $\mathcal{C}$ is *epimorphically closed* if $S \in \mathcal{C}$ and $\alpha: S \to T$ is epi implies that $T \in \mathcal{C}$. A class of semigroups is *saturated* (*absolutely closed*) if all its members are saturated (absolutely closed).

It is clear that every absolutely closed class is saturated, but the converse is false: the variety of normal bands is saturated [13], although the 2×2 rectangular band is not absolutely closed [17, Theorem 2.9].

Now let $\mathcal{C}$ be a morphically closed class of semigroups, that is, a class such that $S\alpha \in \mathcal{C}$ whenever $S \in \mathcal{C}$ and $\alpha: S \to T$ is a morphism. (Any variety is in particular morphically closed.) It is easy to see that if $\mathcal{C}$ is also saturated then it is epimorphically closed. The converse is, however false: the variety of commutative semigroups is epimorphically closed [18, Corollary 2.5], but is not saturated, since, for example, the inclusion of an infinite cycle semigroup into an infinite cyclic group is epimorphic.

A most useful characterization of semigroup dominions is provided by Isbell's Zigzag Theorem.

RESULT 1 [18, Theorem 2.3 or 16, Chapter 7, Theorem 2.13]. *Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in \mathrm{Dom}(U, S)$ if and only if $d \in U$ or there is a series of factorizations of d as follows*: $d = u_0 y_1 = x_1 u_1 y_1 = x_1 u_2 y_2 = x_2 u_3 y_2 = \cdots = x_m u_{2m-1} y_m = x_m u_{2m}$ *where* $m \geqslant 1$, $u_i \in U$, $x_i, y_i \in S$ *with* $u_0 = x_1 u_1$, $u_{2i-1} y_i = u_{2i} y_{i+1}$, $x_i u_{2i} = x_{i+1} u_{2i+1}$ $(1 \leqslant i \leqslant m-1)$ *and* $u_{2m-1} y_m = u_{2m}$.

Such a series of factorizations is called a *zigzag* in S over U with *value d*, *length m and spine* $u_0, u_1, u_2, \ldots, u_{2m}$. If m and n are positive integers and α is a function from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$ then $f(x_1, x_2, \ldots, x_m) = x_{\alpha(1)} x_{\alpha(2)} \cdots x_{\alpha(n)}$ is a typical semigroup product in the variables $x_1, x_2, \ldots, x_m$. The length of f will be denoted by $|f|$ and the number of occurrences of a variable x_i in f will be denoted by $|x_i|_f$. Whenever the variables $x_1, x_2, \ldots, x_m$ are introduced it will be understood that they are all distinct.

The notations and conventions of Clifford and Preston [4] and Howie [16] will be used throughout without explicit reference.

The general question of when are epis onto and what properties are preserved by epis have been studied in semigroup theory, ring theory and elsewhere [2]. For example, in [7] Gardner showed that epis are onto for regular rings. A proof that the variety of commutative rings is epimorphically closed can be found in [1] although Gardner has shown that certain identities weaker than commutativity are not preserved by epimorphisms of rings [8]. The author has shown [12] that the absolutely closed varieties of semigroups are those consisting entirely of Clifford semigroups or entirely of right groups or entirely of left groups. All saturated varieties of commutative semigroups have been determined jointly by the author and N. M. Khan ([14] and [19]) while in showing that all varieties of commutative semigroup are epimorphically closed [19] Khan has generalised the classic result of Isbell [18, Corollary 2.5] that commutativity is preserved by epis. However finding a complete determination of all saturated and epimorphically closed varieties of semigroups remains an open problem.

2. An example

We will construct a semigroup S, which satisfies no non-trivial identity, and which is dominated by a subsemigroup U, which satisfies an identity ϕ if and only if both sides of ϕ contain a repeated variable.

Let F be the free semigroup on the countably infinitely many generators $\{x_1, x_2, \dots, a_1, a_2, \dots, y_1, y_2, \dots\}$. Let A be the subsemigroup of F generated by $\{a_1, a_2, \dots\}$. Let ρ_0 be the relation on F consisting of the pairs (wv, w) and (vw, w) for all words v of A, and all words w of A contained a repeated letter, together with the pairs define by the zigzags: $x_n = a_{6n-2}y_{2n} = x_{2n}a_{6n-1}y_{2n} = x_{2n}a_{6n}$ for all $n = 1, 2, \dots$ and $y_n = a_{6n+1}y_{2n+1} = x_{2n+1}a_{6n+2}y_{2n+1} = x_{2n+1}a_{6n+3}$ for all $n = 0, 1, 2, \dots$, that is, $(x_n, a_{6n-2}y_{2n})$, $(a_{6n-2}, x_{2n}a_{6n-1})$, $(a_{6n-1}y_{2n}, a_{6n})$ for all $n = 1, 2, \dots$ and $(y_n, a_{6n+1}y_{2n+1})$, $(a_{6n+1}, x_{2n+1}a_{6n+2})$, $(a_{6n+2}y_{2n+1}, a_{6n+3})$ for all $n = 0, 1, 2, \dots$; we note that y_0 is a symbol denoting a_1y_1 and is not a generator of F. Let ρ be the congruence generated by ρ_0 and put $S = F/\rho$ and $U = A\rho^{\natural}$. By construction $\mathrm{Dom}(U, S) = S$, and note that all words of A with a repeated letter are in the same ρ-class which forms the 0 of U. We next show that $U \neq S$ by showing that $y_0\rho \notin U$.

First we introduce some convenient defintions. For an arbitrary elementary ρ_0-transition $puq \to pvq$ with $p, q \in F^1$ we call u the *base* and v the *replacement* of the transition. Elementary transitions of the type $pwvq \to pwq$ or $pvwq \to pwq$, where $p, q \in F^1$, and their reversals, will be known as *zero transitions*. By a *forward transition* we will mean one of the type $pa_{6n-2}q \to px_{2n}a_{6n-1}q$, $pa_{6n-1}y_{2n}q \to pa_{6n}q$, $pa_{6n+1}q \to px_{2n+1}a_{6n+2}q$ or $pa_{6n+2}y_{2n+1}q \to pa_{6n+3}q$ while the corresponding reversals will be called *backward transitions*. Transitions of the type $px_nq \to pa_{6n-2}y_{2n}q$ and $py_nq \to pa_{6n+1}y_{2n+1}q$ will be called *upward transitions* and their reversals will be called *downward transitions*. Collectively, upward and forward transitions will be called *positive transitions* while backward and downward transitions will be called *negative transitions*. A sequence of elementary transitions I will be called *positive* of it consists entirely of positive transitions. A set of the form $\{a_{3n-2}, a_{3n-1}, a_{3n}\}$ $n = 1, 2, \ldots,$ will be called a *companion set* and each member of the set is a *companion* of the other two. The companion sets correspond to the spines of the above zigzags.

LEMMA 2. *Suppose* $w\rho a_1 y_1$ *and let* I: $a_1 y_1 \to \cdots \to w' \to w$ *be a shortest possible sequence of elementary* ρ_0*-transitions form* $a_1 y_1$ *to* w. *Then the following conditions are satisfied*:

(i) w *contains no repeated letter*;

(ii) w *does not contain two members from any one companion set*;

(iii) I *is positive*;

(iv) *no two transitions of* I *have the same base and any base of a transition in* I *does not occur in* w;

(v) *there is a factorization* $w = w_1 w_2 w_3$ *of* w *in* F^1 *such that*

(a) $w_2 = x_m$ *or* $w_2 = a_{3m-2} y_m$, *for some* $m \geqslant 1$,

(b) *if* $w_2 = x_m$ *for some* $m \geqslant 1$, *then* $w_3 \neq 1$,

(c) w_3 *is a product of words* a_{3m}, $n = 1, 2, 3, \ldots,$ *and* $a_{3n-1} y_n$, $n = 1, 2, 3, \ldots$;

(vi) *there is a factorization* $w = v_1 v_2 v_3$ *of* w *in* F^1 *such that*

(a) $v_2 = y_m$ *or* $v_2 = x_m a_{3m}$, *for some* $m \geqslant 1$,

(b) *if* $v_2 = y_m$ *for some* $m \geqslant 1$, *then* $v_1 \neq 1$,

(c) v_1 *is a product of words* a_{3n-2}, $n = 1, 2, 3, \ldots,$ *and* $x_n a_{3n-1}$, $n = 1, 2, 3, \ldots$.

REMARK. It follows at once from either conditions (v) or (vi) that $C(w) \nsubseteq A$ and so $y_0 \rho \notin U$ as required.

PROOF. We proceed by induction on $|I|$, the number of transitions in I. If $|I| = 0$ then conditions (i) to (vi) are evidently satisfied. Consider an arbitrary shortest sequence I and suppose the lemma holds for the initial subsequence J: $a_1 y_1 \to \cdots \to w'$ of I with w_1', w_2', w_3' and v_1', v_2', v_3' being subwords of w' satisfying conditions (v) and (vi) respectively.

We shall consider the transition $w' \to w$ which is either (1) a zero transition, (2) an upward transition based on some x_n or y_n, (3a) a forward transition based on a_{3n-2} for some $n \geqslant 1$, (3b) a forward transition based on $a_{3n-1} y_n$ for some $n \geqslant 1$ or (4) a negative transition. We shall show that cases (1) and (4) do not arise while in cases (2), (3a) and (3b) the conditions (i) to (vi) of the lemma continue to hold.

By condition (i) applied to w' the transition $w' \to w$ cannot be a zero transition thereby eliminating case (1).

Next consider case (2) and suppose $w' \to w$ has the form $px_n q \to pa_{6n-2} y_{2n} q$ (the case where $w' \to w$ is based on some y_n is similar). By condition (iv), no transition based on x_n has occurred in J, and since x_n is the base of the unique positive transition which introduces either of the letters a_{6n-2} or y_{2n}, it follows that w has not repeated letters, that is, w satisfies condition (i). Since the unique positive transition which introduces a_{6n-1} is based on a a_{6n-2}, it follows that $a_{6n-1}, a_{6n} \notin C(w)$ and so condition (ii) is satisfied by w, and of course that condition (iii) is satisfied is clear, while condition (iv) follows from the facts that the letters a_{6n-1} and y_{2n} have not appeared in J and w' has no repeated letters, so that $x_n \notin C(w)$. For condition (v) we note that if w_2' is in q then we may take $w_i = w_i'$, $i = 2, 3$, otherwise $w_1' = p$, $w_2' = x_n$ and $w_3' = q$ whence we can take $w_1 = w_1'$, $w_2 = a_{6n-2}y_{2n}$ and $w_3 = w_3'$.

To prove condition (vi) we note that v_3' can act as v_2 if v_2' occurs in p, otherwise v_2' occurs in x_nq whence we may put $v_1 = pa_{6n-2}$, $v_2 = y_{2n}$ and $v_3 = q$.

Next we consider case (3a) where $w' \to w$ has the form $pa_{3n-2}q \to px_na_{3n-1}q$ for some $n \geqslant 1$. Since a_{3n-2} and a_{3n-1} are companions, it follows that $a_{3n-1}$is not a repeated letter by condition (ii). If x_n were repeated, then since J has no negative transitions, this would imply that a forward transition based on a_{3n-2} occurred in J, which contradicts condition (iv), as a_{3n-2} appears in w'. Hence w satisfies condition (i), and conditions (ii) and (iii) are clearly satisfied, condition (ii) also following from the fact that a_{3n-2} and a_{3n-1} are companions. Condition (iv) applied to w' shows that no transition based on a_{3n-2} occurred in J, which implies condition (iv) holds for I. To show condition (v) holds, we note that if w_2' occurs in q then w_2' can serve as w_2 in w, while w_2' cannot occur in p. The remaining case is where $pa_{3n-2}q = pa_{3n-2}y_nq'$ where $y_nq' = q$ and $w_2' = a_{3n-2}y_n$. We then take $w_1 = p$, $w_2 = x_n$ and $w_3 = a_{3n-1}q$. As for condition (vi) v_2' must occur either in p or q and in either case we may take $v_2 = v_2'$.

In case (3b) $w' \to w$ has the form $pa_{3n-1}y_nq \to pa_{3n}q$. Conditions (i) to (iv) then follow in the same way as for case (3a), while for condition (v) w_2' occurs in either p or q and can serve as w_2 in w. As for condition (vi) we note that v_2' occurs either in p, in which case it may serve as v_2, or $pa_{3n-1}y_nq = p'x_na_{3n-1}y_nq$ where $p = p'x_n$ with $v_1' = pa_{3n-1}$, $v_2' = y_n$ and $v_3' = q$ whence we may take $v_1 = p'$, $v_2 = x_na_{3n}$ and $v_3 = q$.

Finally, we consider cases (4) where we suppose $w' \to w$ is a negative transition of the form $puq \to pvq$. In this case, a positive transition of the form $p'vq' \to p'uq'$ for some $p', q' \in F^1$ has occurred in J, and by condition (iv) this is the unique transition of J based on v. Hence no word of J preceding $p'uq'$ contains u. Observe that J contains no transition based on a subword of u, as this would contradict condition (iv) since u occurs in w'. This allows us to construct a new sequence, I': $a_1y_1 \to \cdots \to p'vq' \to \cdots \to w$, whose transitions are based on the corresponding transitions of J, but with the transition $p'vq' \to p'uq'$ deleted. In detail, I' is identical to J up to and including the appearance of the word $p'vq'$, and the words in I' appearing after $p'vq'$ correspond to the words of J appearing

https://doi.org/10.1017/S1446788700024629 Published online by Cambridge University Press

after $p'uq'$, except that in the words of I' the subword v appears instead of u. However, $|I|=|I|-2$, contradicting our choice of I and so we conclude case (4) does not arise, thus completing the proof.

LEMMA 3. *The semigroup U satisfies an identity ϕ if and only if both sides of ϕ contain a repeated variable.*

PROOF. An identity ϕ in which both sides contain a repeated variable is satisfied by U, as both sides become 0 upon substitution of the variables of ϕ with any members of U.

Conversely, the subsemigroup of U generated by $\{a_2, a_5, \ldots, a_{3n-1}, \ldots\}$ is a relatively free semigroup on countably infinitely many generators satisfying all identities for which both sides contain a repeated variable, for if w is a product of this set without repeats, there are no non-trivial ρ_0-transitions from w. This completes the proof.

Since U is properly epimorphically embedded in S it follows that no variety containing U is saturated. This observation together with Lemma 3, implies that any variety $\mathcal{V}$ which only admits non-trivial identities for which both sides contain a repeated variable contains U, and so is not saturated. We will now show that any such $\mathcal{V}$ is not epimorphically closed (unless it is the variety of all semigroups) by showing that S generates the variety of all semigroups. We prove this by showing that the subsemigroup S' of S generated by y_0 and $a_2 y_0$ is a free semigroup on two generators, and so contains a free semigroup on countably infinitely many generators [6, Theorem 1], which satisfies no non-trivial identity.

LEMMA 4. *Let w be a word F, u an arbitrary product of $a_2 y_0$ and y_0 which we may write as*

$$(a_2 y_0)^{m(1)} y_0^{n(1)} \cdots (a_2 y_0)^{m(k)} y_0^{n(k)}$$

where $m(1) \geqslant 0$, $m(i) \geqslant 1$ for all $1 < i \leqslant k$, $n(i) \geqslant 1$ for all $1 \leqslant i < k$, $n(k) \geqslant 0$. Then $w \rho u$ if and only if there is a factorization $w = r_1 s_1 r_2 s_2 \cdots r_k s_k$ such that

(a) *for all $1 \leqslant i \leqslant k$ each r_i admits a factorization $r_i = a_2 p_{i_1} a_2 p_{i_2} \cdots a_2 p_{i_{m(i)}}$ where each $p_{i_j} \rho y_0$ and*

(b) *for all $1 \leqslant i \leqslant k$ each s_i admits a factorization $s_i = q_{i_1} q_{i_2} \cdots q_{i_{n(i)}}$ where each $q_{i_j} \rho y_0$.*

REMARK. The statement of the lemma says that $w \rho u$ if and only if w has the same form as that given for u, with each instance of y_0 replaced by some word ρ-related to y_0.

PROOF. The 'if' part of the statement is immediate. To prove the converse we let I: $u \to \cdots \to w' \to w$ be a sequence of elementary transitions from u to w, and we assume inductively that w' can be factorized in the manner of the statement of the lemma. We establish the lemma by showing that the base of the transition $w' \to w$ s contained in one of the p's or q's occurring in this factorization.

If this were not the case, the base of $w' \to w$ would be one of the following: (1) a word of A containing a repeated letter; (2) $a_2 t$ where t is the first letter of some p; (3) ta_2 where t is the last letter of some p or (4) $t_1 t_2$ where t_1 is the last letter of some $p_{m(k)}$ or q and t_2 is the first letter of the following q.

In general, if $v\rho y_0$ and $v = v'a$ where a is a word in A then by Lemma 2(v) we have $C(a) \subseteq \{a_{3n}, n = 1, 2, 3, \dots\}$ and dually, if $v = av'$ where a is a word of A then by Lemma 2(vi) we have $C(a) \subseteq \{a_{3n-2}, n = 1, 2, 3, \dots\}$. These sets are disjoint and a_2 is not a member of either. From this, and the fact that any word ρ-related to y_0 has no repeated letter (Lemma 2(i)), it follows that w' contains no word of A with a repeated letter, and hence the transition $w' \to w$ is not a zero transition, and therefore case (1) does not arise.

For case (2) to arise we would have $t = y_1$, but no word ρ-related to y_0 begins with y_1 by Lemma 2(vi) so this is impossible. Similarly, case (3) cannot happen, as no word ρ-related to y_0 ends with x_1 by Lemma 2(v). Lastly, case (4) does not arise as no word ρ-related to y_0 begins with an a_{3n-1} by Lemma 2(vi), nor ends with an a_{3n-1} or a_{3n-2} by Lemma 2(v).

LEMMA 5. *The semigroup S satisfies no non-trivial identity.*

PROOF. We show that the subsemigroup S' of S generated by $\{y_0, a_2 y_0\}$ is freely generated by this pair. Let $u, v \in S'$ and suppose $u\rho v$ with $u = (a_1 y_0)^{m(1)} y_0^{n(1)} \cdots (a_2 y_0)^{m(k)} y_0^{n(k)}$, $v = (a_2 y_0)^{s(1)} y_0^{t(1)} \cdots (a_2 y_0)^{s(l)} y_0^{t(l)}$. By Lemma 4, u can be factorized in the form given for v with the y_0's replaced by words ρ-related to y_0. However by Lemma 2(iv), if $p\rho y_0$ and y_0 occurs in p then $p = y_0$, and since $(y_0, a_2) \notin \rho$ each p and occurring in this factorization of u contains y_0, and so equals y_0, which implies $u = v$ as required.

An identity $f(x_1, x_2, \dots, x_n) = g(x_1, x_2, \dots, x_n)$ is *heterotypical* if $C(f) \neq C(g)$; otherwise the identity is *homotypical.*

We can now give the main result of this section.

THEOREM 6. *Suppose $\mathcal{V}$ is a variety not equal to the variety of all semigroups. Then $\mathcal{V}$ is epimorphically closed only if*

(E) each set of identities which define $\mathcal{V}$ contains a non-trivial identity for which at least one side contains no repeated variable. Condition (E) is equivalent to the condition that $\mathcal{V}$ admits a nontrivial homotypical identity of the form $x_1 x_2 \cdots x_n = f(x_1, x_2, \dots, x_n)$.

Furthermore $\mathcal{V}$ is saturated only if

(S) each set of identities which define $\mathcal{V}$ contains a non-trivial identity, not a permutation identity, for which at least one side contains no repeated variable. Condition (S) is equivalent to the condition that $\mathcal{V}$ admits a homotypical identity of the form $x_1x_2\cdots x_n = f(x_1, x_2,\ldots,x_n)$ with $|x_i|_f > 1$ for some variable x_i.

PROOF. Suppose $\mathcal{V}$ is an epimorphically closed variety which is defined by a set of identities I, and suppose further that for each non-trivial member of I both sides contain a repeated variable. Then by Lemma 3, $U \in \mathcal{V}$, which implies $S \in \mathcal{V}$, and so by Lemma 5 we see that $\mathcal{V}$ is the variety of all semigroups.

To show the equivalence of condition (E) and the given condition, we let $\mathcal{V}$ be a variety admitting a non-trivial identity ϕ for which at least one side contains no repeated variable, $x_1x_2\cdots x_n = f(x_1, x_2,\ldots,x_m)$. Replace each variable on the right, which does not occur on the left by x_1, to get an identity ϕ': $x_1x_2\cdots x_n = f'(x_1, x_2,\ldots,x_n)$. Next suppose x_i is a variable which occurs on the left of ϕ', but not on the right. It follows that the value of $x_1x_2\cdots x_n$ is independent of the value assigned to x_i, so we get $x_1x_2\cdots x_i\cdots x_n = x_1x_2\cdots(x_ix_{i+1}\cdots x_nx_i)x_{i+1}\cdots x_n = f'(x_1, x_2,\ldots,x_n)x_ix_{i+1}\cdots x_n$, on replacing x_i by $x_ix_{i+1}\cdots x_nx_i$ in ϕ'. Therefore ϕ implies the identity $x_1x_2\cdots x_n = f'(x_1, x_2,\ldots,x_n)x_ix_{i+1}\cdots x_n$, and repeating this procedure for each such x_i eventually yields a homotypical identity of the required type. Conversely take any variety $\mathcal{V}$ admitting a non-trivial homotypical identity $x_1x_2\cdots x_n = f(x_1, x_2,\ldots,x_n)$. Then by Lemma 3, $U \notin \mathcal{V}$ and it follows again by Lemma 3, that any defining set of identities for $\mathcal{V}$ must contain one of the required type.

To prove the next statement, we consider the example in [14] of a non-saturated, commutative semigroup, which satisfies every identity for which both sides contain a repeated variable. If $\mathcal{V}$ is any saturated variety, this semigroup is not included in $\mathcal{V}$, and so each set of defining identities for $\mathcal{V}$ must include a non-trivial identity, not a permutation identity, for which at least one side contains no repeated variable. The equivalence of condition (S) to the stated condition can now be proved in the same way as we proved the equivalence of condition (E) and the other condition.

3. Semigroups admitting a heterotypical identity

The following is a list of definitions of six 'nil' conditions on a semigroup S with a zero 0.

(N) S is *nil* if given $x \in S$ there exists a positive integer $n = n(x)$ such that $x^n = 0$.

(NB) S is *nil-bounded* if there exists a positive integer n such that $x^n = 0$ for all $x \in S$.

(LT) S is *left T-nilpotent* if given a list $x_1, x_2, x_3, \ldots$ of elements of S there exists a positive integer $n = (x_1, x_2, x_3, \ldots)$ such that $x_1 x_2 \cdots x_n = 0$.

(RT) S is *right T-nilpotent* if given a list $x_1, x_2, x_3, \ldots$ of elements of S there exists a positive integer $n = n(x_1, x_2, x_3, \ldots)$ such that $x_n x_{n-1} \cdots x_1 = 0$.

(T) is *T-nilpotent* if S is both left and right T-nilpotent.

(NP) S is *nilpotent* if there exists a positive integer n such that $S^n = 0$.

For S satisfying condition (NB), the least integer n such that $x^n = 0$ for all $x \in S$ we will call the *index* of S.

The implications that exists between these conditions are as follows: (NP) implies (T), which implies both (LT) and (RT), which each imply (N), and (NP) also implies (NB), which implies (N). For commutative semigroups, conditions (LT), (RT) and (T) are clearly equivalent, but otherwise the list of implications is the same. It is routine to construct counter examples to prove there are no implications except those given above. The only case of some difficulty involves showing that the conditions (LT) and (RT) are not equivalent. The semigroup S with zero 0 and non-zero elements the ordered pairs of positive integers (i, j), such that $i < j$, with multiplication according to the rule that

$$(i, j)(r, s) = \begin{cases} (i, s) & \text{if } j = r, \\ 0 & \text{if } j \neq r \end{cases}$$

is the right T-nilpotent but not left T-nilpotent. This example first appears in [21]. The origin of the concept of T-nilpotency is in ring theory; see for example [1] or [9].

A variety $\mathcal{V}$ is called *heterotypical* if it admits a heterotypical identity; otherwise $\mathcal{V}$ is *homotypical*. The structure of the members of a heterotypical variety is described by the following result due to Chrislock [3].

RESULT 7. *For a semigroup S the following are equivalent*:

(1) *S satisfies a heterotypical identity*;

(2) *S satisfies a heterotypical identity of the form $(x^m y^m x^m)^m = x^m$ for some $m \geqslant 1$*;

(3) *S is an ideal extension of a completely simple semigroup whose structure group satisfies $x^r = 1$, (for some $r \geqslant 1$), by a nil-bounded semigroup of index r.*

https://doi.org/10.1017/S1446788700024629 Published online by Cambridge University Press

Bulazewska and Krempa [1] have shown that left [right] T-nilpotent rings are saturated. We now give some similar results for semigroups, which were derived jointly with T. E. Hall, and which when combined with Result 7 will give a partial converse to Theorem 6.

THEOREM 8. *An ideal extension of a saturated semigroup by a T-nilpotent semigroup is saturated.*

PROOF. Let I be an ideal of a semigroup U, such that I is saturated, U/I is T-nilpotent and suppose U is properly epimorphically embedded in a semigroup S. Let $d \in S\setminus U$, and suppose Z is a zigzag with value d, and shortest possible length m, in S over U. By the zigzag theorem we may successively rewrite $u_{2i}y_{i+1}$, $i = 0, 1, \ldots, m-1$ as $u_{2i}y_{i+1} = u_{2i}u_{2i}^{(1)}y_{i+1}^{(1)} = u_{2i}u_{2i}^{(1)}u_{2i}^{(2)}y_{i+1}^{(2)} = \cdots$, where each $u_{2i}^{(j)} \in U$, $y_{i+1}^{(j)} \in S$, and by left T-nilpotency we have for some n_i, $u_{2i}y_{i+1} = u_{2i}[u_{2i}^{(1)}u_{2i}^{(2)} \cdots u_{2i}^{(n_i)}]y_{i+1}^{(n_i)}$, where the bracketed term is an element of the ideal I of U which we rename s_i, we rename $y_{i+1}^{(n_i)}$ as y'_{i+1}. Using right T-nilpotency we can rewrite each x_iu_{2i-1}, $i = 1, 2, \ldots, m$ as $x'_it_iu_{2i-1}$ where $t_i \in I$, and so we can construct a new zigzag Z' with value d, whose spine consists entirely of elements of the ideal I:

$$\begin{aligned}
d &= (u_0s_1)y'_1 = x'_1(t_1u_1s_1)y'_1 = x'_1(t_1u_2s_2)y'_2 = \cdots = x'_i(t_iu_{2i-1}s_i)y'_i \\
&= x'_i(t_iu_{2i}s_{i+1})y'_{i+1} = x'_{i+1}(t_{i+1}u_{2i+1}s_{i+1})y'_{i+1} = \cdots \\
&= x'_m(t_mu_{2m-1}s_m)y'_m = x'_m(t_mu_{2m})
\end{aligned}$$

where $u_0s_1 = x'_1t_1u_1s_1$, $x'_it_iu_{2i}s_{i+1} = x'_{i+1}t_{i+1}u_{2i+1}s_{i+1}$ for all $1 \leqslant i \leqslant m-1$, $t_iu_{2i-1}s_iy'_i = t_iu_{2i}s_{i+1}y'_{i+1}$ for all $1 \leqslant i \leqslant m-1$ and $t_mu_{2m-1}s_my'_m = t_mu_{2m}$, and the bracketed terms form the spine of the zigzag Z'.

Since Z has shortest possible length, clearly $x_1, x_2, \ldots, x_m, y_1, y_2, \ldots, y_m \in S\setminus U$, so the new zigzag Z' shows that d is in the dominion of I in $\langle I \cup (S\setminus U)\rangle$, the subsemigroup generated by $I \cup (S\setminus U)$. Thus I is epimorphically embedded in $\langle I \cup (S\setminus U)\rangle$, a contradiction. The proof is complete.

It is natural to ask whether or not this theorem is true if the condition of T-nilpotence is replaced by one of the weaker conditions of left T-nilpotence of nil-boundedness. The answer in the latter case is 'no', because in [14] the author has given an example of a commutative semigroup satisfying $x^2 = 0$, which is not saturated. However the question of whether or an ideal extension of a saturated semigroup by a left T-nilpotent semigroup is necessarily saturated remains unanswered.

THEOREM 9. *An ideal extension of* (a) *a regular saturated semigroup, or* (b) *a finite saturated semigroup, by a left T-nilpotent semigroup is saturated. In particular, left T-nilpotent semigroups are saturated.*

PROOF. *Case* (a). Let U be an ideal extension of a regular saturated semigroup I, such that U/I is a left T-nilpotent semigroup. Using left T-nilpotency as in the proof of Theorem 8 we can replace each $u_{2i}y_{i+1}$ by $u_{2i}s_{i+1}y'_{i+1}$. This gives a new zigzag Z' with value d, for which each element of the spine is a member of the ideal I, except perhaps the final one, u_{2m}. The final pair of lines of Z' are $d = x_m u_{2m-1}s_m y'_m = x_m u_{2m}$ with $u_{2m-1}s_m y'_m = u_{2m}$. We rename $u_{2m-1}s_m$ as s and since I is regular and $s \in I$, there exists $s' \in V(s)$, and so we have $u_{2m} = sy'_m = ss'(sy'_m) = ss'u_{2m} \in I$. Hence the spine of Z' does consist entirely of members of I. In the fashion of the proof of Theorem 8 we can now derive the contradiction that I dominates the subsemigroup of S generated by $I \cup (S\backslash U)$.

REMARK. Note that the proof of the above result goes through under weaker conditions than regularity: namely that for each $s \in I$ there exists an $\bar{s} \in I$ such that $\bar{s}s = s$.

PROOF (continued). *Case* (b). Let U be an ideal extension of a finite saturated semigroup I, such that U/I is left T-nilpotent. Proceed as in (a), and construct a zigzag Z' with value d for which each element of the spine is a member of I, except perhaps for the final one, u_{2m}. The final two lines of Z' have the form $x_m s_{2m-1} y_m = x_m u_{2m}$ with $s_{2m-1}y_m = u_{2m}$, $s_{2m-1} \in I$. Using the zigzag theorem we successively factorize x_m to get

$$x_m = x_m^{(1)}t_1 = x_m^{(2)}t_2t_1 = \cdots = x_m^{(i)}t_it_{i-1}\cdots t_1 = \cdots.$$

Consider the corresponding sequence in I: $t_1s_{2m-1}, t_2t_1s_{2m-1}, \ldots, t_it_{i-1}\cdots t_1s_{2m-1}, \cdots$. Since I is finite there exist some i, j, $i<j$ such that $t_jt_{j-2}\cdots t_1s_{2m-1} = t_it_{i-1}\cdots t_1s_{2m-1}$, which implies that $t_it_{i-1}\cdots t_1s_{2m-1} = t^pt_it_{i-1}\cdots t_1s_{2m-1}$, where $t = t_jt_{j-1}\cdots t_{i+1}$ and p is any positive integer. Since U/I is nil-bounded (left T-nilpotent in fact) there exists a positive integer p such that $t^p = s \in I$. This allows us to construct a new zigzag Z'' with value d, which is the same as Z', except the final two lines of Z'' are $x_m^{(i)}t_it_{i-1}\cdots t_1s_{2m-1}y_m = x_m^{(i)}t_it_{i-1}\cdots t_1u_{2m}$ with $t_it_{i-1}\cdots t_1s_{2m-1}y_m = t_it_{i-1}\cdots t_1u_{2m}$. This last spine member has a left identity s in I, and so is itself a member of I. The proof is now complete as in Case (a).

COROLLARY 10. *Any finite semigroup S satisfying a heterotypical identity is saturated.*

PROOF. By Result 7, S is an ideal extension of a completely simple semigroup by a nil-bounded semigroup. However, since the six listed nil conditions are equivalent for finite semigroups [22, Theorem 1.2], we can replace 'nil-bounded' by 'nilpotent' in the preceding statement. Since completely simple semigroups are saturated [11, Theorem 1], the result now follows by Theorem 8.

The next result is known for commutative semigroups [17, Corollary 3.4].

COROLLARY 11. *A semigroup S is saturated if S^n is saturated for some $n > 1$.*

PROOF. Since S is a nilpotent extension of the ideal S^n, the result follows by Theorem 8.

LEMMA 12. *Let S be a semigroup satisfying a homotypical identity ϕ: $x_1x_2 \cdots x_n = f(x_1, x_2, \ldots, x_n)$ for which $f(x_1, x_2, \ldots, x_n)$ contains a repeated variable. Then given $a \in S^n$ there exists $c, d \in S$, $e \in E(S)$ such that $a = ced$.*

PROOF. Note that $|f| > n$, and so ϕ may be written in the form $x_1x_2 \cdots x_n = v_1w_1$ where $|v_1| = n$, $|w_1| \geq 1$ (where $|w|$ means the length of the word w). We next apply ϕ to v_1 and by repetition of this process we produce a sequence of identities all implied by ϕ:

$$\begin{aligned}
\phi = \phi_1&: x_1x_2 \cdots x_nx_n = v_1w_1,\\
\phi_2&: x_1x_2 \cdots x_n = v_2w_2w_1,\\
&\vdots\\
\phi_i&: x_1x_2 \cdots x_n = v_iw_iw_{i-1} \cdots w_1, \ldots,
\end{aligned}$$

where $|v_i| = n$ for all $i = 1, 2, \ldots$ and $|w_i| \geq 1$ for all $i = 1, 2, \ldots$. Note that $C(v_1) \supseteq C(v_2) \supseteq \cdots$. Eventually we reach ϕ_k where k is the least integer such that $C(v_k) = C(v_l)$ for all $l \geq k$. Next let $r \geq k$ be the least integer such that there exists an integer m such that $v_r = v_{r+m}$. Then we have

$$\begin{aligned}
\phi_r&: x_1x_2 \cdots x_n = v_rw_rw_{r-1} \cdots w_1,\\
\phi_{r+m}&: x_1x_2 \cdots x_n = v_r(w_{r+m}w_{r+m-1} \cdots w_{r+1})w_rw_{r-1} \cdots w_1,\\
\phi_{r+sm}&: x_1x_2 \cdots x_n = v_r(w_{r+m}w_{r+m-1} \cdots w_{r+1})^s w_rw_{r-1} \cdots w_1,
\end{aligned}$$

for all positive integers s.

Replacing all variables of ϕ by the single variable x we see that S satisfies $x^n = x^{|f|}$ and hence there exists an integer t, $n \leq t < |f|$, such that $a^t \in E(S)$ for all $a \in S$. Now take $a \in S^n$, so that $a = a_1a_2 \cdots a_n$ say, for some $a_1, a_2, \ldots, a_n \in S$. Applying the identity ϕ_{r+tm} to $a_1a_2 \cdots a_n$ yields $a = cb^td$ where $c, b, d \in S$. By the above comment $b^t \in E(S)$, thus completing the proof.

LEMMA 13. *The globally idempotent members of a variety of semigroups $\mathcal{V}$ form a subvariety if and only if they are completely regular.*

PROOF. Suppose the globally idempotent members of $\mathcal{V}$ form a subvariety $\mathcal{V}'$. Then $\mathcal{V}'$ does not include the two element null semigroup, and so admits an identity of the form $x = x^{1+n}$ for some $n \geqslant 1$, which implies that all members of $\mathcal{V}'$ are unions of groups.

Conversely let $\mathcal{V}'$ be the class of all globally idempotent members of $\mathcal{V}$ and suppose each such member is completely regular. Take $S \in \mathcal{V}'$ and $a \in S$. Since S is completely regular there exists $e \in E(S)$ such that $ae = ea = a$. The monogenic semigroup generated by $a, \langle a \rangle$ cannot be infinite for if it were $\mathcal{V}$ would contain the subsemigroup T of S, $T = \{e, a, a^2, \ldots\}$ which is globally idempotent but not completely regular. Now let U be a subsemigroup of S and take $u \in U$. Since S is a union of groups, $\langle u \rangle$ is a periodic subsemigroup of a group and so itself a group. Therefore U is completely regular. Since morphic images and direct products of completely regular semigroups are completely regular, it follows that the class of completely regular semigroups in $\mathcal{V}$, namely $\mathcal{V}'$, is a subvariety, as required.

COROLLARY 14. *A semigroup variety $\mathcal{V}$ consists entirely of nilpotent extensions of completely regular semigroups if and only if $\mathcal{V}$ admits an identity of the form $x_1x_2 \cdots x_n = (x_1x_2 \cdots x_n)^{1+m}$ for some $m, n \geqslant 1$.*

PROOF. Suppose $\mathcal{V}$ consists entirely of nilpotent extensions of completely regular semigroups but that there is no bound on the indices of the nilpotent members of $\mathcal{V}$. Then there exists a sequence of nilpotent semigroups in $\mathcal{V}$, $S_1, S_2, \ldots$ such that index of $S_i <$ index of S_{i+1} for $i = 1, 2, \ldots$. Let $T = S_1 \times S_2 \times \cdots$. No power of T is globally idempotent, but $T \in \mathcal{V}$, a contradiction. Hence there exists an n such that S^n is completely regular for all $S \in \mathcal{V}$. It follows from Lemma 13 that $\mathcal{V}$ satisfies $x_1x_2 \cdots x_n = (x_1x_2 \cdots x_n)^{1+m}$ for some $m \geqslant 1$. The converse is immediate.

THEOREM 15. *Let $\mathcal{V}$ be a variety of semigroups. The following are equivalent:*

(i) *$\mathcal{V}$ is a saturated heterotypical variety;*

(ii) *$\mathcal{V}$ is heterotypical and admits a homotypical identity of the form $x_1x_2 \cdots x_n = f(x_1, x_2, \ldots, x_n)$ with $|x_i|_f > 1$ for some variable x_i;*

(iii) *$\mathcal{V}$ is heterotypical and admits an identity of the form $x_1x_2 \cdots x_n = (x_1x_2 \cdots x_n)^{1+m}$ for some $m, n \geqslant 1$;*

https://doi.org/10.1017/S1446788700024629 Published online by Cambridge University Press

(iv) *there exists a positive integer n such that S^n is a completely simple semigroup for all $S \in \mathcal{V}$;*

(v) *$\mathcal{V}$ consists entirely of nilpotent extensions of completely simple semigroups.*

PROOF. By the proof of Corollary 14 we have that conditions (iv) and (v) are equivalent. We prove (iii) and (iv) are equivalent.

If $\mathcal{V}$ is heterotypical and admits the identity $x_1x_2 \cdots x_n = (x_1x_2 \cdots x_n)^{1+m}$, then for any $S \in \mathcal{V}$ we have S^n is a union of groups, and so is a semilattice of completely simple semigroups [16, Chapter IV, Theorem 17]. Since $S^n \in \mathcal{V}$ it follows that S^n satisfies a heterotypical identity, and so this semilattice is trivial. Therefore S^n is a completely simple semigroup for all $S \in \mathcal{V}$. Conversely, if S^n is completely simple for all $S \in \mathcal{V}$, it follows by Lemma 13 that $\mathcal{V}' = \{S^n: S \in \mathcal{V}\}$ is a subvariety of $\mathcal{V}$. Moreover, $\mathcal{V}'$ is heterotypical as it includes only trivial semilattices, and $\mathcal{V}'$ satisfies $x = x^{1+m}$ for some $m \geqslant 1$ as it excludes null semigroups. Therefore $\mathcal{V}$ is heterotypical and admits the identity $x_1x_2 \cdots x_n = (x_1x_2 \cdots x_n)^{1+m}$.

We now show that (i) implies (ii), (ii) implies (iv) and (iv) implies (i).

(i) implies (ii). This follows immediately from Theorem 6.

(ii) implies (iv). Suppose $\mathcal{V}$ is heterotypical and admits the identity ϕ: $x_1x_2 \cdots x_n = f(x_1, x_2, \ldots, x_n)$ with $|x_i|_f > 1$ for some variable x_i. Given any $S \in \mathcal{V}$, then by Result 7, S is an ideal extension of a completely simple semigroup T, by a nilbounded semigroup U. It suffices to show that U satisfies the identity $x_1x_2 \cdots x_n = 0$.

Now ϕ is an identity of the type described in Lemma 12. As in the proof of Lemma 12, we construct the sequence $\{\phi_i\}_{i \in \mathbf{N}}$. In particular, ϕ implies the identity ϕ_{r+sm}: $x_1x_2 \cdots x_n = v_r(w_{r+m}w_{r+m-1} \cdots w_{r+1})^s w_r w_{r-1} \cdots w_1$ where s is the index of U, from which it follows that U satisfies the identity $x_1x_2 \cdots x_n = 0$ as required.

(iv) implies (i). It follows from Corollary 11 that $\mathcal{V}$ is saturated. If $\mathcal{V}$ were homotypical it would contain the variety of semilattices, but for any non-trivial semilattice S we have $S = S^n$ (for all $n \in \mathbf{Z}^+$) and S is not completely simple. Therefore we conclude that $\mathcal{V}$ is a saturated heterotypical variety, thus completing the proof.

REMARK. N. M. Khan has independently proved that if a variety $\mathcal{V}$ admits a heterotypical identity for which one side has no repeated variable, then $\mathcal{V}$ is saturated.

The saturated commutative varieties form a sublattice of the lattice of all commutative varieties [14, Theorem 5]. The characterizations given in Result 7 (2) and Theorem 15 (iii) allow the method of [14] to be used to prove that the saturated heterotypical varieties form a sublattice of the lattice of all heterotypical varieties.

4. Saturated homotypical varieties

In general it is not true that condition (S) of Theorem 6 is sufficient to ensure that a homotypical variety $\mathcal{V}$ is saturated as the author has found an example of a band which is not saturated [15]. In our next theorem we consider a condition (T), more restrictive than (S), which does ensure that a homotypical variety is saturated. The proof leads us to characterise several types of varieties which consist of nilpotent extensions of semigroups from a particular saturated class of semigroups $\mathcal{C}$. By Theorem 8, these varieties are saturated.

Finally we show that a condition (S'), less restrictive than (T), ensures that a homotypical variety $\mathcal{V}$ is saturated. However the approach here is manipulative and does not reveal the structural information of the previous theorems of this section.

THEOREM 16. *A sufficient condition for a homotypical variety $\mathcal{V}$ to be saturated is* (T) *$\mathcal{V}$ admits an identity ϕ: $x_1x_2\cdots x_n = f(x_1, x_2,\dots,x_n)$ for which $|x_i|_f > 1$ for some $1 \leqslant i \leqslant n$, and such that f neither begins with x_1 nor ends with x_n.*

PROOF. Suppose $\mathcal{V}$ is a homotypical variety satisfying condition (T). By Corollary 11, it suffices to show that S^n is saturated. We will in fact show that S^n is a semilattice of groups and so is certainly saturated [16, Chapter VII, Theorem 2.14].

Take any $a \in S^n$, by Lemma 12 there exist $c, d \in S$, $e \in E(S)$ such that $a = ced$. We show that $a = ea = ae$. Since $n \geqslant 2$ we may write $a = ce^{n-2}(ed)$. Then $f(c, e, e,\dots, ed)$ begins with e, since f does not begin with x_1, and hence $a = ea$. Dually $a = (ce)e^{n-2}d$ and $f(ce, e, e,\dots,d)$ ends in e, so $a = ae$ as claimed.

Now let x_i be a variable which appears at least twice, say k times, in $f(x_1, x_2,\dots,x_n)$. Then $a = f(e, e,\dots,a, e,\dots,e)$, where a occurs in the ith position, and this in turn yields $a = a^k$ for all $a \in S^n$. Hence S^n is a union of groups, and so is a semilattice completely simple semigroups [16, Chapter 4, Theorem 1.7]. It remains only to show that if T is a completely simple semigroup satisfying ϕ, then T is a group. Suppose $e, f \in E(T)$ and hat $e\mathcal{L}f$, so that $e = ef = ef^{n-1}$. Then $e = f(e, f, f,\dots,f)$, which is a product in e and f beginning with f, and so equals f. Hence $e = f$, and dually, if $e\mathcal{R}f$, $e, f \in E(T)$, then $e = f$. Therefore T has a unique idempotent, and so is a group.

In view of this result it is natural to ask what varieties consist entirely of nilpotent extensions of semilattices of groups. Of course, by Theorem 8, all such varieties are saturated.

LEMMA 17. *Let $\mathcal{V}$ be a saturated variety. If $S \in \mathcal{V}$ is an inverse semigroup then S is a semilattice of groups.*

PROOF. If S were not a semilattice of groups then it would follow that the five element combinatorial Brandt semigroup, B_2 is a member of $\mathcal{V}$ [10, Result 5]. However, B_2 has a properly epimorphically embedded subsemigroup, U [18, Example 3.1], and since $U \in \mathcal{V}$ we have a contradiction.

REMARK. The above proof was told to the author by T. E. Hall. We denote the product of n distinct variables $x_1x_2 \cdots x_n$ by X_n.

THEOREM 18. *For any semigroup variety $\mathcal{V}$ the following are equivalent.*

(i) *$\mathcal{V}$ admits identities of the form $X_n = X_n^{1+m}$, $X_n^m Y_n^m = Y_n^m X_n^m$.*

(ii) *There exists a positive integer n such that S^n is a semilattice of groups for all $S \in \mathcal{V}$.*

(iii) *$\mathcal{V}$ consists entirely of nilpotent extensions of inverse semigroups.*

PROOF. Clearly we have (ii) implies (iii) while the reverse implication follows from Theorem 8, Lemma 17 and the argument used in Corollary 14 to show that the indices of the nilpotent extensions are bounded by some integer n.

Observe that a semigroup variety consists entirely of semilattice of groups if and only if it admits identities of the form $x = x^{1+m}$ and $x^m y^m = y^m x^m$. From this it is clear that (i) implies (ii) while the reverse implication also follows from this observation together with Lemma 13.

A special case of the above result allows us interesting characterization.

THEOREM 19. *For any semigroup variety $\mathcal{V}$ the following are equivalent.*

(i) *$\mathcal{V}$ admits identities of the form $X_n = X_n^{1+m}$ and ψ: $x_1x_2 \cdots x_k = x_{1\pi}x_{2\pi} \cdots x_{k\pi}$, where $k > 1$ and π is a permutation on $\{1, 2, \ldots, k\}$, such that $1\pi \neq 1$, $k\pi \neq k$.*

(ii) *$\mathcal{V}$ satisfies condition (S) and admits a permutation identity as in* (i).

(iii) *There exists an integer n such that S^n is a commutative semilattice of the groups.*

(iv) *$\mathcal{V}$ consists entirely of nilpotent extensions of commutative semilattices of groups.*

PROOF. (i) implies (ii). This is immediate.

(ii) implies (iii). By Lemma 12, if $a \in S^n$ there exist $c, d \in S$, $e \in E(S)$ such that $a = ced = ce^{k-1}d$. By applying ψ to ce^{k-1} and $e^{k-1}d$ in turn we conclude that $a = ea = ae$. As in the proof of Theorem 16, we conclude that S^n is a semilattice of completely simple semigroups and that these completely simple semigroups are groups. These groups also satisfy ψ, and so are abelian. Hence S^n is a semilattice of abelian groups and hence is commutative as it is a strong semilattice of abelian groups [16, IV, Theorem 2.1].

(iii) is equivalent to (iv). This follows as in Theorem 18.

(iii) implies (i). From (ii) implies (i) in Theorem 18, we have that $\mathcal{V}$ admits an identity $X_n = X_n^{1+m}$ for some $m, n \geqslant 1$. Further, since S^n is commutative for all $S \in \mathcal{V}$, we have $\mathcal{V}$ admits the identity $x_1x_2 \cdots x_{2n} = x_{n+1}x_{n+2} \cdots x_{2n}x_1x_2 \cdots x_n$, which is of the form required.

A corollary of the result, via Theorem 8, is that if a variety $\mathcal{V}$ satisfies condition (S) and admits a permutation identity of the form in Theorem 18, then $\mathcal{V}$ is saturated. N. M. Khan has shown [20, Theorem 3.1] that this remains true for an arbitrary, non-trivial permutation identity.

Finally we characterize a class of saturated varieties which include all those of Theorems 18 and 19.

THEOREM 20. *For any semigroup variety $\mathcal{V}$ the following are equivalent.*

(i) *$\mathcal{V}$ admits identities of the form $X_n = X_n^{1+m}$, $X_n^mY_n^mZ_n^mW_n^m = X_n^mZ_n^mY_n^mW_n^m$.*

(ii) *There exists an integer n such that S^n is a semilattice of rectangular groups with normal subband.*

(iii) *$\mathcal{V}$ consists entirely of nilpotent extensions of generalised inverse semigroups.*

REMARK. A generalised inverse semigroup is a regular semigroup whose idempotents form a normal band. Generalised inverse semigroups are saturated [13, Theorem 4].

PROOF. (i) implies (ii). By the proof of Corollary 14 there exists an integer n such that S^n is completely regular for all $S \in \mathcal{V}$. The second identity implies that the idempotents of S^n form a normal subband and that each completely simple subsemigroup of S is orthodox, and so is a rectangular group [4, Section 3.2, Exercise 2(b)].

(ii) implies (i). Since S^n is completely regular for each $S \in \mathcal{V}$, we have by Corollary 14 that $\mathcal{V}$ admits the identity $X_n = X_n^{1+m}$ for some $m \geqslant 1$. That the second identity is admitted by $\mathcal{V}$ follows from the fact that the idempotents of each $S \in \mathcal{V}$ satisfy the identity $xyzw = xzyw$.

(ii) implies (iii) is immediate.

(iii) implies (ii). Once more, by the argument of Corollary 14, there exists an integer n such that S^n is a generalised inverse semigroup for all $S \in \mathcal{V}$. Moreover, the maximum inverse semigroup image of S^n is completely regular by Lemma 17. It follows from the Hall-Yamada characterization of orthodox semigroups [16, Chapter VI, Theorem 4.6] that S^n is also completely regular and so S^n is a semilattice of rectangular groups.

We next prove a technical lemma which is based on an argument due to N. M. Khan [19, Theorem 3.1], but our hypotheses are much more general than those of the original.

LEMMA 21. *Let U be a proper subsemigroup of a semigroup S. Let $x, y \in S$ such that for any integer m there exist $a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_m \in U$ and $x', y' \in S$ such that $x = x'a_1a_2 \cdots a_m$, $y = b_1b_2 \cdots b_my'$. Suppose that for all $x, y \in S$ as above, and all $u, v \in U$ the following conditions are satisfied*:

(i) *$xuy = xuwuy$ for some $w \in U^1$,*

(ii) *$xuvy = xvs(u)y = xs'(v)uy$ where $s(u)$ $[s'(v)]$ is a product of members of U containing u $[v]$.*

Then U is not epimorphically embedded in S.

PROOF. We repeatedly use the fact that if x and y are members of S satisfying the above factorization condition then so are xa and by for all $a, b \in U$. This will be done without comment.

Suppose to the contrary that $\mathrm{Dom}(U, S) = S$. Let $d \in S \setminus U$ and take a zigzag in S over U with value d of minimum length m as in Result 1 (the minimality of m implies that $x_i, y_i \in S \setminus U$ for all $i = 1, 2, \ldots, m$). Then

$$d = x_1u_1y_1 = x_1u_1w_1u_1y_1,$$

as $x_1, y_1 \in S \setminus U$ and so satisfy the factorization condition by Result 1 and the assumption that $\mathrm{Dom}(U, S) = S$, whence the second line follows by hypothesis (i) for some $w_1 \in U^1$. Hence we have

$$\begin{aligned} d &= x_1u_1w_1u_2y_2 \qquad \text{(by the zigzag equations)} \\ &= x_1u_2s_1(u_1)y_2, \end{aligned}$$

by hypothesis (ii) applied to $x_1(u_1w_1)u_2y_2$. (Strictly, we might write $s_1(u_1w_1)$ but it is enough to have that u_1 occurs in the product $s_1(u_1w_1)$ so we abbreviate to $s_1(u_1)$.)

Hence we have

$$\begin{aligned} d &= x_2u_3s_1(u_1)y_2 && \text{(by the zigzag equations)} \\ &= x_2u_3w_2u_3s_1(u_1)y_2 && \text{(by applying (i) to } x_2u_3(s_1(u_1)y_2)) \\ &= x_2u_3w_2s_2'(s_1(u_1))u_3y_2 && \text{(by applying the second part of (ii)} \\ & && \text{to } (x_2u_3w_2)u_3s_1(u_1)y_2). \end{aligned}$$

We amalgamate the product $w_2s_2'(s_1(u_1))$ notationally by writing simply $s_2(u_1)$. We continue working down the zigzag,

$$\begin{aligned} d &= x_2u_3s_2(u_1)u_4y_3 \\ &= x_2u_4s_3(u_1, u_3)y_3 \quad \text{for some } s_3(u_1, u_3) \in U \end{aligned}$$

such that both u_1 and u_3 occur in a factorization of the product $s_3(u_1, u_3)$. Again (ii) is used to establish this. Eventually we reach

$$d = x_mu_{2m-1}s_m(u_1, u_3, u_5, \ldots, u_{2m-3})u_{2m-1}y_m$$

ending the first stage of the argument.

Now going up the zigzag we get

$$\begin{aligned} d &= x_{m-1}u_{2m-2}s_m(u_1, u_3, u_5, \ldots, u_{2m-3})u_{2m-1}y_m \\ &= x_{m-1}u_{2m-3}t_{m-1}(u_1, u_3, u_5, \ldots, u_{2m-5})u_{2m-1}y_m \end{aligned}$$

where $t_{m-1} \in U$ which allows a factorization in which each of $u_1, u_3, u_5, \ldots, u_{2m-5}$ occur. To obtain this line we again have employed the commutativity hypothesis.

Continuing we get

$$\begin{aligned} d &= x_{m-2}u_{2m-4}t_{m-1}u_{2m-1}y_m \\ &= x_{m-2}u_{2m-5}t_{m-2}(u_1, u_3, u_5, \ldots, u_{2m-7})u_{2m-1}y_m \\ &\;\;\vdots \\ &= x_1u_1t_1a_{2m-1}y_m \quad \text{where } t_1 \in U \\ &= u_0t_1u_{2m} \in U, \quad \text{a contradiction as required.} \end{aligned}$$

LEMMA 22. *Let $\mathcal{V}$ be a variety satisfying condition (S). Then there exists $k \geqslant 1$ such that for any $U \in \mathcal{V}$, $u \in U$ and $e \in E(U)$, $eue = (eue)^{1+rk}$ for all $r = 0, 1, 2, \ldots$.*

PROOF. The semigroup U satisfies an identity of the form $x_1x_2\cdots x_n = f(x_1, x_2, \ldots, x_n)$ with $|x_i|_f > 1$ for some i. Substituting eue for x_i and e for all other variables in the identity gives $eue = (eue)^{1+k}$ where $k = |x_i|_f - 1$, from which the result follows.

LEMMA 23. *Suppose that U is a subsemigroup of a semigroup S and that U satisfies the identity*

$$x_1x_2\cdots x_m = f_1(x_1, x_2,\dots,x_i)f_2(x_{i+1}, x_{i+2},\dots,x_m) \qquad \text{where } 0 \leqslant i < m-1,$$

and that f_2 does not begin with x_{i+1}. Then for all $x \in S$ satisfying the factorization condition of Lemma 21, and all $c \in U$, $e \in E(U)$, $xce = xece$.

PROOF. We assume $i \geqslant 1$, since the $i = 0$ case is similar. Factorize x over U as $x = x'u_1u_2\cdots u_i = x'u'$ say. Then

$$\begin{aligned} xce &= x'u'ce = x'f_1(u_1, u_2,\dots,u_i)f_2(c, e, e,\dots,e)\\ &= x'f_1(u_1, u_2,\dots,u_i)f_2(ec, e, e,\dots,e)\\ &\qquad\qquad\qquad\qquad (\text{because } f_2 \text{ does not begin with } x_{i+1})\\ &= x_1'u'ece = xece \qquad \text{as required.} \end{aligned}$$

LEMMA 24. *Supose $\mathcal{V}$ is a variety satisfying condition (S) and admitting an identity of the form*

$$\begin{aligned} x_1x_2\cdots x_m &= f_1(x_1, x_2,\dots,x_i)f_2(x_{i+1},\dots,x_m)\\ &= g_1(x_1, x_2,\dots,x_j)g_2(x_{j+1}\cdots x_m) \end{aligned}$$

with $0 \leqslant i < m-1$, $1 < j \leqslant m$, such that f_2 does not begin with x_{i+1} and g_1 does not end in x_j. (Note g_1g_2 is a factorization of f_1f_2 and not a different word.) Let $U \in \mathcal{V}$ and U be epimorphically embedded in a semigroup S. Then for all $x, y \in S\backslash U$ and $u \in U^n$ there exists $e \in E(U)$ such that $xuy = xu(eue)^{k-1}uy$ for some $k > 1$.

PROOF. By Lemma 12 there exist $c, d \in U$, $e \in E(U)$ such that $u = ced$. Hence

$$\begin{aligned} xuy &= xcedy = xecedy \qquad (\text{by Lemma 23})\\ &= xecedey \qquad (\text{by the dual of 23})\\ &= xeuey = x(eue)^{1+k}y \qquad (\text{for some } k > 1, \text{ by Lemma 22})\\ &= xeu(eue)^{k-1}uey = xu(eue)^{k-1}uy\\ &\qquad\qquad\qquad\qquad (\text{by the reverse of 23 and its dual}). \end{aligned}$$

LEMMA 25. *Let $\mathcal{V}$, S and U be as in 24. Let $x, y \in S\backslash U$ and $u, v \in U^n$. Then $xuvy = xvs(u)y = xs'(v)uy$ for some $s(u), s'(v) \in U^n$ such that u occurs in some factorization of $s(u)$ over U, and v occurs in some factorization of $s'(v)$ over U.*

PROOF. By Lemma 12 there exist $c, d \in U$, $e \in E(U)$ such that $v = ced$, whence

$$\begin{aligned} xuvy = xucedy &= xeucedy \qquad \text{(by 23)} \\ &= xeucedey \qquad \text{(by the dual of 23)} \\ &= xeuecedey \qquad \text{(by 23)} \\ &= xeuevey. \end{aligned}$$

Now, for the identity $x_1x_2 \cdots x_m = f_1(x_1, x_2, \ldots, x_i)f_2(x_{i+1}, \ldots, x_m)$ we have, by hypothesis, that the first variable of the word f_2 is x_k say where $k \neq i + 1$. Now we have

$$\begin{aligned} eueve &= f_1(e, e, \ldots, e)f_2(x_{i+1} = eu, x_k = ev, x_l = e \text{ for all } l \neq i+1, k) \\ &= evs(eu) \end{aligned}$$

where $s(eu) \in U$ affords a factorization in which eu occurs at least once. Hence $xuvy = xeuevey = xevs(eu)y = xvs(eu)y$ (using the reverse of 23 to absorb the idempotent e into $s(eu)$). Therefore $xuvy = xvs(u)y$, where $s(u) = s(eu)$, as required. The second part of the lemma follows by the dual argument.

Collecting these lemmas together gives us our main result of this section.

THEOREM 26. *A sufficient condition for a variety $\mathcal{V}$ to be saturated is*
(S') $\mathcal{V}$ satisfies condition (S) of Theorem 26 and admits an identity of the form

$$(*) \qquad \begin{aligned} x_1x_2 \cdots x_m &= f_1(x_1, x_2, \ldots, x_i)f_2(x_{i+1}, \ldots, x_m) \\ &= g_1(x_1, x_2, \ldots, x_j)g_2(x_{j+1}, \ldots, x_m) \end{aligned}$$

with $0 \leq i < m - 1$, $1 < j \leq m$, such that f_2 does not begin with x_{i+1} and g_1 does not end with x_j.

PROOF. Let $U \in \mathcal{V}$ and suppose U is properly epimorphically embedded in a semigroup S. Then, as in the proof of Theorem 8, we have that U^n is properly epimorphically embedded in the subsemigroup of S generated by $S \backslash U \cup U^n$. However Lemmas 24 and 25 allow us to invoke Lemma 21, and hence derive a contradiction.

COROLLARY. 27. *A variety $\mathcal{V}$ admitting a non-trivial permutation identity is saturated if and only if it satisfies condition (S).*

PROOF. Any non-trivial permutation identity is of the kind described in Theorem 26 whence the result follows.

REMARK. N. M. Khan first proved that permutative varieties satisfying condition (S) are saturated in [20, Theorem 31].

The following corollary is also a particular case of the Theorem 26, but does provide a clear generalisation of Theorem 16 which is the case where $i = 0, j = n$.

COROLLARY 28. *Let $\mathcal{V}$ be a variety admitting an identity of the form*

$$x_1 x_2 \cdots x_n = f_1(x_1, x_2, \ldots, x_i) f_2(x_{i+1}, \ldots, x_j) f_3(x_{j+1}, \ldots, x_n)$$

where $0 \leqslant i$, $i + 1 < j \leqslant n$, such that the right hand side contains a repeated variable and f_2 does not begin with x_{i+1} nor end in x_j. Then $\mathcal{V}$ is saturated.

Acknowledgement

I thank my supervisor, Tom Hall, for his sustained help in the preparation of this paper.

References

[1] A. Bulaszewska and J. Krempa, 'On epimorphisms in the category of all associative rings', *Bull Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys.* **23** (1975).

[2] W. Burgess, 'The meaning of mono and epi in some familiar categories', *Canad. Math. Bull.* **8** (1965).

[3] J. L. Chrislock, 'A certain class of identities on semigroups,' *Proc. Amer. Math. Soc.* **21** (1969), 189–190.

[4] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys No. 7 (Amer. Math. Soc., Providence, R.I., Vol. I, 1961; Vol II, 1967).

[5] K. Drbohlav, 'A note on epimorphisms in algebraic categories', *Comment. Math. Univ. Carolinae* **4** (1963).

[6] T. Evans, 'An embedding theorem for semigroups', *Amer. J. Math.* **76** (1954), 399–413.

[7] B. J. Gardner, 'Epimorphisms of regular rings', *Comment. Math. Univ. Carolinae* **16** (1975).

[8] B. J. Gardner, 'A note on ring epimorphisms and polynomial identities', *Comment. Math. Univ. Carolinae* **20** (1979).

[9] B. J. Gardner, 'Some aspects of T-nilpotence', *Pacific J. Math.* **53**, (1974).

[10] T. E. Hall, 'Inverse semigroup varieties with the amalgamation property', *Semigroup Forum* **16** (1978), 37–51.

[11] T. E. Hall and P. R. Jones, 'Epis are onto for finite regular semigroups', *Proc. Edinburgh Math. Soc.*, to appear.

[12] P. M. Higgins, 'The determination of all absolutely closed varieties of semigroups', *Proc. Amer. Math. Soc.*, to appear.

[13] P. M. Higgins, 'Epis are onto for generalised inverse semigroups', *Semigroup Forum* **23** (1981), 255–259.

[14] P. M. Higgins, 'The commutative varieties of semigroups for which epis are onto', *Proc. Edinburgh Math. Soc.*, to appear.

[15] P. M. Higgins, 'A semigroup with an epimorphically embedded subband', submitted.

[16] J. M. Howie, *An introduction to semigroup theory*, London Math. Soc. Monographs 7 (Academic Press, 1976).

[17] J. M. Howie and J. R. Isbell, 'Epimorphisms and dominions II', *J. Algebra* **6** (1967), 7–21.

[18] J. R. Isbell, 'Epimorphisms and dominions', *Proceedings of the Conference on Categorical Algebra*, La Jolla, 1965 (Lange and Springer, Berlin, 1966, 232–246).

[19] N. M. Khan, 'Epimorphisms, dominions and varieties of semigroups', *Semigroup Forum* **25** (1982), 331–337.

[20] N. M. Khan, 'On saturated and epimorphically closed permutative varieties and consequences of permutation identities', submitted.

[21] W. D. Munn, 'Semigroups satisfying mininal conditions', *Proc. Glasgow Math. Assoc.* **3** (1957), 145–152.

[22] T. Tamura, 'The theory of construction of finite semigroups III. Finite unipotent semigroups', *Osaka Math. J.* **10** (1958), 191–204.

Department of Mathematics
Monash University
Clayton, Victoria 3168
Australia

https://doi.org/10.1017/S1446788700024629 Published online by Cambridge University Press